

# Finite Horizon Robustness Analysis of LTV Systems Using Integral Quadratic Constraints

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International Workshop on Robust LPV Control  
Techniques and Anti-Windup Design  
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# Performance Adaptive Aeroelastic Wing (PAAW)

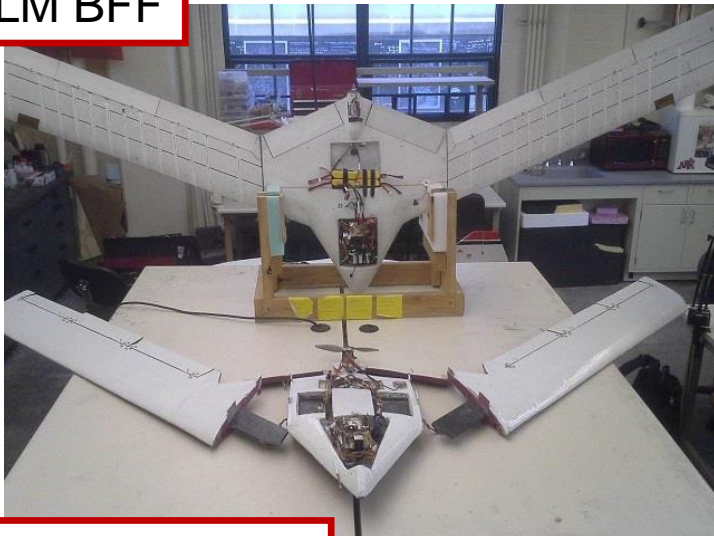
- Goal: Suppress flutter, control wing shape and alter shape to optimize performance
  - Funding: NASA NRA NNX14AL36A
  - Technical Monitor: Dr. Jeffrey Ouellette
  - Two years of testing at UMN followed by two years of testing on NASA's X-56 Aircraft



*Schmidt & Associates*



LM BFF



LM/NASA X-56

UMN Mini-Mutt

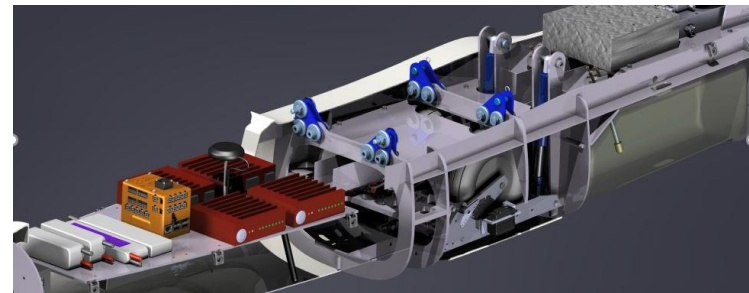
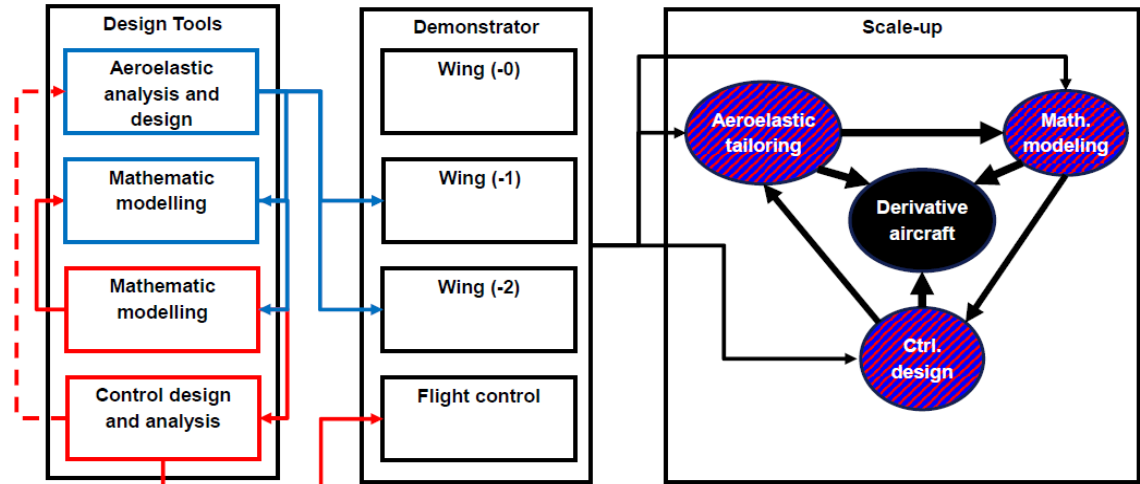
# The FlexOp Project

Flutter Free FLight Envelope eXpansion for ecOnomical Performance improvement

*Balint Vanek ([vanek@sztaki.hu](mailto:vanek@sztaki.hu)), coordinator  
Institute for Computer Science and Control, HAS*

## Approach

- \*Move towards methods and tools enabling multidisciplinary design analysis and optimization in the aeroservoelastic domain
- \*Validate the developed tools with the demonstrator



# Aeroservoelasticity (ASE)

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## Efficient aircraft design

- Lightweight structures
- High aspect ratios



Source: [www.flightglobal.com](http://www.flightglobal.com)

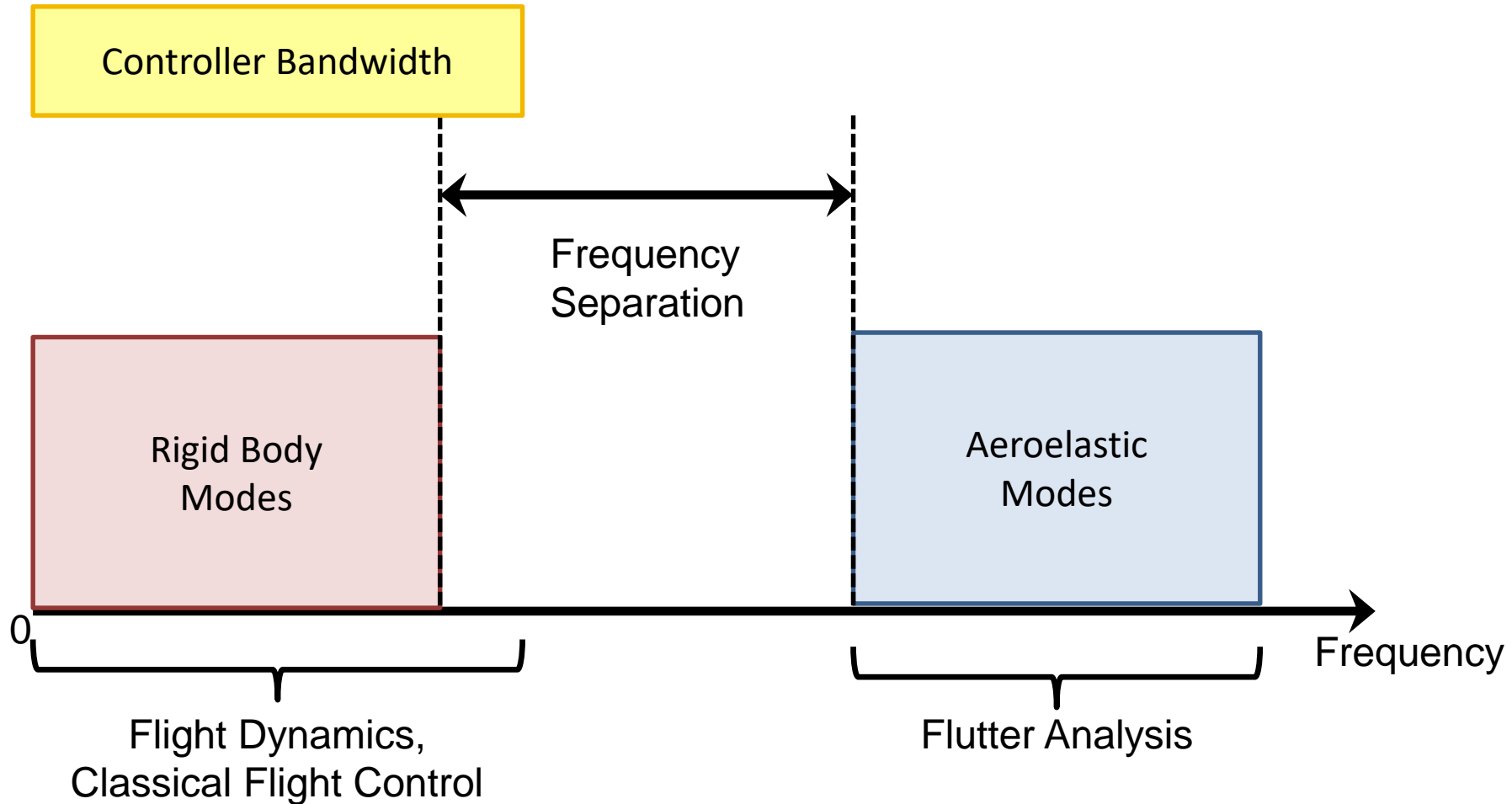
# Flutter

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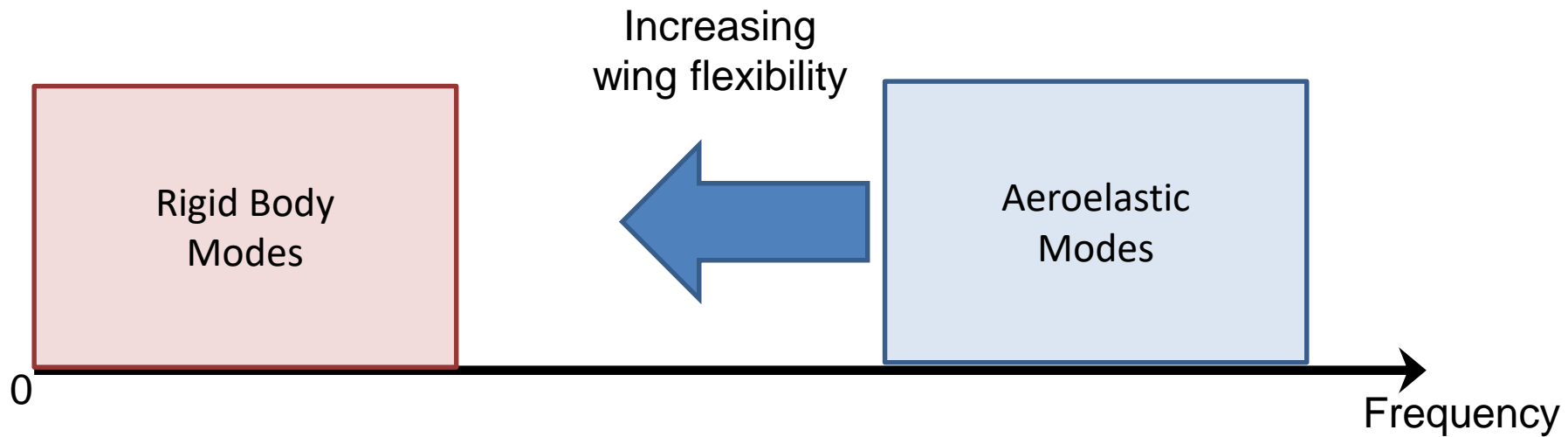
Source: NASA Dryden Flight Research

