Finite Horizon Robustness Analysis of LTV Systems Using Integral Quadratic Constraints

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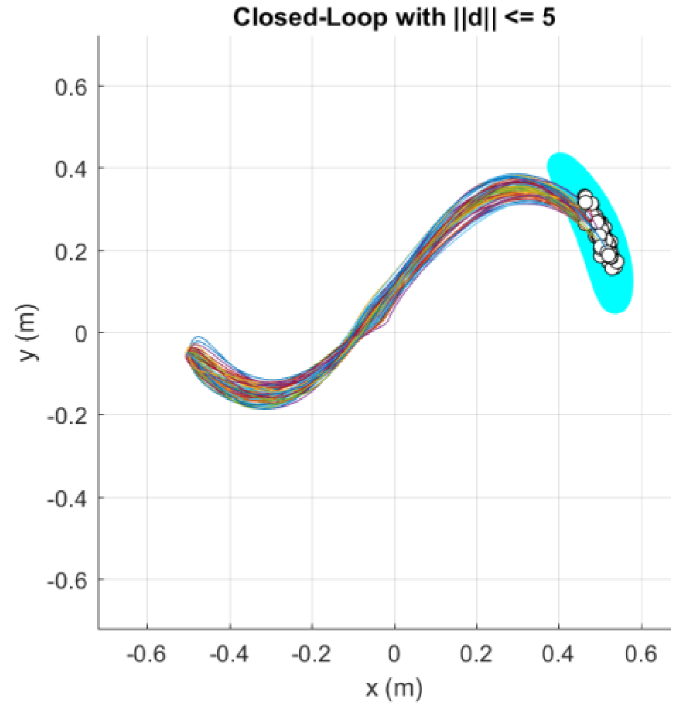
Analysis Objective

Goal: Assess the robustness of linear time-varying (LTV) systems on finite horizons.

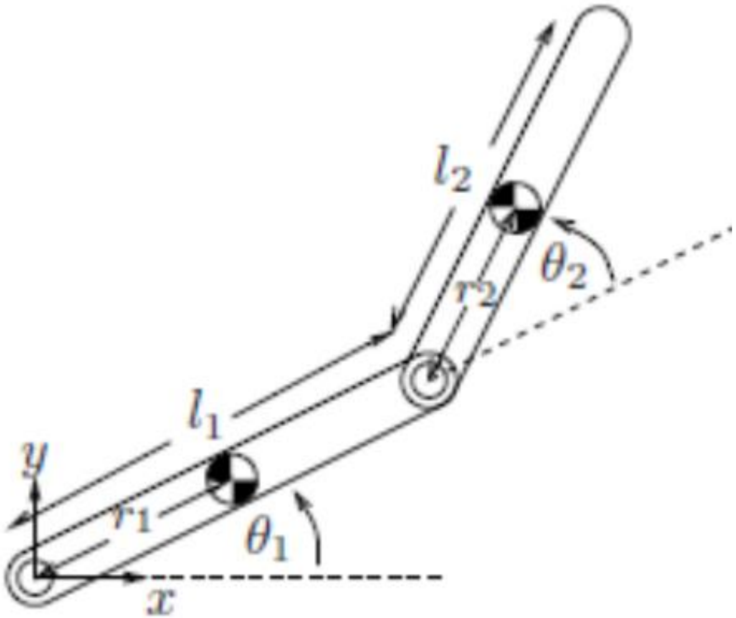
Approach: Classical Gain/Phase Margins focus on (infinite horizon) stability and frequency domain concepts.

Instead focus on:

- Finite horizon metrics, e.g. induced gains and reachable sets.
- Effect of disturbances and model uncertainty (D-scales, IQCs, etc).
- Time-domain analysis conditions.



Two-Link Robot Arm



Two-Link Diagram [MZS]

Nonlinear dynamics [MZS]:

$$\dot{\eta} = f(\eta, \tau, d)$$

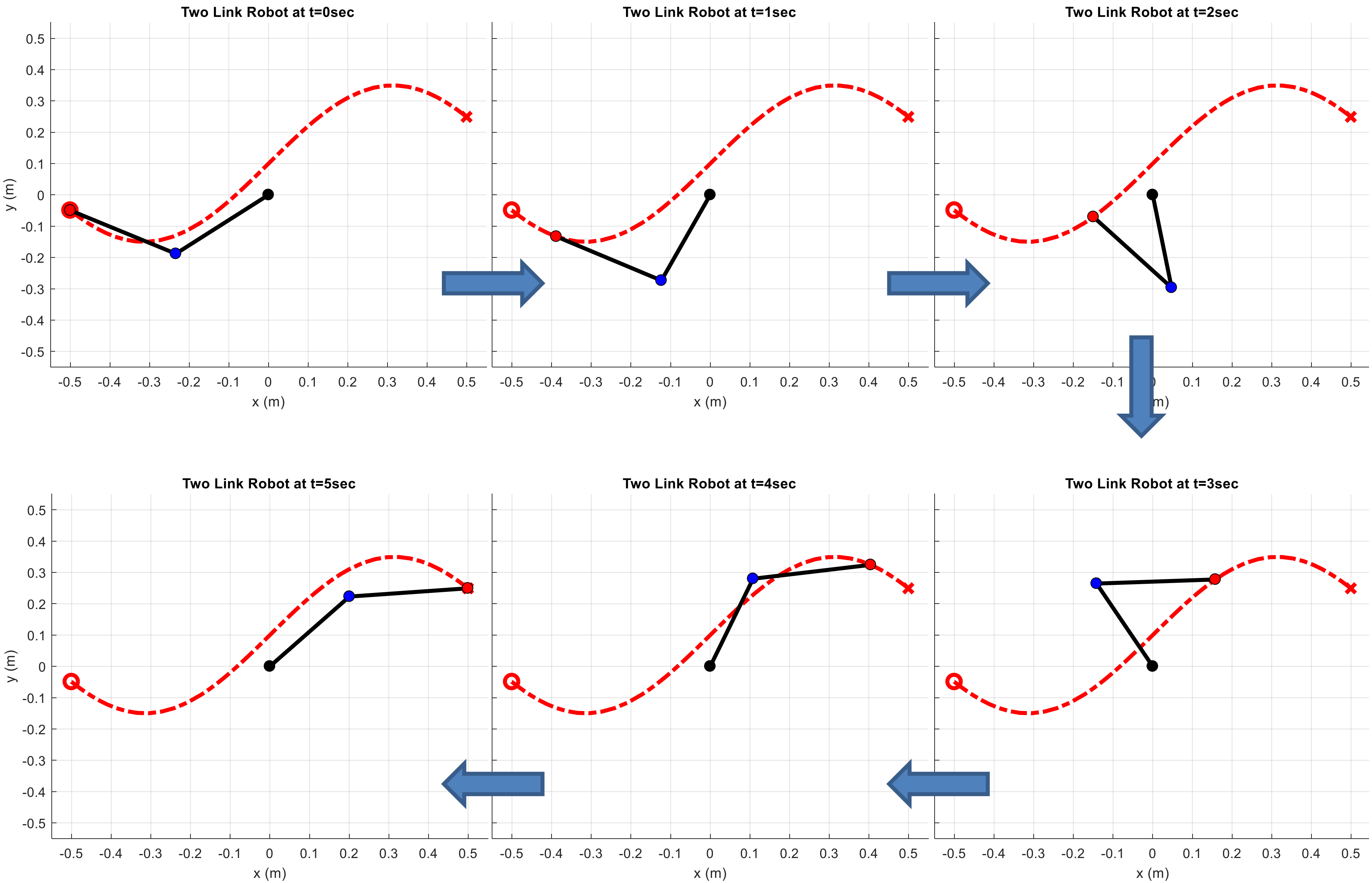
where

$$\eta = [\theta_1, \dot{\theta}_1, \theta_2, \dot{\theta}_2]^T$$
$$\tau = [\tau_1,\ \tau_2\ ]^T$$
$$d = [d_1,\ d_2\ ]^T$$

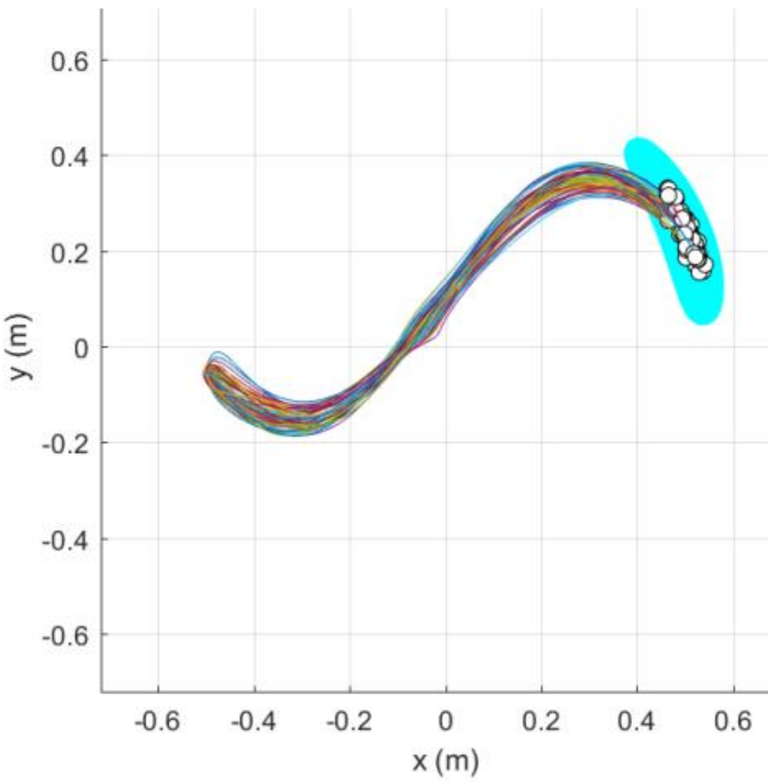
τ and d are control torques and disturbances at the link joints.

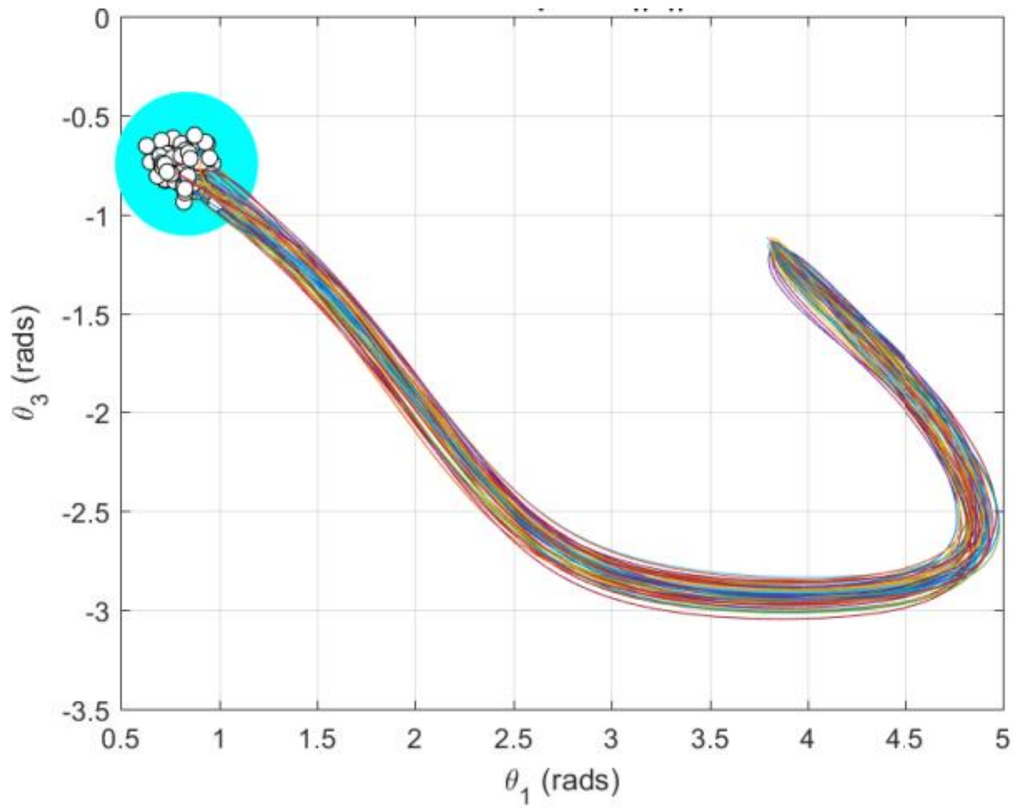
[MZS] R. Murray, Z. Li, and S. Sastry. *A Mathematical Introduction to Robot Manipulation*, 1994.