# Classical Approach

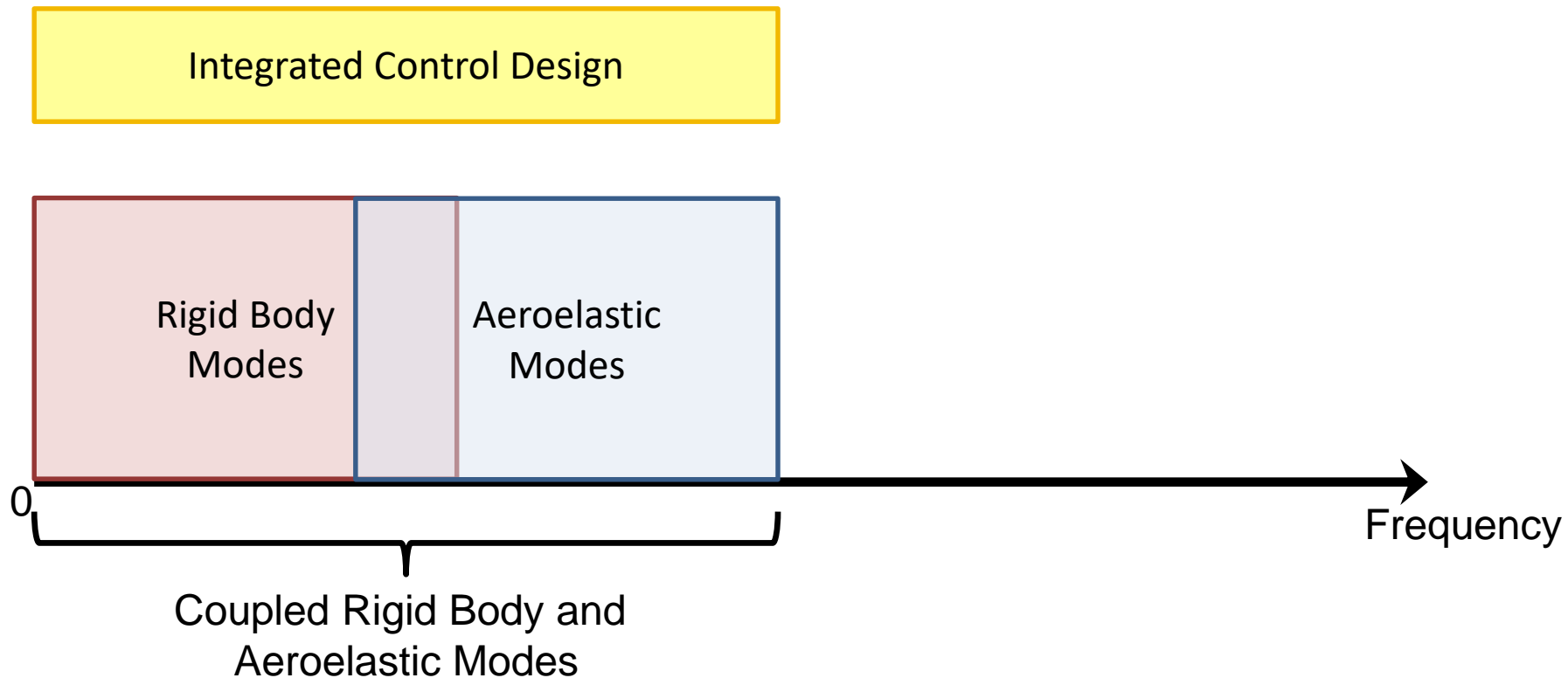


# Flexible Aircraft Challenges

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# Flexible Aircraft Challenges





# Modeling and Control for Flex Aircraft

## 1. Parameter Dependent Dynamics

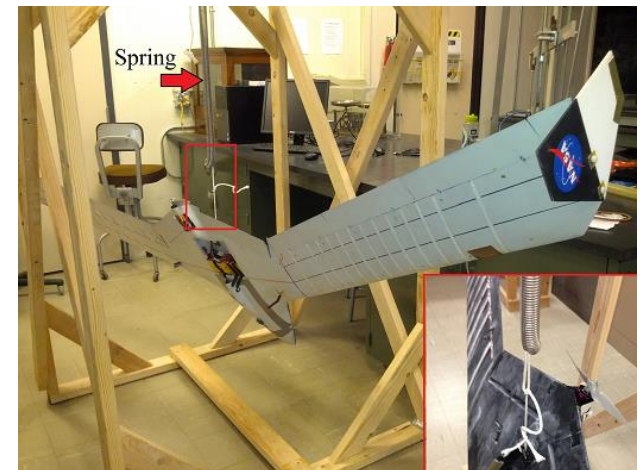
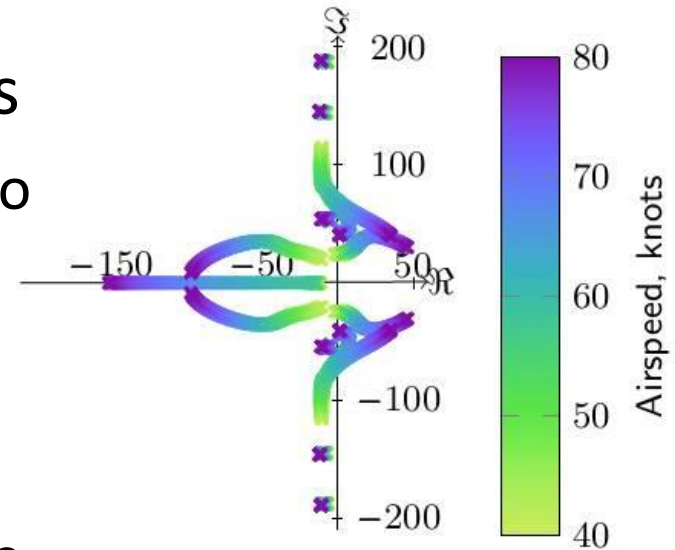
- Models depend on airspeed due to structural/aero interactions
- LPV is a natural framework.

## 2. Model Reduction

- High fidelity CFD/CSD models have many (millions) of states.

## 3. Model Uncertainty

- Use of simplified low order models OR reduced high fidelity models
- Unsteady aero, mass/inertia & structural parameters



# Current PAAW Aircraft



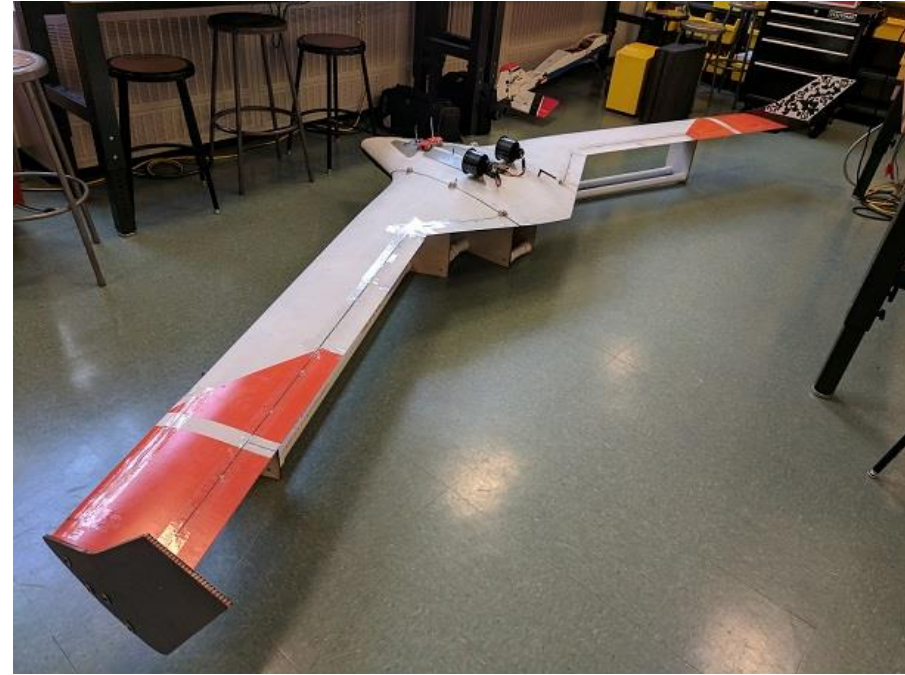
## **mAEWing1**

*10 foot wingspan*

*~14 pounds*

*Laser-scan replica of BFF*

*4 aircraft, >50 flights*



## **mAEWing2**

*14 foot wingspan*

*~42 pounds*

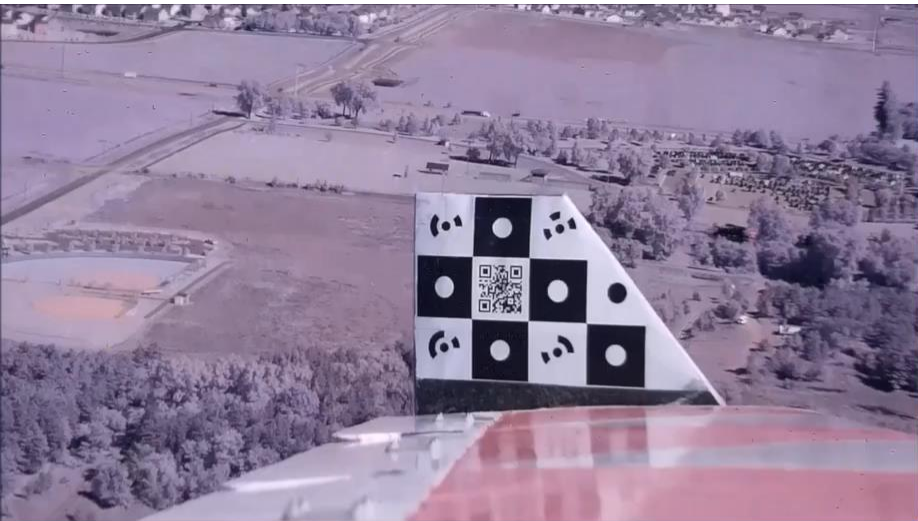
*Half-scale X-56*

*Currently ground testing*

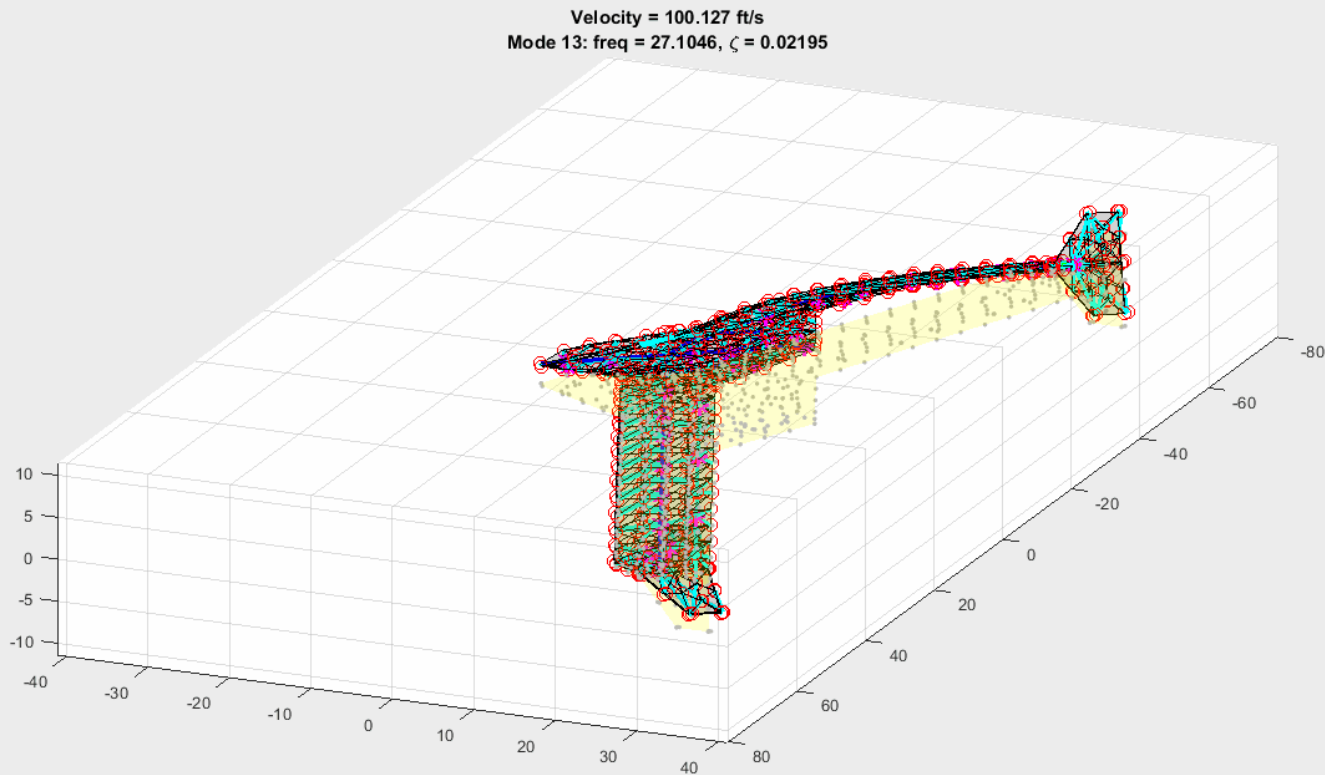
# mAEWing1 and 2



# Open-Loop Flutter



# Animated Mode Shape



*The BFF mode (genesis at SWB1) at a velocity near the flutter point.  
The coupling of SWB1 and short period is apparent*

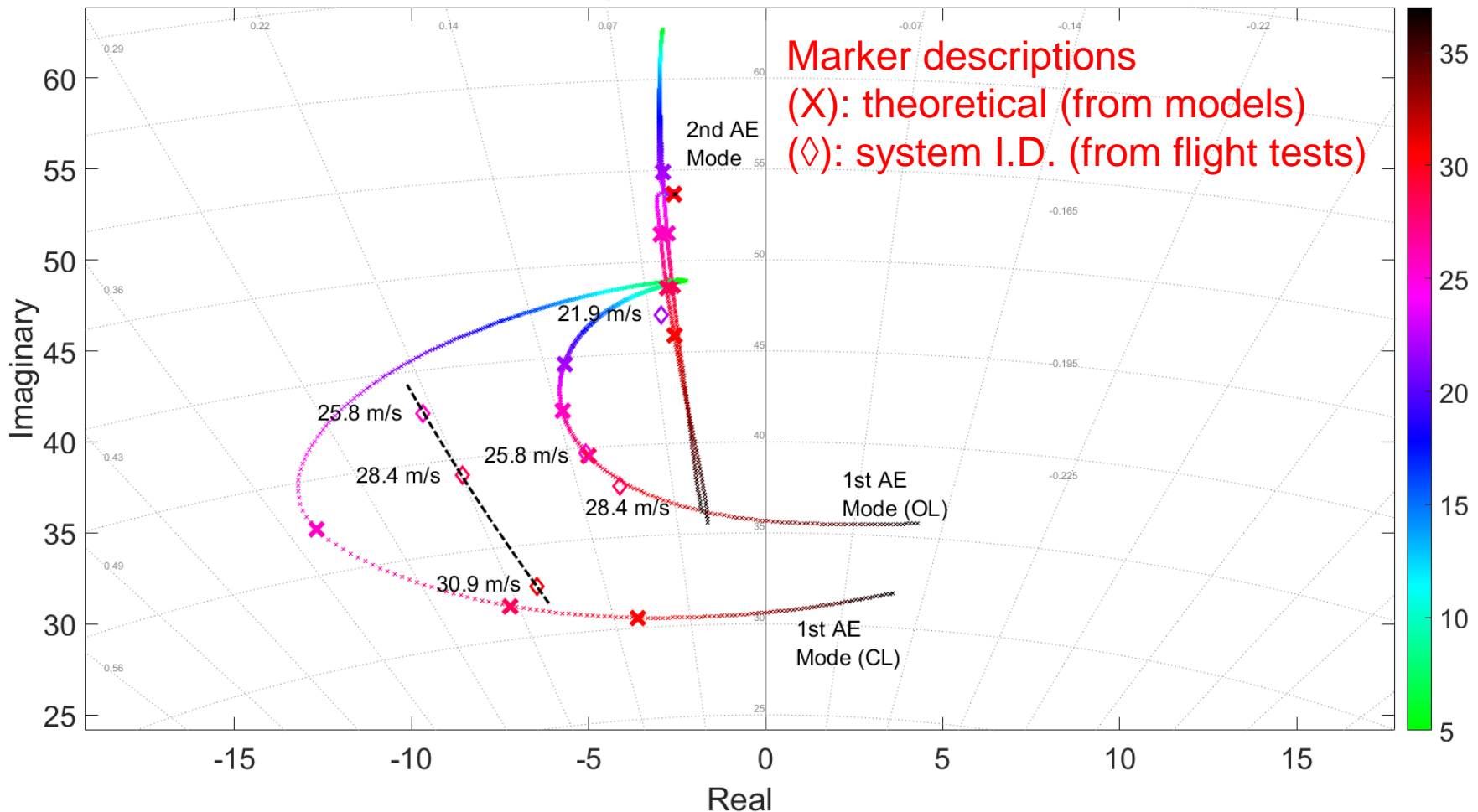
# In Flight Mode Shape

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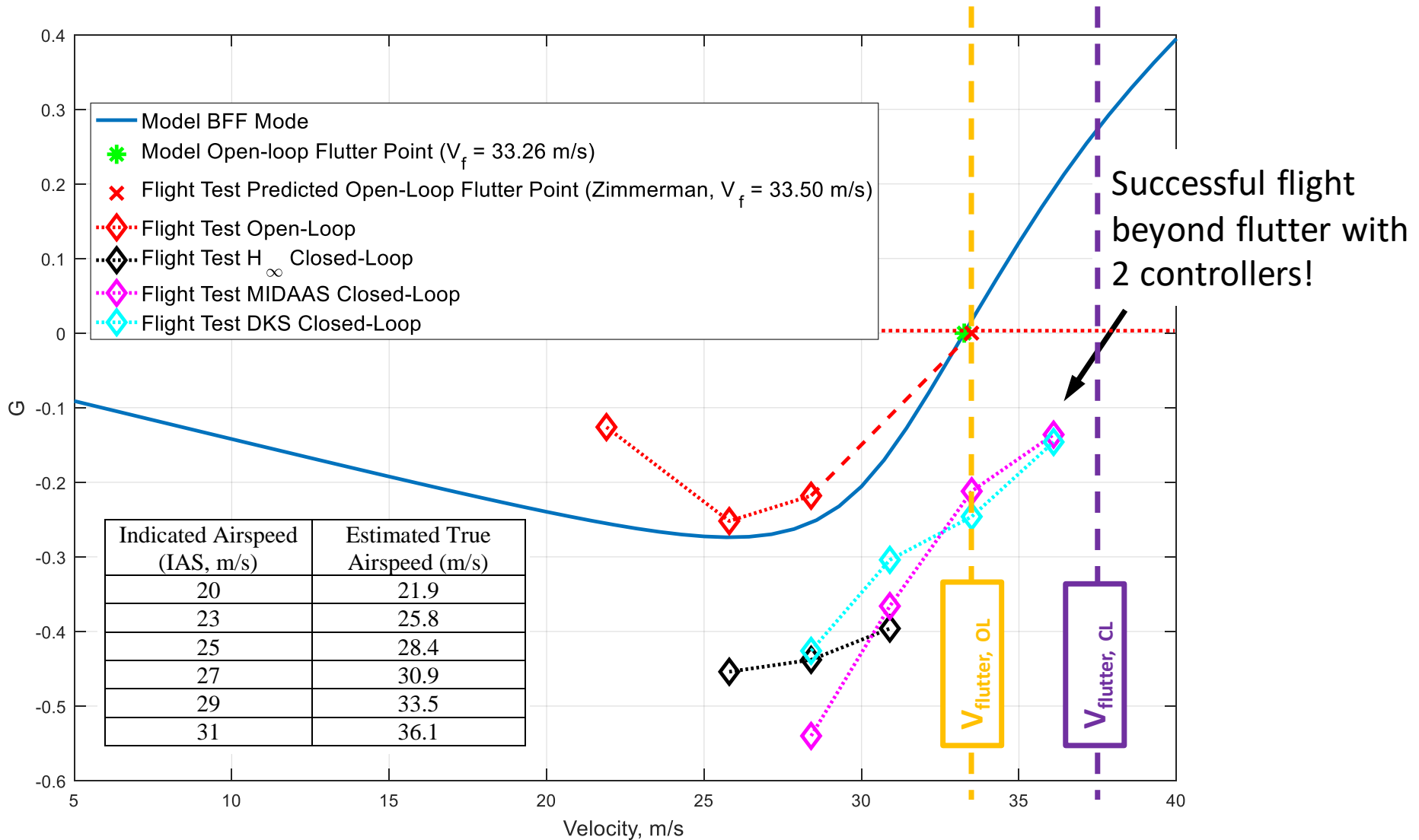
# Pole Map for H-Inf Controller

Map of Poles and Zeros



Hinf Design procedure due to Julian Theis ('16 AIAA, '18 Phd) with re-tuning by Kotikalpudi, et al ('18 Aviation).

# Flight Test Summary





# Outline

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- Motivation for LTV Analysis
- Nominal LTV Performance
- Robust LTV Performance
- Examples
- Conclusions

# Outline

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- **Motivation for LTV Analysis**
- Nominal LTV Performance
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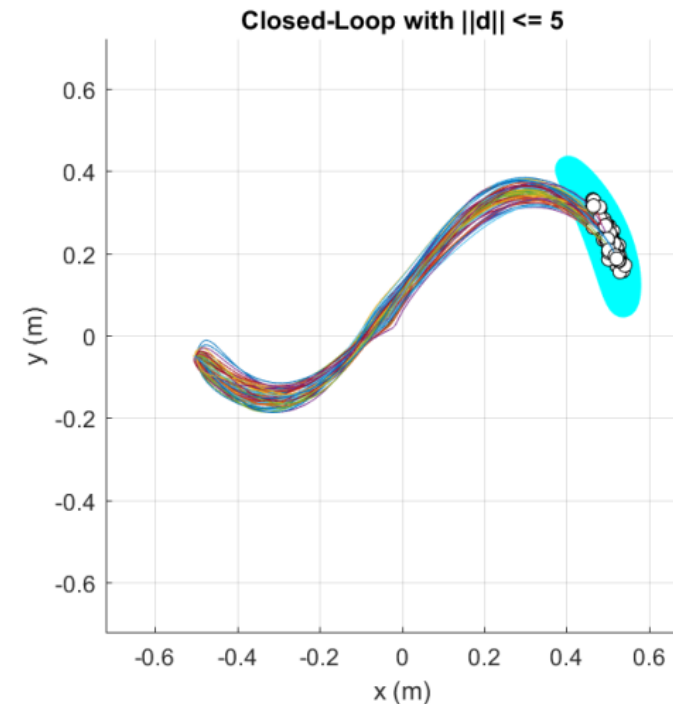
# Analysis Objective

**Goal:** Assess the robustness of linear time-varying (LTV) systems on finite horizons.

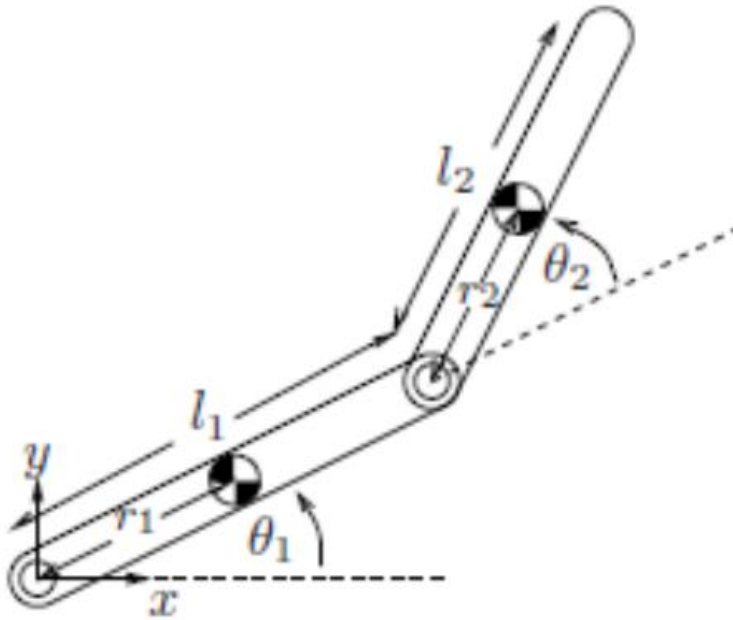
**Approach:** Classical Gain/Phase Margins focus on (infinite horizon) stability and frequency domain concepts.