Nominal Trajectory (Cartesian Coords.)



Effect of Disturbances / Uncertainty





Cartesian Coords.

Joint Angles

Overview of Analysis Approach

Nonlinear dynamics:

$$\dot{\eta} = f(\eta, \tau, d)$$

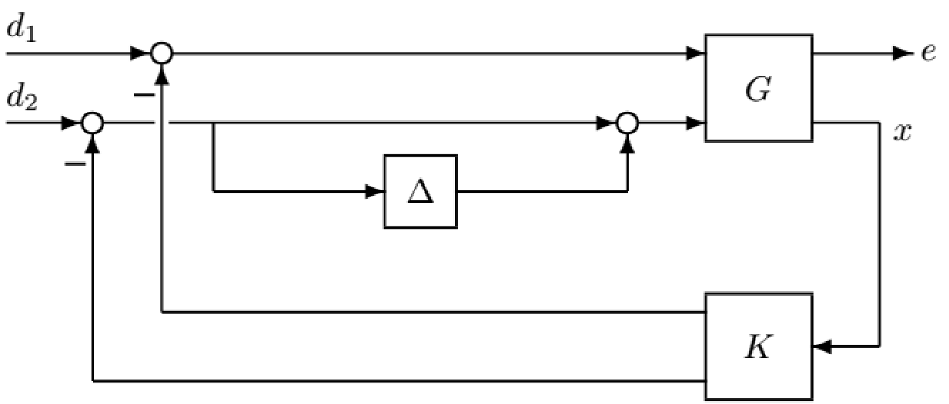
Linearize along a (finite –horizon) trajectory $(\bar{\eta}, \bar{\tau}, d = 0)$

$$\dot{x} = A(t)x + B(t)u + B(t)d$$

Compute bounds on the terminal state *x(T)* or other quantity *e(T) = C x(T)* accounting for disturbances and uncertainty.

Comments:

- The analysis can be for open or closed-loop.
- LTV analysis complements the use of Monte Carlo simulations.



Outline

- Nominal LTV Performance
- Robust LTV Performance
- Examples
- Conclusions

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Finite-Horizon LTV Performance

Finite-Horizon LTV System *G* defined on [0,T]

$$\dot{x}(t) = A(t)x(t) + B(t)d(t)$$
$$e(t) = C(t)x(t) + D(t)d(t)$$

Induced L_2 Gain

$$\|G\|_{2,[0,T]} := \sup \left\{ \frac{\|e\|_{2,[0,T]}}{\|d\|_{2,[0,T]}} \;\middle|\; x(0) = 0, 0 \neq d \in \mathcal{L}_{2,[0,T]} \right\}$$

L_2-to-Euclidean Gain

$$\|G\|_{E,[0,T]} := \sup \left\{ \frac{\|e(T)\|_2}{\|d\|_{2,[0,T]}} \;\middle|\; x(0) = 0, 0 \neq d \in \mathcal{L}_{2,[0,T]} \right\}$$

The L_2-to-Euclidean gain requires *D(T)=0* to be well-posed.

The definition can be generalized to estimate ellipsoidal bounds on the reachable set of states at *T.*

General (Q,S,R,F) Cost

Cost function *J* defined by *(Q,S,R,F)*

$$J(d) := x(T)^T F x(T) + \int_0^T \begin{bmatrix} x(t) \\ d(t) \end{bmatrix}^T \begin{bmatrix} Q(t) & S(t) \\ S(t)^T & R(t) \end{bmatrix} \begin{bmatrix} x(t) \\ d(t) \end{bmatrix} dt$$

Subject to: LTV Dynamics with *x(0)=0*

Example: Induced L_2 Gain

Select *(Q,S,R,F)* as:

$$Q(t) := C(t)^T C(t),\ S(t) := C(t)^T D(t),\ R(t) := D(t)^T D(t) - \gamma^2 I_{n_d},\ \text{and } F := 0.$$

Cost Function *J* is:

$$J(d) = \|e\|^2_{2,[0,T]} - \gamma^2 \|d\|^2_{2,[0,T]}$$

⇒ $J(d) \leq 0$ for all $d \in \mathcal{L}_2[0,T]$ if and only if $\|G\|_{2,[0,T]} \leq \gamma$.

General (Q,S,R,F) Cost

Cost function *J* defined by *(Q,S,R,F)*

$$J(d) := x(T)^T F x(T) + \int_0^T \begin{bmatrix} x(t) \\ d(t) \end{bmatrix}^T \begin{bmatrix} Q(t) & S(t) \\ S(t)^T & R(t) \end{bmatrix} \begin{bmatrix} x(t) \\ d(t) \end{bmatrix} dt$$

Subject to: LTV Dynamics with *x(0)=0*

Example: L_2-to-Euclidean Gain

Select *(Q,S,R,F)* as:

$$Q(t) := 0,\ S(t) := 0,\ R(t) := -\gamma^2 I_{n_d},\ \text{and}\ F := C^T(T)C(T).$$

Cost Function *J* is:

$$J(d) = \|e(T)\|_2^2 - \gamma^2 \|d\|_{2,[0,T]}^2$$

⇒ $J(d) \leq 0$ for all $d \in \mathcal{L}_2[0,T]$ if and only if $\|G\|_{E,[0,T]} \leq \gamma$.