Instead focus on:

- Finite horizon metrics, e.g. induced gains and reachable sets.
- Effect of disturbances and model uncertainty (D-scales, IQCs, etc).
- Time-domain analysis conditions.



# Two-Link Robot Arm



Two-Link Diagram [MZS]

Nonlinear dynamics [MZS]:

$$\dot{\eta} = f(\eta, \tau, d)$$

where

$$\eta = [\theta_1, \dot{\theta}_1, \theta_2, \dot{\theta}_2]^T$$

$$\tau = [\tau_1, \tau_2]^T$$

$$d = [d_1, d_2]^T$$

$\tau$  and  $d$  are control torques and disturbances at the link joints.

[MZS] R. Murray, Z. Li, and S. Sastry. *A Mathematical Introduction to Robot Manipulation*, 1994.

# Overview of Analysis Approach

Nonlinear dynamics:

$$\dot{\eta} = f(\eta, \tau, d)$$

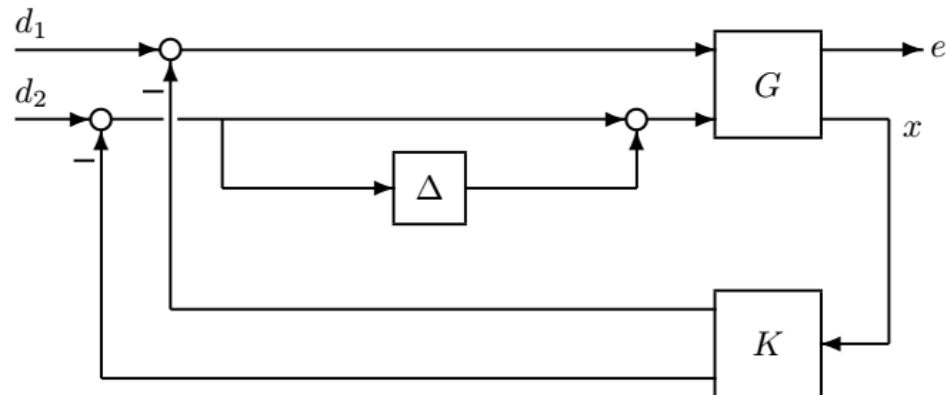
Linearize along a (finite –horizon) trajectory  $(\bar{\eta}, \bar{\tau}, d = 0)$

$$\dot{x} = A(t)x + B(t)u + B(t)d$$

Compute bounds on the terminal state  $x(T)$  or other quantity  $e(T) = C x(T)$  accounting for disturbances and uncertainty.

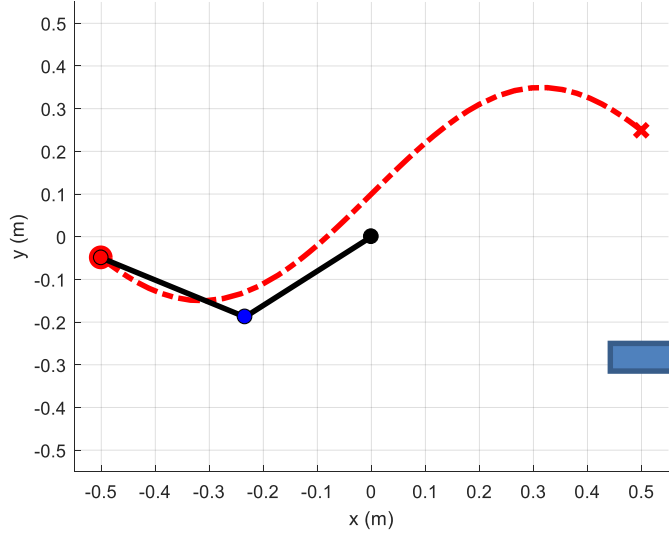
Comments:

- The analysis can be for open or closed-loop.
- LTV analysis complements the use of Monte Carlo simulations.