Strict Bounded Real Lemma

Theorem 1. *Assume $R(t) \prec 0$ for all $t \in [0,T]$. The following are equivalent:*

1. $\exists \epsilon > 0$ *such that* $J(d) \le -\epsilon \|d\|^2_{2,[0,T]}$ $\forall d \in \mathcal{L}_2[0,T]$.
2. *There exists a differentiable function Y on $[0,T]$ such that $Y(T) = F$ and*

$$\dot{Y} + A^T Y + YA + Q - (YB+S)R^{-1}(YB+S)^T = 0$$

This is a **Riccati Differential Equation (RDE)**.

3. *There exists $\epsilon > 0$ and a differentiable function P on $[0,T]$ such that $P(T) \succeq F$ and*

$$\dot{P} + A^T P + PA + Q - (PB+S)R^{-1}(PB+S)^T \preceq -\epsilon I$$

This is a strict **Riccati Differential Inequality (RDI)**.

This is a generalization of results contained in:

*Tadmor, Worst-case design in the time domain. *MCSS*, 1990 .

*Ravi, Nagpal, and Khargonekar. H_∞ control of linear time-varying systems. *SIAM JCO*, 1991.

*Green and Limebeer. *Linear Robust Control*, 1995.

*Chen and Tu. The strict bounded real lemma for linear time-varying systems. JMAA, 2000.

Proof: 3→1

By Schur complements, the RDI is equivalent to:

$$\begin{bmatrix} \dot{P} + A^T P + PA & PB \\ B^T P & 0 \end{bmatrix} + \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \preceq -\tilde{\epsilon} I$$

This is an LMI in *P*. It is also equivalent to a dissipation inequality with the storage function $V(x,t) := x^T P(t) x$.

$$\dot{V} + \begin{bmatrix} x \\ d \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} x \\ d \end{bmatrix} \leq -\tilde{\epsilon} \begin{bmatrix} x \\ d \end{bmatrix}^T \begin{bmatrix} x \\ d \end{bmatrix}$$

Integrate from *t=0* to *t=T*:

$$V(x(T),T) - V(x(0),0) + \int_0^T \begin{bmatrix} x(t) \\ d(t) \end{bmatrix}^T \begin{bmatrix} Q(t) & S(t) \\ S(t)^T & R(t) \end{bmatrix} \begin{bmatrix} x(t) \\ d(t) \end{bmatrix} dt \leq -\tilde{\epsilon} \left\| \left[\begin{smallmatrix} x \\ d \end{smallmatrix} \right] \right\|_{2,[0,T]}^2$$

Apply *x(0)=0* and *P(T)≥F:*

$$J(d) \leq -\epsilon \|d\|_{2,[0,T]}^2$$

Strict Bounded Real Lemma

Theorem 1. *Assume $R(t) \prec 0$ for all $t \in [0,T]$. The following are equivalent:*

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3. *There exists $\epsilon > 0$ and a differentiable function P on $[0,T]$ such that $P(T) \succeq F$ and*

$$\dot{P} + A^T P + PA + Q - (PB+S)R^{-1}(PB+S)^T \preceq -\epsilon I$$

   This is a strict **Riccati Differential Inequality (RDI)**.

Comments:

*For nominal analysis, the RDE can be integrated. If the solution exists on [0,T] then nominal performance is achieved. This typically involves bisection, e.g. over γ, to find the best bound on a gain.

*For robustness analysis, both the RDI and RDE will be used to construct an efficient numerical algorithm.

Outline

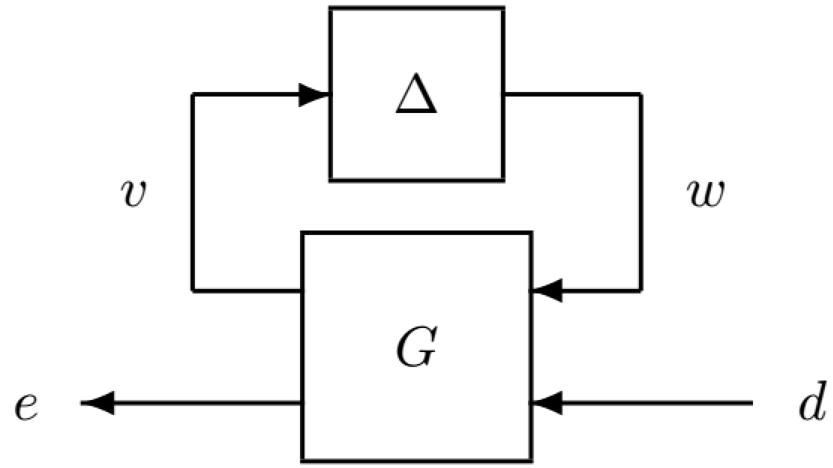
- Nominal LTV Performance
- **Robust LTV Performance**
- Examples
- Conclusions

Uncertainty Model

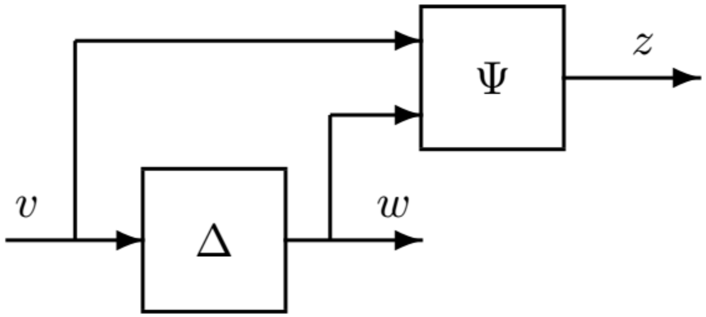
- Standard LFT Model, $F_u(G,\Delta)$, where G is LTV:

$$
\begin{aligned}
\dot{x}_G(t) &= A_G(t)\, x_G(t) + B_{G1}(t)\, w(t) + B_{G2}(t)\ d(t) \\
v(t) &= C_{G1}(t)\, x_G(t) + D_{G11}(t)\, w(t) + D_{G12}(t)\ d(t) \\
e(t) &= C_{G2}(t)\, x_G(t) + D_{G21}(t)\, w(t) + D_{G22}(t)\ d(t)
\end{aligned}
$$

Δ is block structured and used to model parametric / dynamic uncertainty and nonlinear perturbations.



Integral Quadratic Constraints (IQCs)



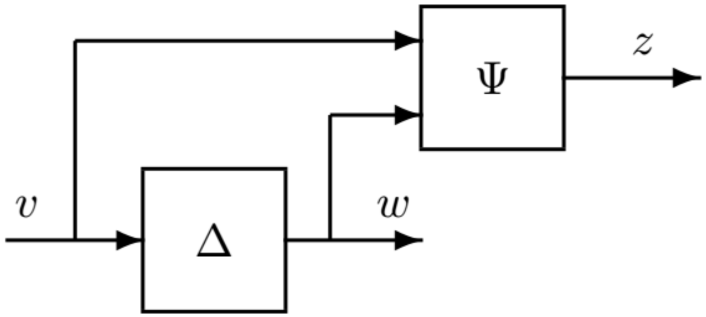
$\int_0^T z(t)^T M(t) z(t)\, dt \geq 0$

Definition 2. *Let* $\Psi \in \mathbb{RH}_\infty^{n_z \times (n_v + n_w)}$ *and* $M : [0,T] \to \mathbb{S}^{n_z}$ *with* M *piecewise continuous. A bounded, causal operator* $\Delta : \mathbf{L}_2^{n_v}[0,T] \to \mathbf{L}_2^{n_w}[0,T]$ *satisfies the time domain IQC defined by* (Ψ, M) *if the following inequality holds for all* $v \in \mathcal{L}_2^{n_v}[0,T]$ *and* $w = \Delta(v)$*:*

$$\int_0^T z(t)^T M(t) z(t)\, dt \geq 0 \qquad (1)$$

where z *is the output of* Ψ *driven by inputs* (v, w) *with zero initial conditions* $x_\psi(0) = 0$.

Integral Quadratic Constraints (IQCs)



$$\int_0^T z(t)^T M(t) z(t)\, dt \geq 0$$

Comments:

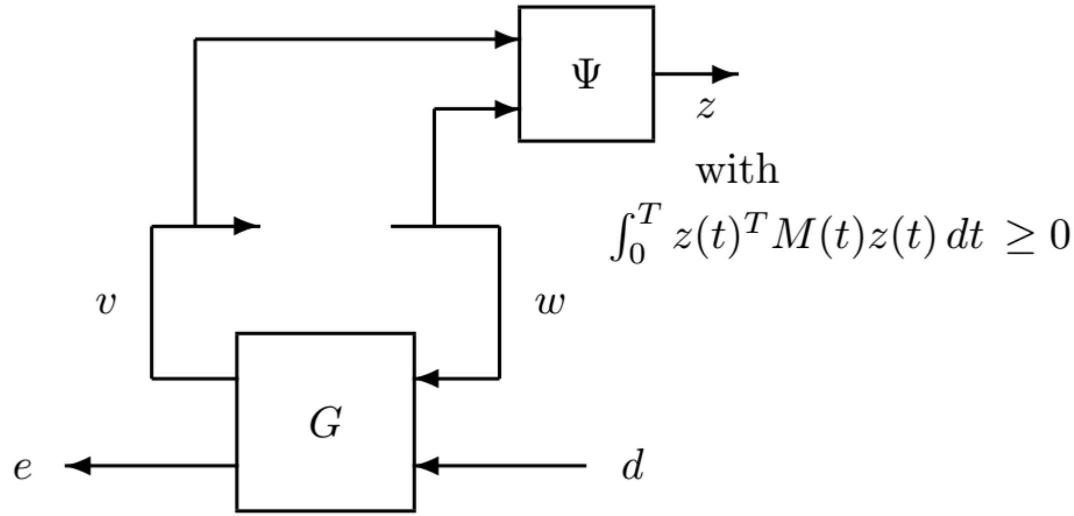
*A library of IQC for various uncertainties / nonlinearities is given in [MR]. Many of these are given as frequency domain inequalities.

*Time-domain IQCs that hold over finite horizons are called hard.

*This generalizes D and D/G scales for LTI and parametric uncertainty. It can be used to model the I/O behavior of nonlinear elements.

[MR] Megretski and Rantzer. System analysis via integral quadratic constraints, TAC, 1997.

Robustness Analysis



The robustness analysis is performed on the extended (LTV) system of *(G, Ψ)* using the constraint on *z*.

$$\begin{bmatrix} \dot{x}(t) \\ z(t) \\ e(t) \end{bmatrix} = \begin{bmatrix} \mathcal{A}(t) & \mathcal{B}_1(t) & \mathcal{B}_2(t) \\ \mathcal{C}_1(t) & \mathcal{D}_{11}(t) & \mathcal{D}_{12}(t) \\ \mathcal{C}_2(t) & \mathcal{D}_{21}(t) & \mathcal{D}_{22}(t) \end{bmatrix} \begin{bmatrix} x(t) \\ w(t) \\ d(t) \end{bmatrix}$$

Robustness Analysis: Induced L_2 Gain

Theorem 3. *Assume Δ satisfies the IQC defined by (Ψ, M). If there exists $\epsilon > 0$, $\gamma > 0$ a differentiable function P on $[0,T]$ and such that $P(T) \succeq 0$ and for all $t \in [0,T]$*

$$\begin{bmatrix} \dot{P} + \mathcal{A}^T P + P\mathcal{A} & P\mathcal{B}_1 & P\mathcal{B}_2 \\ \mathcal{B}_1^T P & 0 & 0 \\ \mathcal{B}_2^T P & 0 & -\gamma^2 I \end{bmatrix} + (\cdot)^T \begin{bmatrix} \mathcal{C}_2 & \mathcal{D}_{21} & \mathcal{D}_{22} \end{bmatrix} + (\cdot)^T M \begin{bmatrix} \mathcal{C}_1 & \mathcal{D}_{11} & \mathcal{D}_{12} \end{bmatrix} \preceq -\epsilon I$$

then $\|F_u(G,\Delta)\|_{2,[0,T]} < \gamma$.