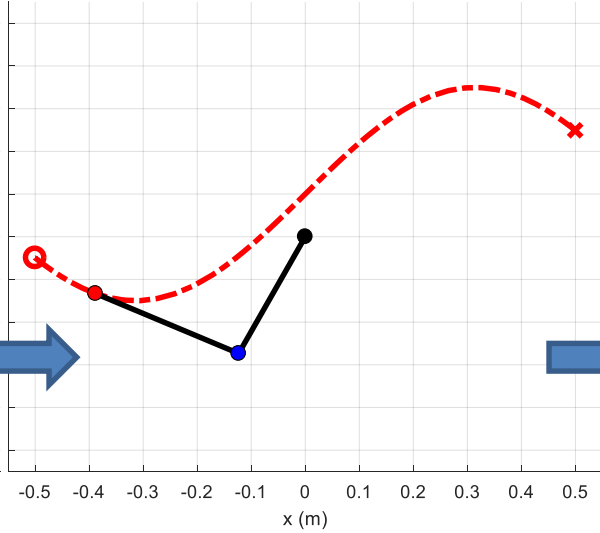


# Nominal Trajectory (Cartesian Coords.)

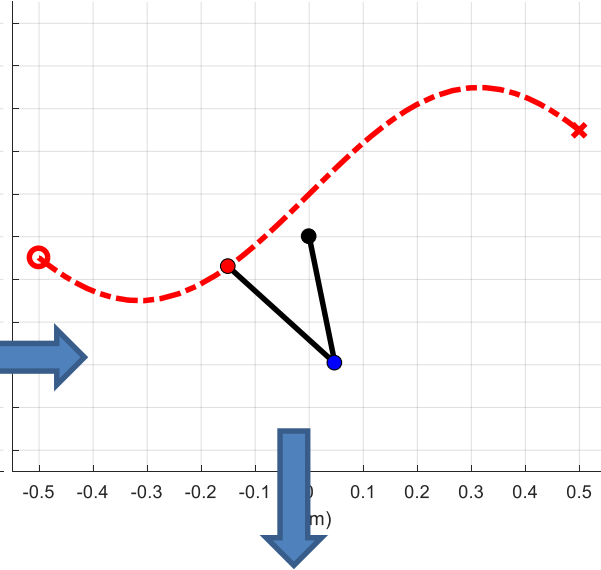
Two Link Robot at t=0sec



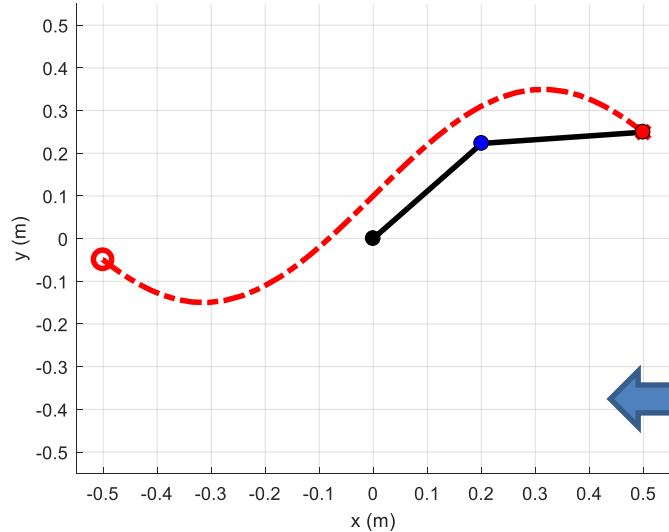
Two Link Robot at t=1sec



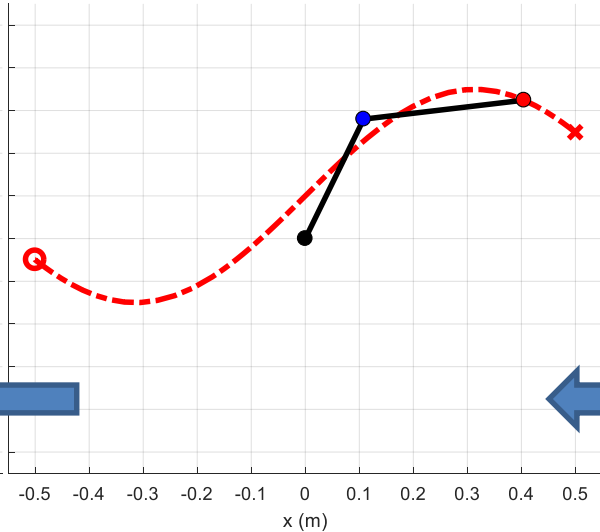
Two Link Robot at t=2sec



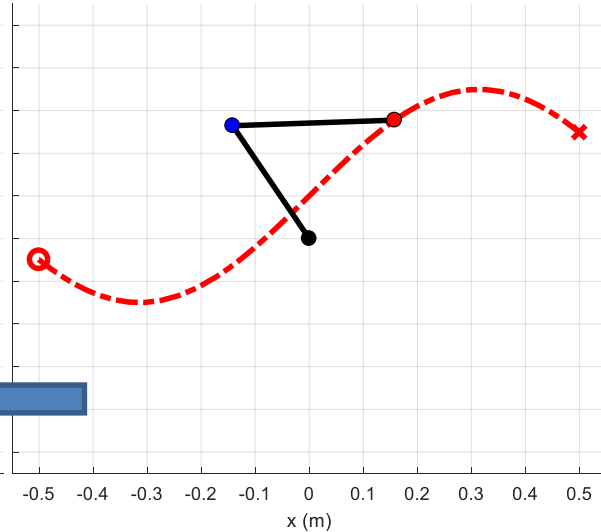
Two Link Robot at t=5sec



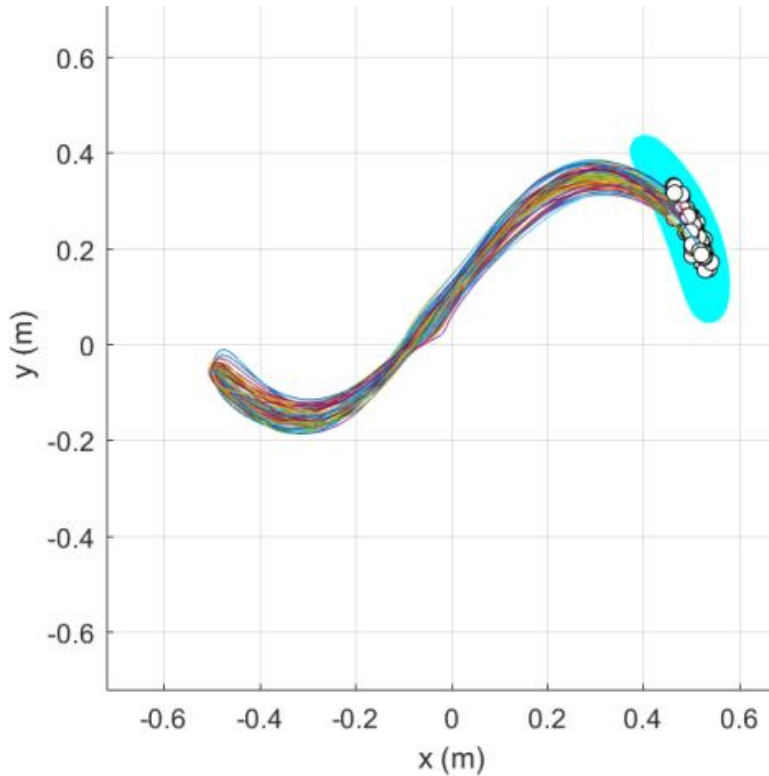
Two Link Robot at t=4sec



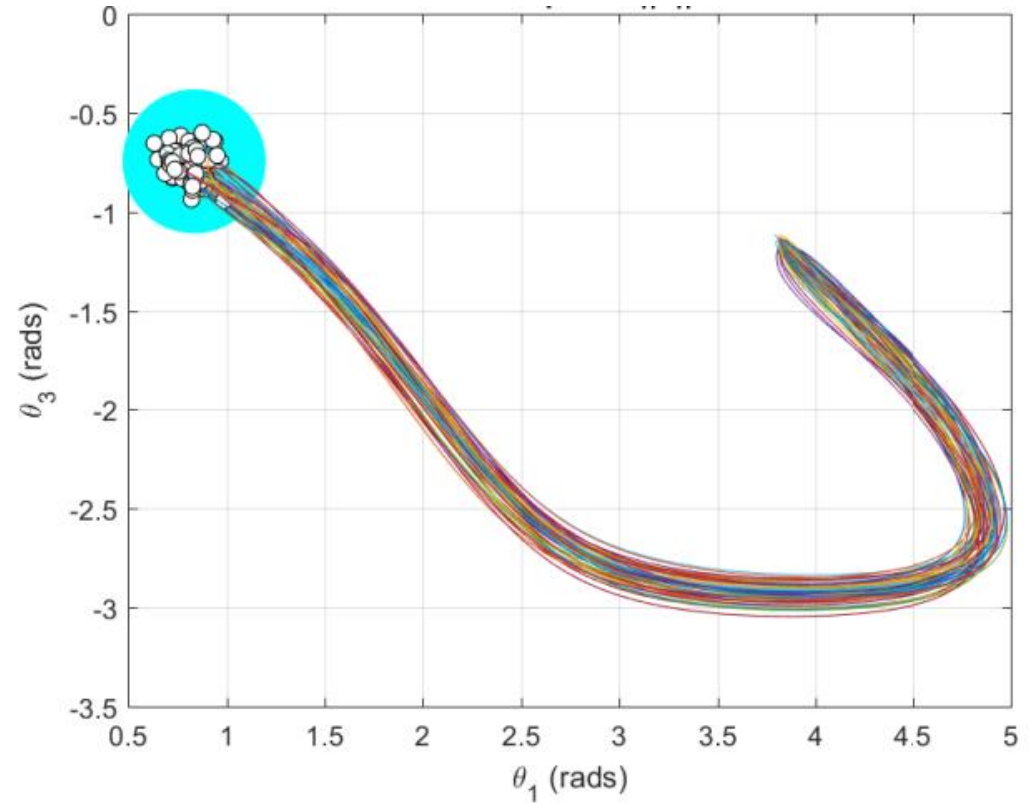
Two Link Robot at t=3sec



# Effect of Disturbances / Uncertainty



**Cartesian Coords.**



**Joint Angles**

# Outline

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- Motivation for LTV Analysis
- **Nominal LTV Performance**
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# Finite-Horizon LTV Performance

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Finite-Horizon LTV System  $G$  defined on  $[0,T]$

$$\dot{x}(t) = A(t)x(t) + B(t)d(t)$$

$$e(t) = C(t)x(t) + D(t)d(t)$$

Induced  $L_2$  Gain

$$\|G\|_{2,[0,T]} := \sup \left\{ \frac{\|e\|_{2,[0,T]}}{\|d\|_{2,[0,T]}} \mid x(0) = 0, 0 \neq d \in \mathcal{L}_{2,[0,T]} \right\}$$

$L_2$ -to-Euclidean Gain

$$\|G\|_{E,[0,T]} := \sup \left\{ \frac{\|e(T)\|_2}{\|d\|_{2,[0,T]}} \mid x(0) = 0, 0 \neq d \in \mathcal{L}_{2,[0,T]} \right\}$$

The  $L_2$ -to-Euclidean gain requires  $D(T)=0$  to be well-posed.

The definition can be generalized to estimate ellipsoidal bounds on the reachable set of states at  $T$ .

# General (Q,S,R,F) Cost

Cost function  $J$  defined by  $(Q,S,R,F)$

$$J(d) := x(T)^T F x(T) + \int_0^T \begin{bmatrix} x(t) \\ d(t) \end{bmatrix}^T \begin{bmatrix} Q(t) & S(t) \\ S(t)^T & R(t) \end{bmatrix} \begin{bmatrix} x(t) \\ d(t) \end{bmatrix} dt$$

Subject to: LTV Dynamics with  $x(0)=0$

Example: Induced  $L_2$  Gain

Select  $(Q,S,R,F)$  as:

$$Q(t) := C(t)^T C(t), S(t) := C(t)^T D(t), R(t) := D(t)^T D(t) - \gamma^2 I_{n_d}, \text{ and } F := 0.$$

Cost Function  $J$  is:

$$J(d) = \|e\|_{2,[0,T]}^2 - \gamma^2 \|d\|_{2,[0,T]}^2$$

➡  $J(d) \leq 0$  for all  $d \in \mathcal{L}_2[0, T]$  if and only if  $\|G\|_{2,[0,T]} \leq \gamma$ .

# General (Q,S,R,F) Cost

Cost function  $J$  defined by  $(Q,S,R,F)$

$$J(d) := x(T)^T F x(T) + \int_0^T \begin{bmatrix} x(t) \\ d(t) \end{bmatrix}^T \begin{bmatrix} Q(t) & S(t) \\ S(t)^T & R(t) \end{bmatrix} \begin{bmatrix} x(t) \\ d(t) \end{bmatrix} dt$$

Subject to: LTV Dynamics with  $x(0)=0$

Example:  $L_2$ -to-Euclidean Gain

Select  $(Q,S,R,F)$  as:

$$Q(t) := 0, \quad S(t) := 0, \quad R(t) := -\gamma^2 I_{n_d}, \quad \text{and} \quad F := C^T(T)C(T).$$

Cost Function  $J$  is:

$$J(d) = \|e(T)\|_2^2 - \gamma^2 \|d\|_{2,[0,T]}^2$$

➡  $J(d) \leq 0$  for all  $d \in \mathcal{L}_2[0, T]$  if and only if  $\|G\|_{E,[0,T]} \leq \gamma$ .

# Strict Bounded Real Lemma

**Theorem 1.** Assume  $R(t) \prec 0$  for all  $t \in [0, T]$ . The following are equivalent:

1.  $\exists \epsilon > 0$  such that  $J(d) \leq -\epsilon \|d\|_{2, [0, T]}^2 \quad \forall d \in \mathcal{L}_2[0, T]$ .

2. There exists a differentiable function  $Y$  on  $[0, T]$  such that  $Y(T) = F$  and

$$\dot{Y} + A^T Y + Y A + Q - (Y B + S) R^{-1} (Y B + S)^T = 0$$

This is a **Riccati Differential Equation (RDE)**.

3. There exists  $\epsilon > 0$  and a differentiable function  $P$  on  $[0, T]$  such that  $P(T) \succeq F$  and

$$\dot{P} + A^T P + P A + Q - (P B + S) R^{-1} (P B + S)^T \preceq -\epsilon I$$

This is a strict **Riccati Differential Inequality (RDI)**.

This is a generalization of results contained in:

\*Tadmor, Worst-case design in the time domain. *MCSS*, 1990 .

\*Ravi, Nagpal, and Khargonekar.  $H_\infty$  control of linear time-varying systems. *SIAM JCO*, 1991.

\*Green and Limebeer. *Linear Robust Control*, 1995.

\*Chen and Tu. The strict bounded real lemma for linear time-varying systems. *JMAA*, 2000.

# Proof: 3 $\rightarrow$ 1

By Schur complements, the RDI is equivalent to:

$$\begin{bmatrix} \dot{P} + A^T P + PA & PB \\ B^T P & 0 \end{bmatrix} + \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \preceq -\tilde{\epsilon} I$$

This is an LMI in  $P$ . It is also equivalent to a dissipation inequality with the storage function  $V(x, t) := x^T P(t)x$ .

$$\dot{V} + \begin{bmatrix} x \\ d \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} x \\ d \end{bmatrix} \leq -\tilde{\epsilon} \begin{bmatrix} x \\ d \end{bmatrix}^T \begin{bmatrix} x \\ d \end{bmatrix}$$

Integrate from  $t=0$  to  $t=T$ :

$$V(x(T), T) - V(x(0), 0) + \int_0^T \begin{bmatrix} x(t) \\ d(t) \end{bmatrix}^T \begin{bmatrix} Q(t) & S(t) \\ S(t)^T & R(t) \end{bmatrix} \begin{bmatrix} x(t) \\ d(t) \end{bmatrix} dt \leq -\tilde{\epsilon} \|[x; d]\|_{2, [0, T]}^2$$

Apply  $x(0)=0$  and  $P(T) \geq F$ :

$$J(d) \leq -\epsilon \|d\|_{2, [0, T]}^2$$

# Strict Bounded Real Lemma

**Theorem 1.** Assume  $R(t) \prec 0$  for all  $t \in [0, T]$ . The following are equivalent:

1.  $\exists \epsilon > 0$  such that  $J(d) \leq -\epsilon \|d\|_{2, [0, T]}^2 \quad \forall d \in \mathcal{L}_2[0, T]$ .
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3. There exists  $\epsilon > 0$  and a differentiable function  $P$  on  $[0, T]$  such that  $P(T) \succeq F$  and

$$\dot{P} + A^T P + P A + Q - (P B + S) R^{-1} (P B + S)^T \preceq -\epsilon I$$

This is a strict **Riccati Differential Inequality (RDI)**.

## Comments:

\*For nominal analysis, the RDE can be integrated. If the solution exists on  $[0, T]$  then nominal performance is achieved. This typically involves bisection, e.g. over  $\gamma$ , to find the best bound on a gain.

\*For robustness analysis, both the RDI and RDE will be used to construct an efficient numerical algorithm.

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- **Robust LTV Performance**
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# Uncertainty Model

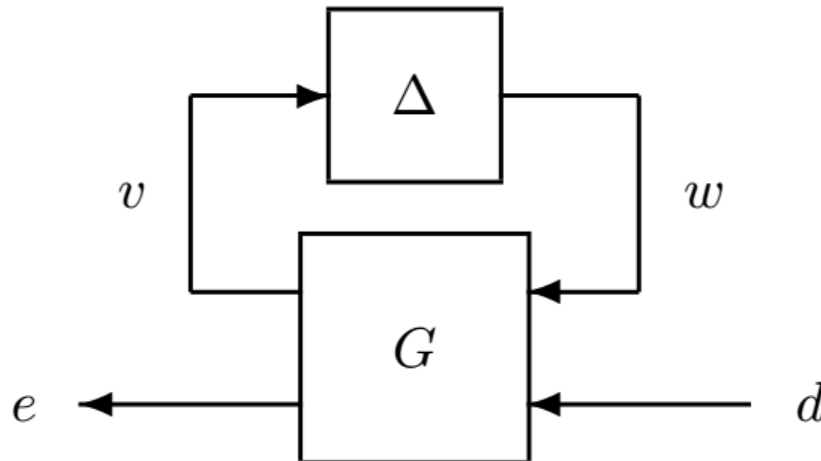
- Standard LFT Model,  $F_{\Delta}(G, \Delta)$ , where  $G$  is LTV:

$$\dot{x}_G(t) = A_G(t) x_G(t) + B_{G1}(t) w(t) + B_{G2}(t) d(t)$$

$$v(t) = C_{G1}(t) x_G(t) + D_{G11}(t) w(t) + D_{G12}(t) d(t)$$

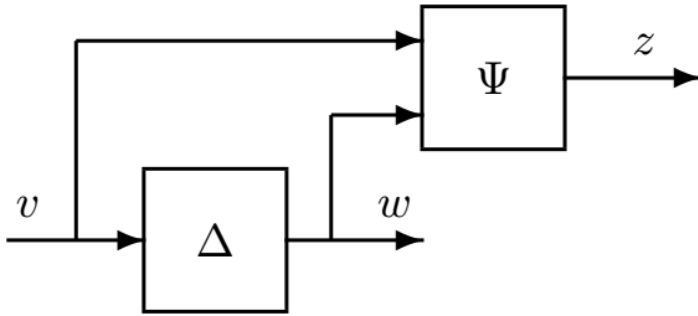
$$e(t) = C_{G2}(t) x_G(t) + D_{G21}(t) w(t) + D_{G22}(t) d(t)$$

$\Delta$  is block structured and used to model parametric / dynamic uncertainty and nonlinear perturbations.