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$$\begin{bmatrix} \dot{P} + \mathcal{A}^T P + P\mathcal{A} & P\mathcal{B}_1 & P\mathcal{B}_2 \\ \mathcal{B}_1^T P & 0 & 0 \\ \mathcal{B}_2^T P & 0 & -\gamma^2 I \end{bmatrix} + (\cdot)^T \begin{bmatrix} \mathcal{C}_2 & \mathcal{D}_{21} & \mathcal{D}_{22} \end{bmatrix} + (\cdot)^T M \begin{bmatrix} \mathcal{C}_1 & \mathcal{D}_{11} & \mathcal{D}_{12} \end{bmatrix} \preceq -\epsilon I$$

then $\|F_u(G,\Delta)\|_{2,[0,T]} < \gamma$.

Proof:

The Differential LMI (DLMI) is equivalent to a dissipation ineq. with storage function $V(x,t) := x^T P(t) x$.

$$\dot{V}(x,t) + z(t)^T M z(t) - (\gamma^2 - \epsilon) d(t)^T d(t) + e(t)^T e(t) \leq 0$$

Integrate and apply the IQC + boundary conditions to conclude that the induced L_2 gain is $\leq\gamma$.

Robustness Analysis: Induced L_2 Gain

Theorem 3. *Assume Δ satisfies the IQC defined by (Ψ, M). If there exists $\epsilon > 0$, $\gamma > 0$ a differentiable function P on $[0,T]$ and such that $P(T) \succeq 0$ and for all $t \in [0,T]$*

$$\begin{bmatrix} \dot{P} + \mathcal{A}^T P + P\mathcal{A} & P\mathcal{B}_1 & P\mathcal{B}_2 \\ \mathcal{B}_1^T P & 0 & 0 \\ \mathcal{B}_2^T P & 0 & -\gamma^2 I \end{bmatrix} + (\cdot)^T \begin{bmatrix} \mathcal{C}_2 & \mathcal{D}_{21} & \mathcal{D}_{22} \end{bmatrix} + (\cdot)^T M \begin{bmatrix} \mathcal{C}_1 & \mathcal{D}_{11} & \mathcal{D}_{12} \end{bmatrix} \preceq -\epsilon I$$

then $\|F_u(G,\Delta)\|_{2,[0,T]} < \gamma$.

Comments:

*A similar result exists for L_2-to-Euclidean or, more generally (Q,S,R,F) cost functions.

*The DLMI can be expressed as a Riccati Differential Ineq. (RDI) by Schur Complements.

*The RDI is equivalent to a related Riccati Differential Eq. (RDE) condition by the strict Bounded Real Lemma.

Robustness Analysis: Induced L_2 Gain

Theorem 3. *Assume* Δ *satisfies the IQC defined by* (Ψ, M)*. If there exists* $\epsilon > 0$, $\gamma > 0$ *a differentiable function* P *on* $[0,T]$ *and such that* $P(T) \succeq 0$ *and for all* $t \in [0,T]$

$$\begin{bmatrix} \dot{P} + \mathcal{A}^T P + P\mathcal{A} & P\mathcal{B}_1 & P\mathcal{B}_2 \\ \mathcal{B}_1^T P & 0 & 0 \\ \mathcal{B}_2^T P & 0 & -\gamma^2 I \end{bmatrix} + (\cdot)^T \begin{bmatrix} \mathcal{C}_2 & \mathcal{D}_{21} & \mathcal{D}_{22} \end{bmatrix} + (\cdot)^T M \begin{bmatrix} \mathcal{C}_1 & \mathcal{D}_{11} & \mathcal{D}_{12} \end{bmatrix} \preceq -\epsilon I$$

then $\|F_u(G,\Delta)\|_{2,[0,T]} < \gamma$.

Comments:

*The DLMI is convex in the IQC matrix *M* but requires gridding on time *t* and parameterization of *P*.

*The RDE form directly solves for *P* by integration (no time gridding) but the IQC matrix *M* enters in a non-convex fashion.

Numerical Implementation

An efficient numerical algorithm is obtained by mixing the LMI and RDE conditions.

Sketch of algorithm:

1. Initialize: Select a time grid and basis functions for *P(t).*
2. Solve DLMI: Obtain finite-dimensional optim. by enforcing DLMI on the time grid and using basis functions.
3. Solve RDE: Use IQC matrix *M* from step 2 and solve RDE. This gives the optimal storage *P* for this matrix *M*.
4. Terminate: Stop if the costs from Steps 2 and 3 are similar. Otherwise return to Step 2 using optimal storage P as a basis function.