# Integral Quadratic Constraints (IQCs)



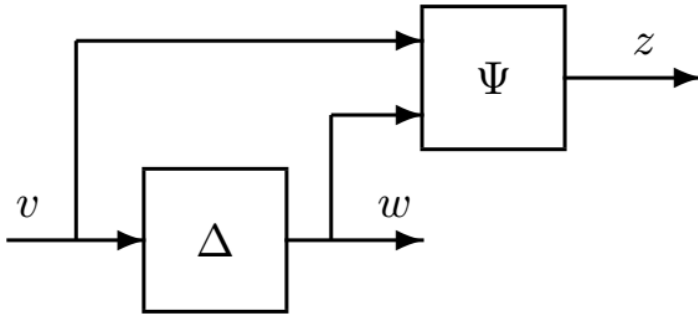
$$\int_0^T z(t)^T M(t) z(t) dt \geq 0$$

**Definition 2.** Let  $\Psi \in \mathbb{RH}_{\infty}^{n_z \times (n_v + n_w)}$  and  $M : [0, T] \rightarrow \mathbb{S}^{n_z}$  with  $M$  piecewise continuous. A bounded, causal operator  $\Delta : \mathbf{L}_2^{n_v}[0, T] \rightarrow \mathbf{L}_2^{n_w}[0, T]$  satisfies the time domain IQC defined by  $(\Psi, M)$  if the following inequality holds for all  $v \in \mathcal{L}_2^{n_v}[0, T]$  and  $w = \Delta(v)$ :

$$\int_0^T z(t)^T M(t) z(t) dt \geq 0 \quad (1)$$

where  $z$  is the output of  $\Psi$  driven by inputs  $(v, w)$  with zero initial conditions  $x_{\psi}(0) = 0$ .

# Integral Quadratic Constraints (IQCs)



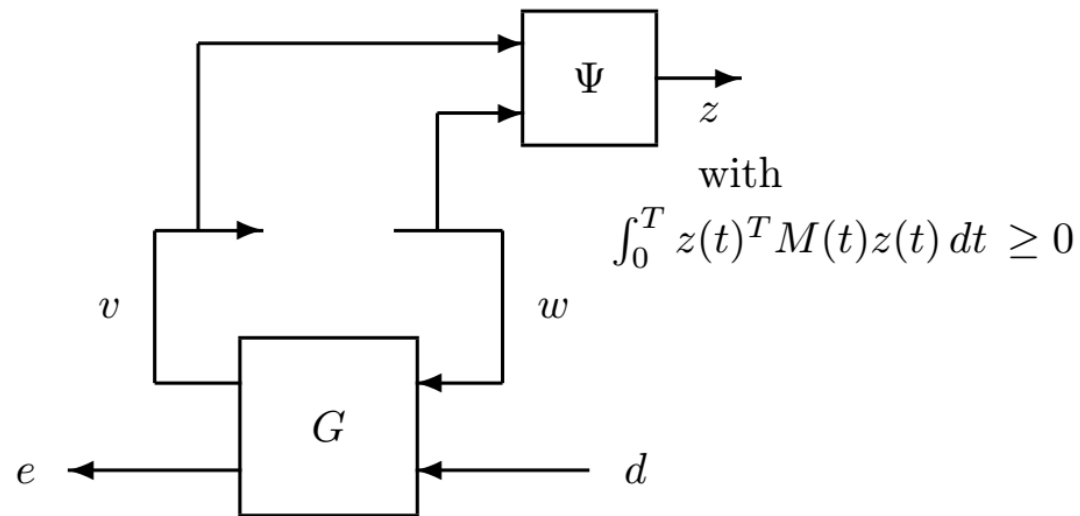
$$\int_0^T z(t)^T M(t) z(t) dt \geq 0$$

## Comments:

- \*A library of IQC for various uncertainties / nonlinearities is given in [MR]. Many of these are given as frequency domain inequalities.
- \*Time-domain IQCs that hold over finite horizons are called hard.
- \*This generalizes D and D/G scales for LTI and parametric uncertainty. It can be used to model the I/O behavior of nonlinear elements.

[MR] Megretski and Rantzer. System analysis via integral quadratic constraints, TAC, 1997.

# Robustness Analysis



The robustness analysis is performed on the extended (LTV) system of  $(G, \Psi)$  using the constraint on  $z$ .

$$\begin{bmatrix} \dot{x}(t) \\ z(t) \\ e(t) \end{bmatrix} = \begin{bmatrix} \mathcal{A}(t) & \mathcal{B}_1(t) & \mathcal{B}_2(t) \\ \mathcal{C}_1(t) & \mathcal{D}_{11}(t) & \mathcal{D}_{12}(t) \\ \mathcal{C}_2(t) & \mathcal{D}_{21}(t) & \mathcal{D}_{22}(t) \end{bmatrix} \begin{bmatrix} x(t) \\ w(t) \\ d(t) \end{bmatrix}$$

# Robustness Analysis: Induced $L_2$ Gain

**Theorem 3.** Assume  $\Delta$  satisfies the IQC defined by  $(\Psi, M)$ . If there exists  $\epsilon > 0$ ,  $\gamma > 0$  a differentiable function  $P$  on  $[0, T]$  and such that  $P(T) \succeq 0$  and for all  $t \in [0, T]$

$$\begin{bmatrix} \dot{P} + \mathcal{A}^T P + P \mathcal{A} & P \mathcal{B}_1 & P \mathcal{B}_2 \\ \mathcal{B}_1^T P & 0 & 0 \\ \mathcal{B}_2^T P & 0 & -\gamma^2 I \end{bmatrix} + (\cdot)^T [\mathcal{C}_2 \quad \mathcal{D}_{21} \quad \mathcal{D}_{22}] + (\cdot)^T M [\mathcal{C}_1 \quad \mathcal{D}_{11} \quad \mathcal{D}_{12}] \preceq -\epsilon I$$

then  $\|F_u(G, \Delta)\|_{2, [0, T]} < \gamma$ .

# Robustness Analysis: Induced $L_2$ Gain

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then  $\|F_u(G, \Delta)\|_{2, [0, T]} < \gamma$ .

## Proof:

The Differential LMI (DLMI) is equivalent to a dissipation ineq. with storage function  $V(x, t) := x^T P(t)x$ .

$$\dot{V}(x, t) + z(t)^T M z(t) - (\gamma^2 - \epsilon)d(t)^T d(t) + e(t)^T e(t) \leq 0$$

Integrate and apply the IQC + boundary conditions to conclude that the induced  $L_2$  gain is  $\leq \gamma$ .

# Robustness Analysis: Induced $L_2$ Gain

**Theorem 3.** Assume  $\Delta$  satisfies the IQC defined by  $(\Psi, M)$ . If there exists  $\epsilon > 0$ ,  $\gamma > 0$  a differentiable function  $P$  on  $[0, T]$  and such that  $P(T) \succeq 0$  and for all  $t \in [0, T]$

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then  $\|F_u(G, \Delta)\|_{2,[0,T]} < \gamma$ .

## Comments:

\*A similar result exists for  $L_2$ -to-Euclidean or, more generally  $(Q, S, R, F)$  cost functions.

\*The DLMI can be expressed as a Riccati Differential Ineq. (RDI) by Schur Complements.

\*The RDI is equivalent to a related Riccati Differential Eq. (RDE) condition by the strict Bounded Real Lemma.

# Robustness Analysis: Induced $L_2$ Gain

**Theorem 3.** Assume  $\Delta$  satisfies the IQC defined by  $(\Psi, M)$ . If there exists  $\epsilon > 0$ ,  $\gamma > 0$  a differentiable function  $P$  on  $[0, T]$  and such that  $P(T) \succeq 0$  and for all  $t \in [0, T]$

$$\begin{bmatrix} \dot{P} + \mathcal{A}^T P + P \mathcal{A} & P \mathcal{B}_1 & P \mathcal{B}_2 \\ \mathcal{B}_1^T P & 0 & 0 \\ \mathcal{B}_2^T P & 0 & -\gamma^2 I \end{bmatrix} + (\cdot)^T [\mathcal{C}_2 \quad \mathcal{D}_{21} \quad \mathcal{D}_{22}] + (\cdot)^T M [\mathcal{C}_1 \quad \mathcal{D}_{11} \quad \mathcal{D}_{12}] \preceq -\epsilon I$$

then  $\|F_u(G, \Delta)\|_{2, [0, T]} < \gamma$ .

## Comments:

\*The DLMI is convex in the IQC matrix  $M$  but requires gridding on time  $t$  and parameterization of  $P$ .

\*The RDE form directly solves for  $P$  by integration (no time gridding) but the IQC matrix  $M$  enters in a non-convex fashion.

# Numerical Implementation

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An efficient numerical algorithm is obtained by mixing the LMI and RDE conditions.

Sketch of algorithm:

1. **Initialize:** Select a time grid and basis functions for  $P(t)$ .
2. **Solve DLMI:** Obtain finite-dimensional optim. by enforcing DLMI on the time grid and using basis functions.
3. **Solve RDE:** Use IQC matrix  $M$  from step 2 and solve RDE. This gives the optimal storage  $P$  for this matrix  $M$ .
4. **Terminate:** Stop if the costs from Steps 2 and 3 are similar. Otherwise return to Step 2 using optimal storage  $P$  as a basis function.



# Outline

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- Motivation for LTV Analysis
- Nominal LTV Performance
- Robust LTV Performance
- **Examples**
- Conclusions

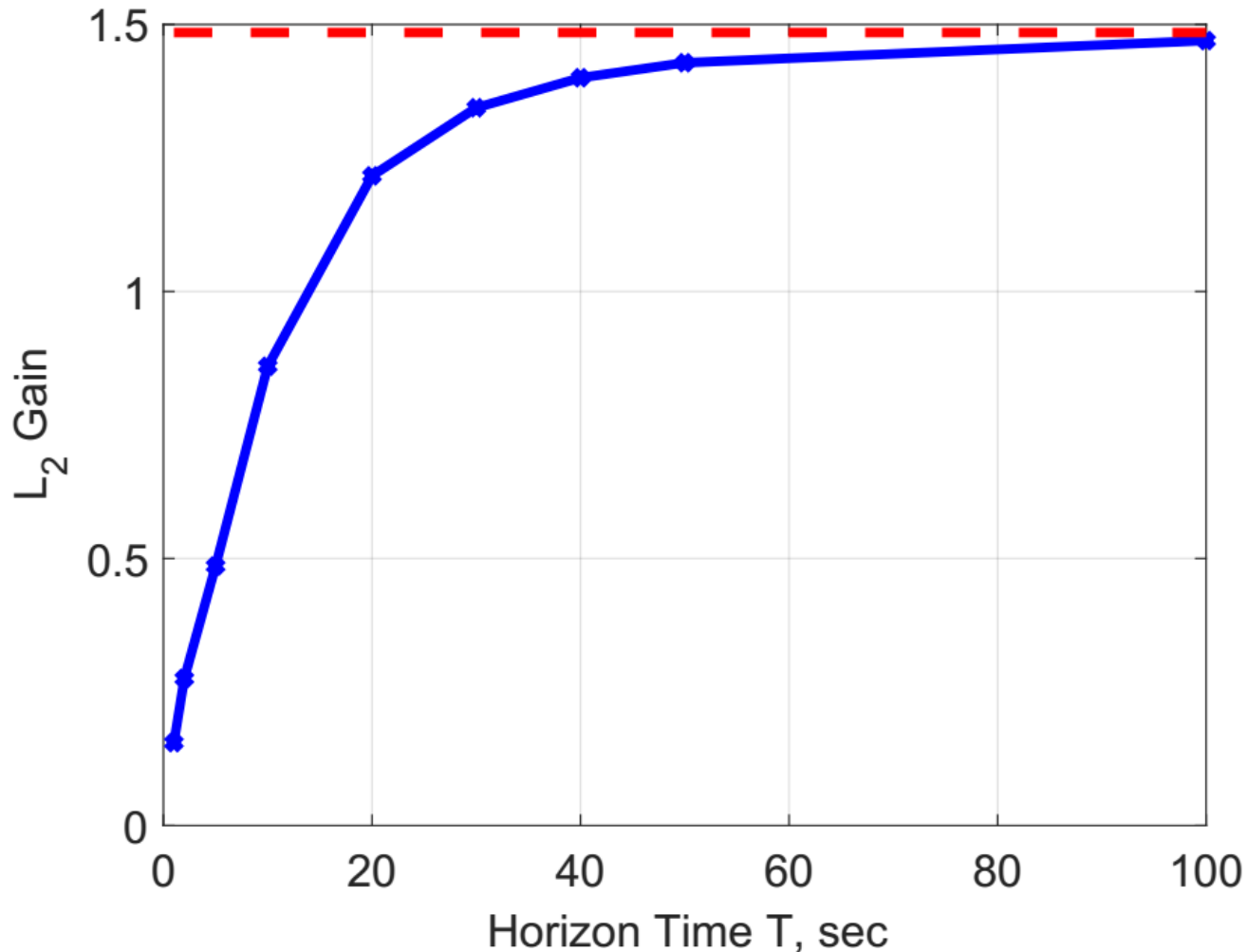
# Example 1: LTI Plant

- Compute the induced  $L_2$  gain of  $Fu(G, \Delta)$  where  $\Delta$  is LTI with  $\|\Delta\| \leq 1$  and  $G$  is:

$$A_G := \begin{bmatrix} -0.8 & -1.3 & -2.1 & -2.5 \\ 2 & -0.9 & -8.4 & 0.7 \\ 2 & 8.6 & -0.5 & 12.5 \\ 2.1 & -0.3 & -12.6 & -0.6 \end{bmatrix} \quad B_G := \begin{bmatrix} -0.6 & 1 \\ 0 & 0.2 \\ 0 & 0.4 \\ -1.3 & -0.2 \end{bmatrix}$$
$$C_G := \begin{bmatrix} -1.4 & 0 & 0.5 & 0 \\ 0 & -0.1 & 1 & 0 \end{bmatrix} \quad D_G := \begin{bmatrix} -0.3 & 0 \\ 0 & 0 \end{bmatrix}$$

- By (standard) mu analysis, the worst-case (infinite horizon)  $L_2$  gain is 1.49.
- This example is used to assess the finite-horizon robustness results.

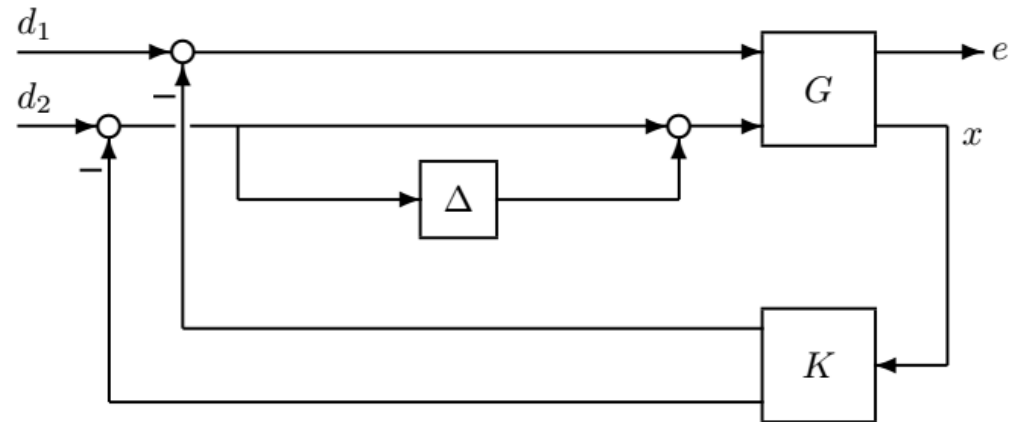
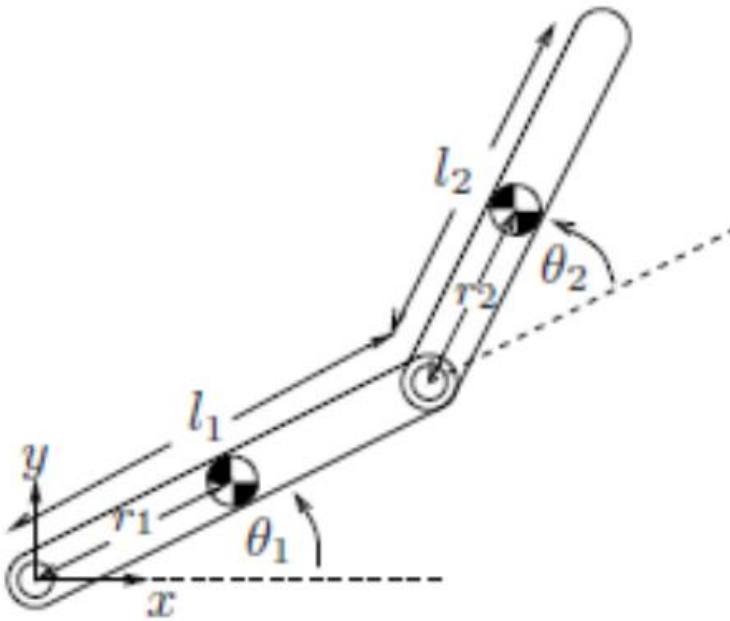
# Example 1: Finite Horizon Results



Total comp. time is 466 sec to compute worst-case gains on nine finite horizons.

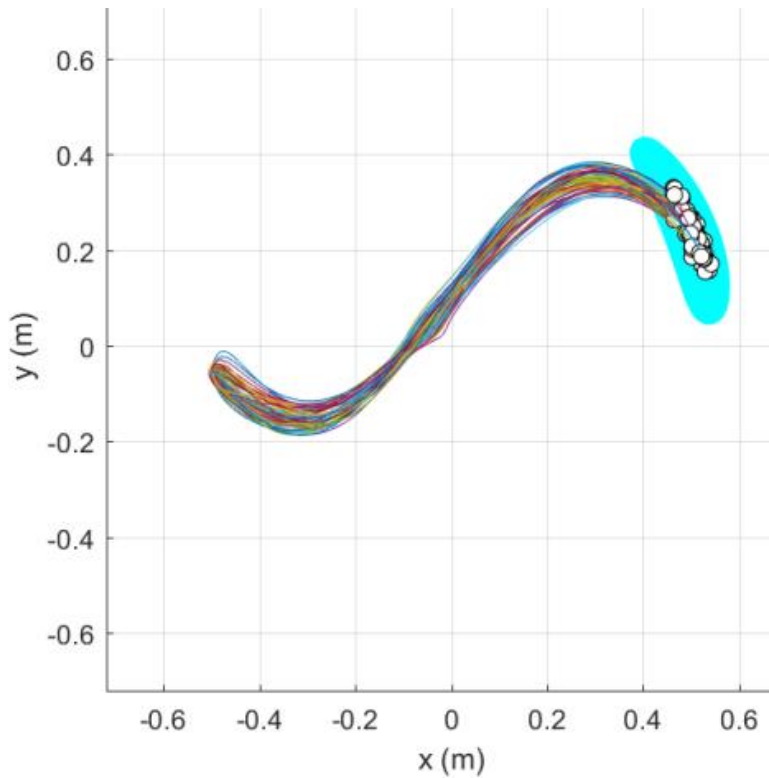
## Example 2: Two-Link Robot Arm

- Assess the worst-case L2-to-Euclidean gain from disturbances at the arm joints to the joint angles.
- LTI uncertainty with  $\|\Delta\| \leq 0.8$  injected at 2nd joint.
- Analysis performed along nominal trajectory in with LQR state feedback.

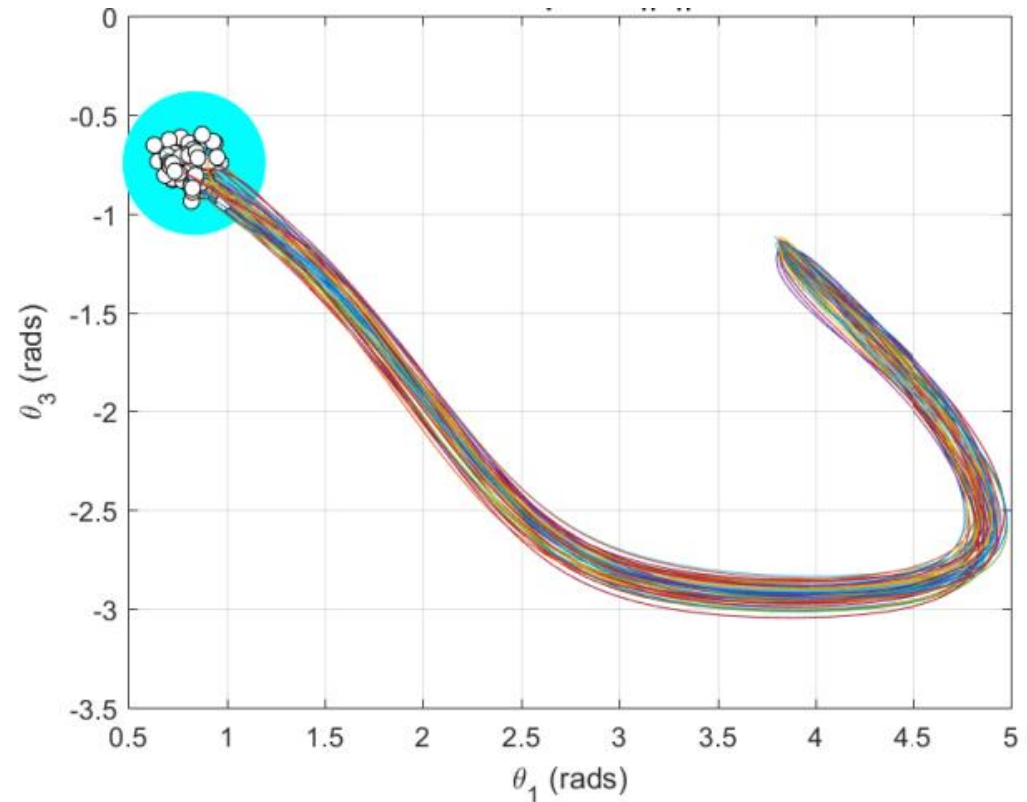


# Example 2: Results

Bound on worst-case  $L_2$ -to-Euclidean gain = 0.0592.  
Computation took 102 seconds.