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Example 1: LTI Plant

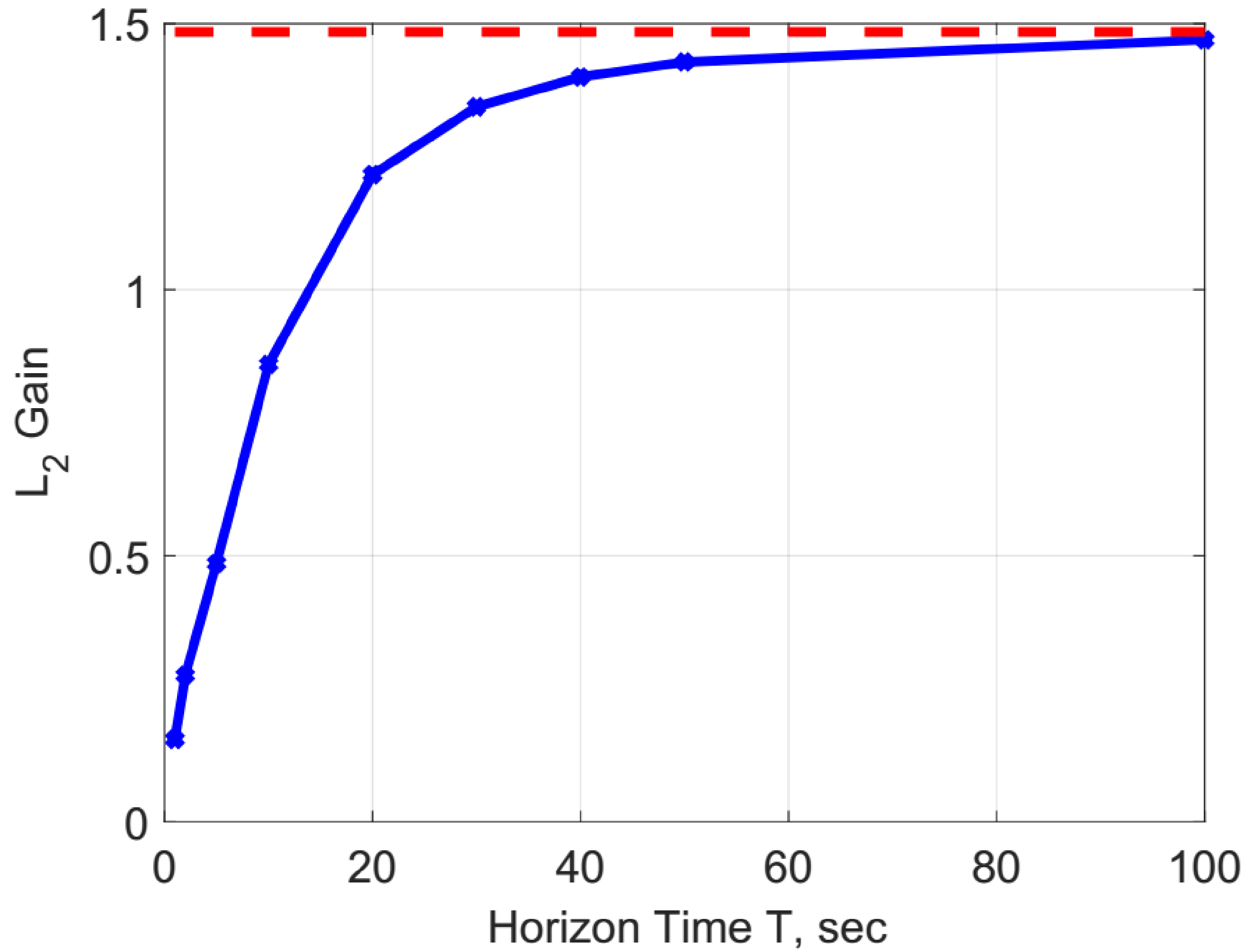
- Compute the induced L_2 gain of $F_u(G,\Delta)$ where Δ is LTI with $\|\Delta\| \leq 1$ and G is:

$$A_G := \begin{bmatrix} -0.8 & -1.3 & -2.1 & -2.5 \\ 2 & -0.9 & -8.4 & 0.7 \\ 2 & 8.6 & -0.5 & 12.5 \\ 2.1 & -0.3 & -12.6 & -0.6 \end{bmatrix} \quad B_G := \begin{bmatrix} -0.6 & 1 \\ 0 & 0.2 \\ 0 & 0.4 \\ -1.3 & -0.2 \end{bmatrix}$$

$$C_G := \begin{bmatrix} -1.4 & 0 & 0.5 & 0 \\ 0 & -0.1 & 1 & 0 \end{bmatrix} \quad D_G := \begin{bmatrix} -0.3 & 0 \\ 0 & 0 \end{bmatrix}$$

- By (standard) mu analysis, the worst-case (infinite horizon) L_2 gain is 1.49.
- This example is used to assess the finite-horizon robustness results.

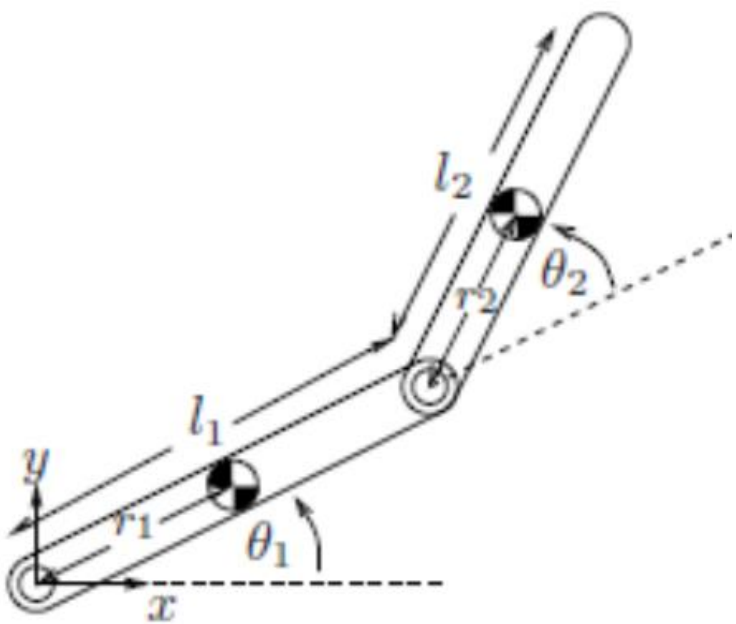
Example 1: Finite Horizon Results

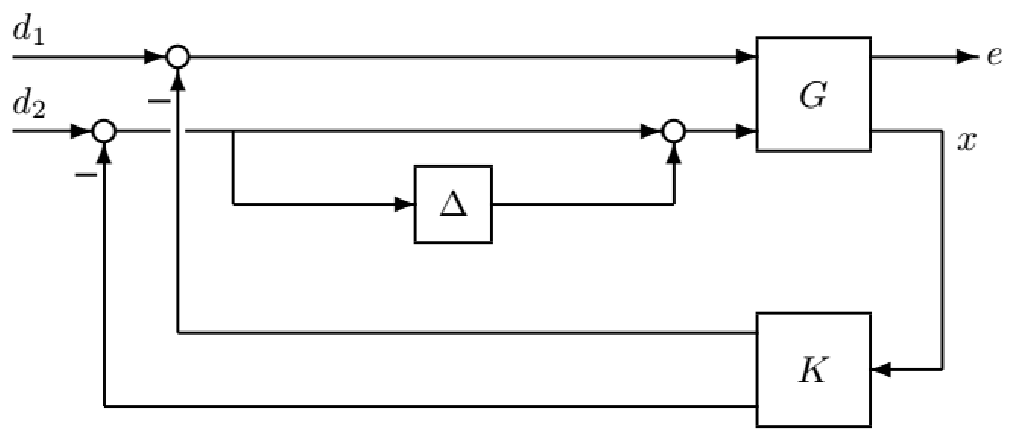


Total comp. time is 466 sec to compute worst-case gains on nine finite horizons.

Example 2: Two-Link Robot Arm

- Assess the worst-case L2-to-Euclidean gain from disturbances at the arm joints to the joint angles.
- LTI uncertainty with $\|\Delta\| \leq 0.8$ injected at 2nd joint.
- Analysis performed along nominal trajectory in with LQR state feedback.

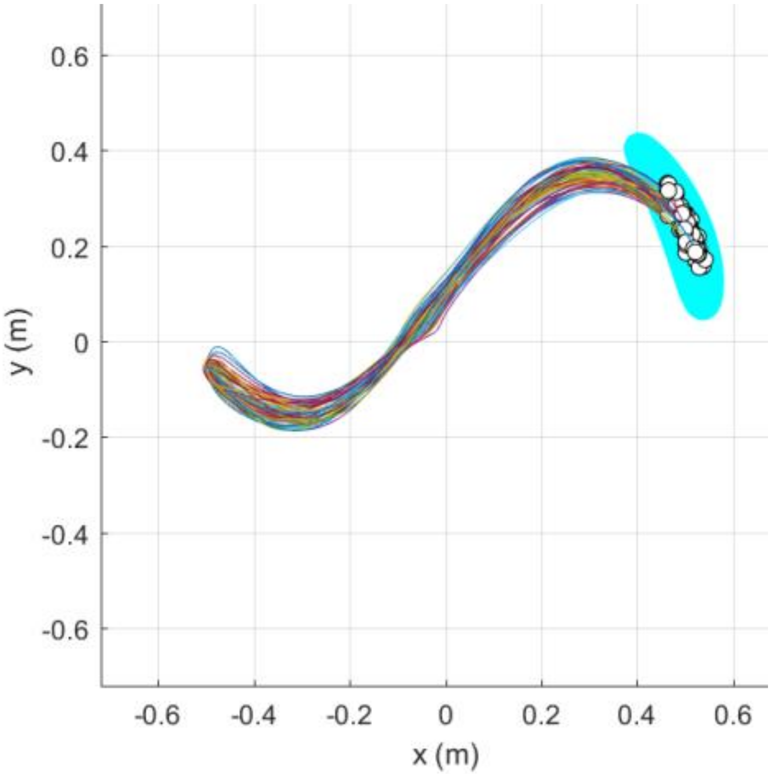




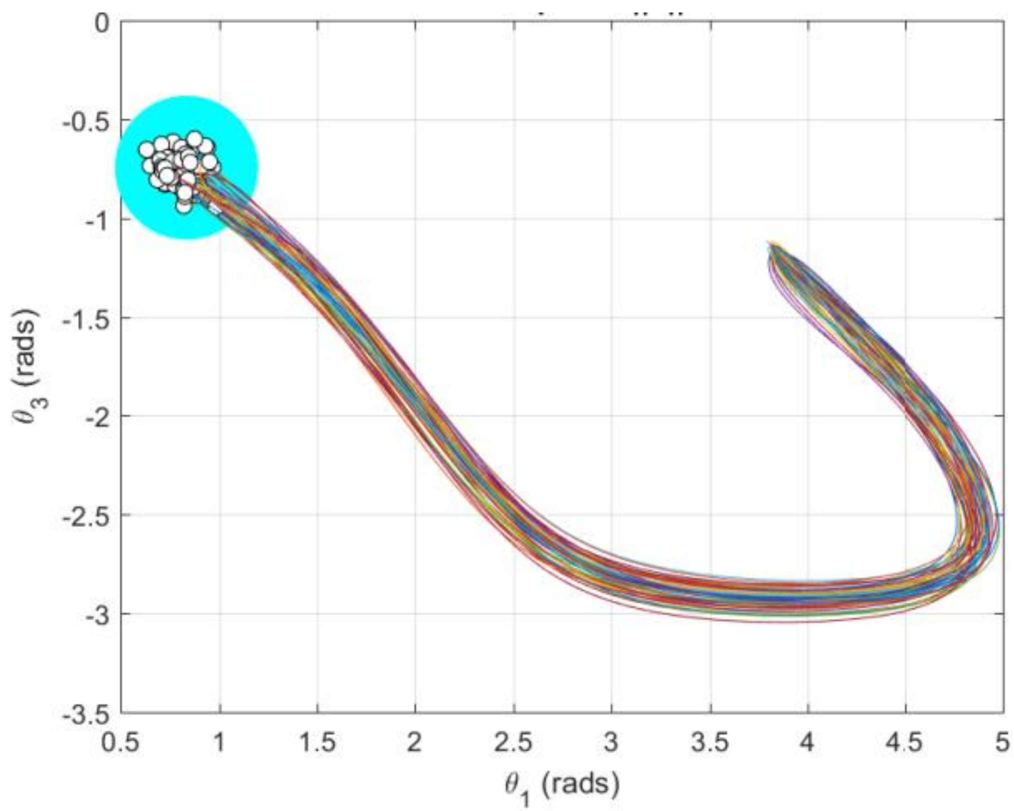
Example 2: Results

Bound on worst-case L_2-to-Euclidean gain = 0.0592.

Computation took 102 seconds.



Cartesian Coords.



Joint Angles

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Conclusions

- Main Result: Bounds on finite-horizon robust performance can be computed using differential equations or inequalities.
  - These results complement the use of nonlinear Monte Carlo simulations.
  - It would be useful to construct worst-case inputs / uncertainties analogous to μ lower bounds.
  - An LTVTools toolbox is in development with β-code of the proposed methods.
- References

  - Moore, Finite Horizon Robustness Analysis Using Integral Quadratic Constraints, MS Thesis, 2015.
  - Moore, Seiler, Meissen, Arcak, Packard, Finite Horizon Robustness Analysis of LTV Systems Using Integral Quadratic Constraints, draft in preparation.