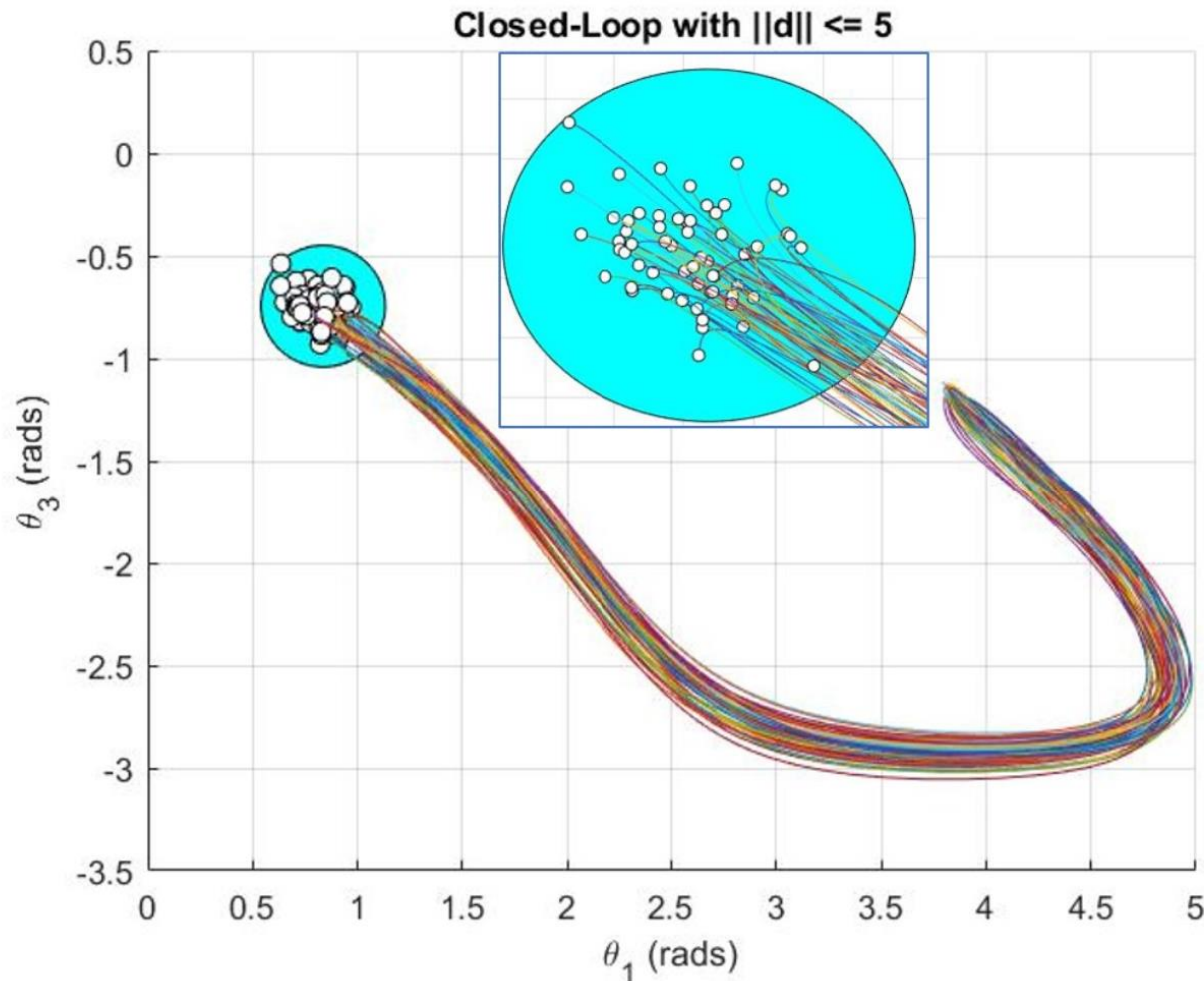


**Cartesian Coords.**



**Joint Angles**

# Worst-case Disturbance / Uncertainty



Numerically robust algorithm to construct the worst-case disturbance (work with A. Iannelli and A. Marcos)

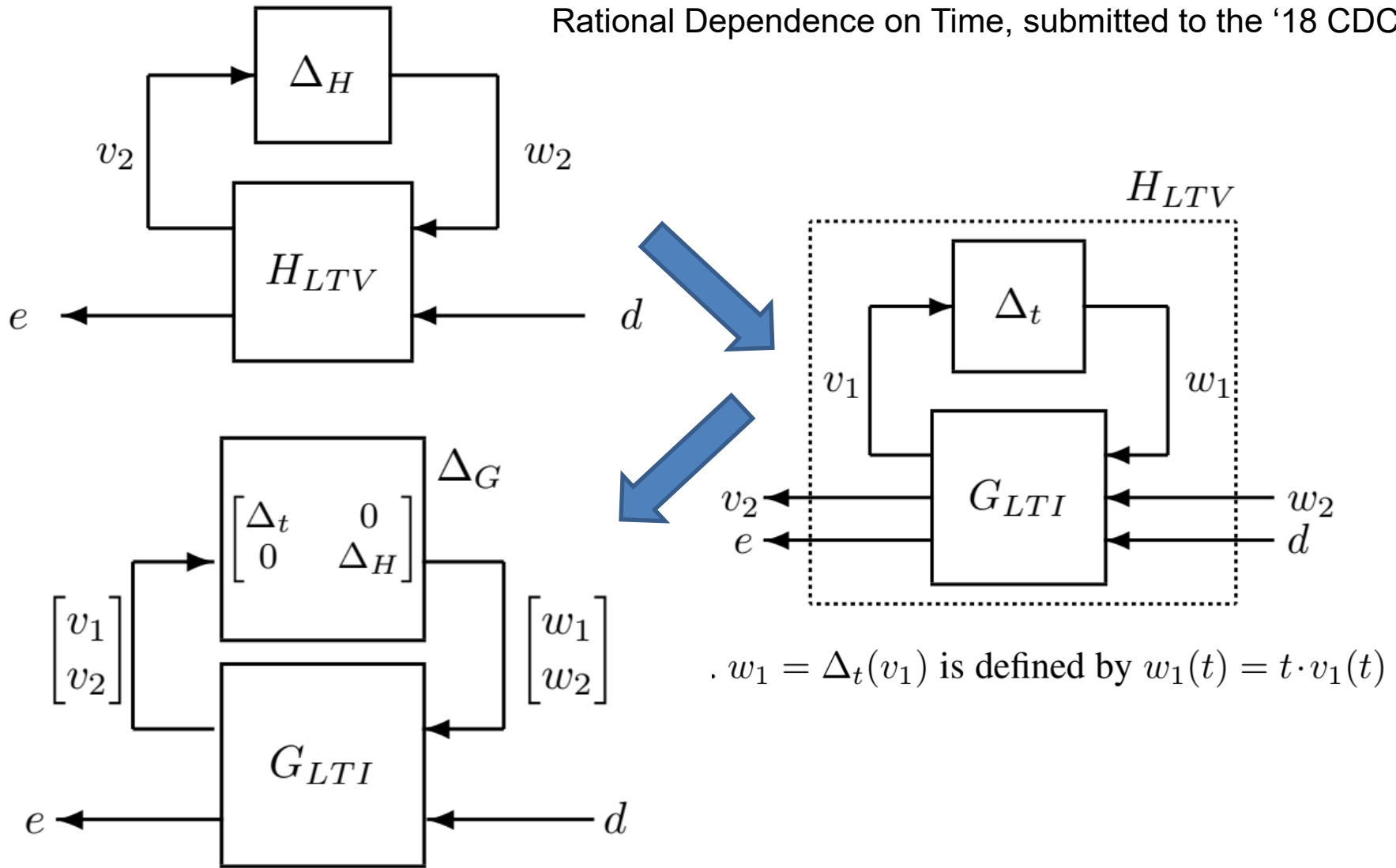
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# Extensions: Rational Dependence on Time

Ref: Seiler, IQC-Analysis of Uncertain LTV Systems With Rational Dependence on Time, submitted to the '18 CDC.





# Dynamic IQC for Time Operator $\Delta_t$

## Swapping Lemma

**Lemma 1.** Let  $F \in \mathbb{RH}_{\infty}^{n_F \times n_{v1}}$  have a realization with state matrices  $(A_F, B_F, C_F, D_F)$ . Then

$$F \Delta_{\delta} = \Delta_{\delta} F - F_C \Delta_{\delta} F_B$$

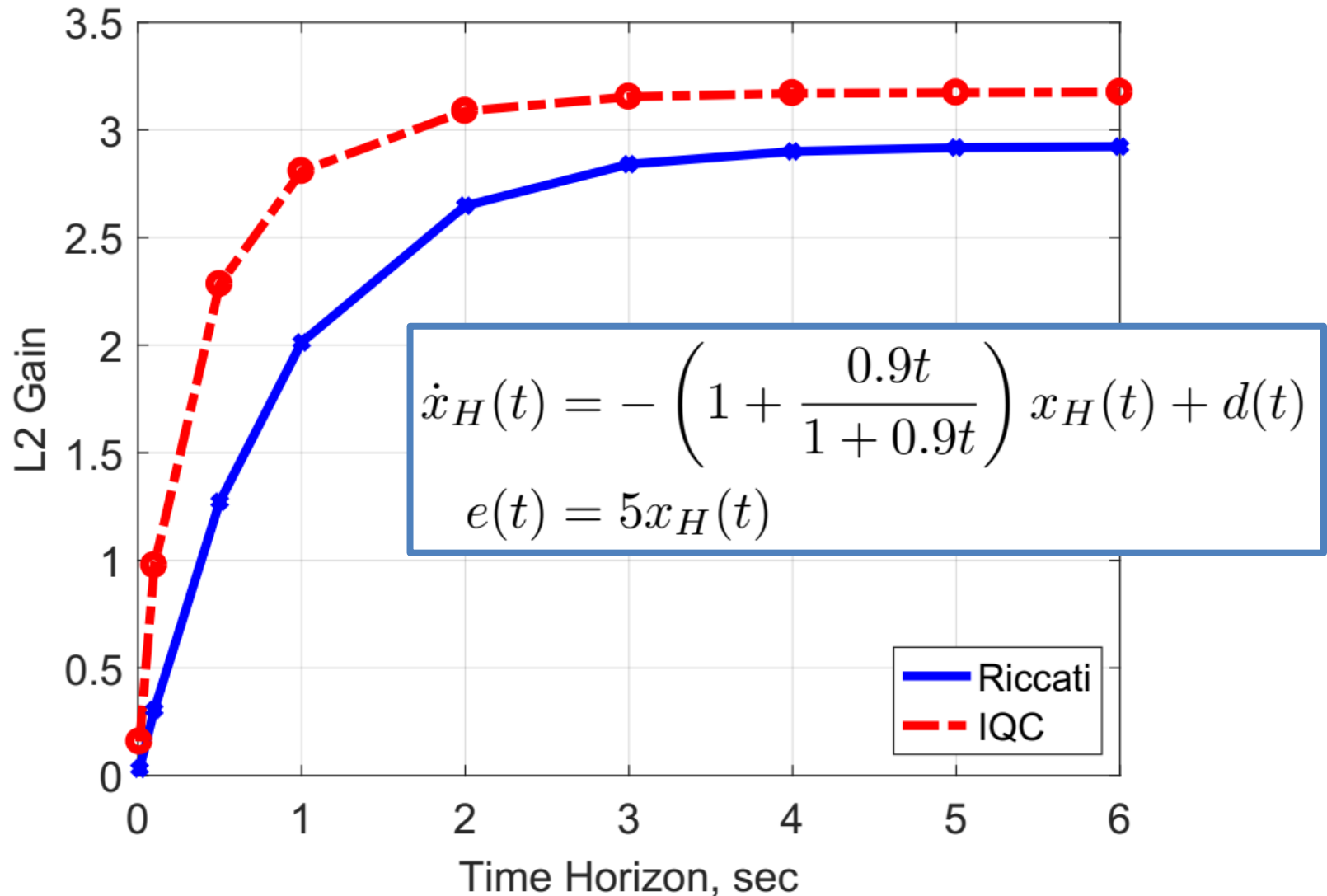
where the state realizations of  $F_B$  and  $F_C$  are given by  $F_B := (A_F, B_F, I, 0)$  and  $F_C := (A_F, I, C_F, 0)$ .

## IQC for $\Delta_t$

$$\Psi := \begin{bmatrix} F & 0 \\ F_C F_B & F \end{bmatrix} \text{ and } M := \begin{bmatrix} 0 & X + Y \\ X + Y^T & 0 \end{bmatrix}$$

for any  $X \succeq 0$  and  $Y = -Y^T$

# Example: Nominal Analysis



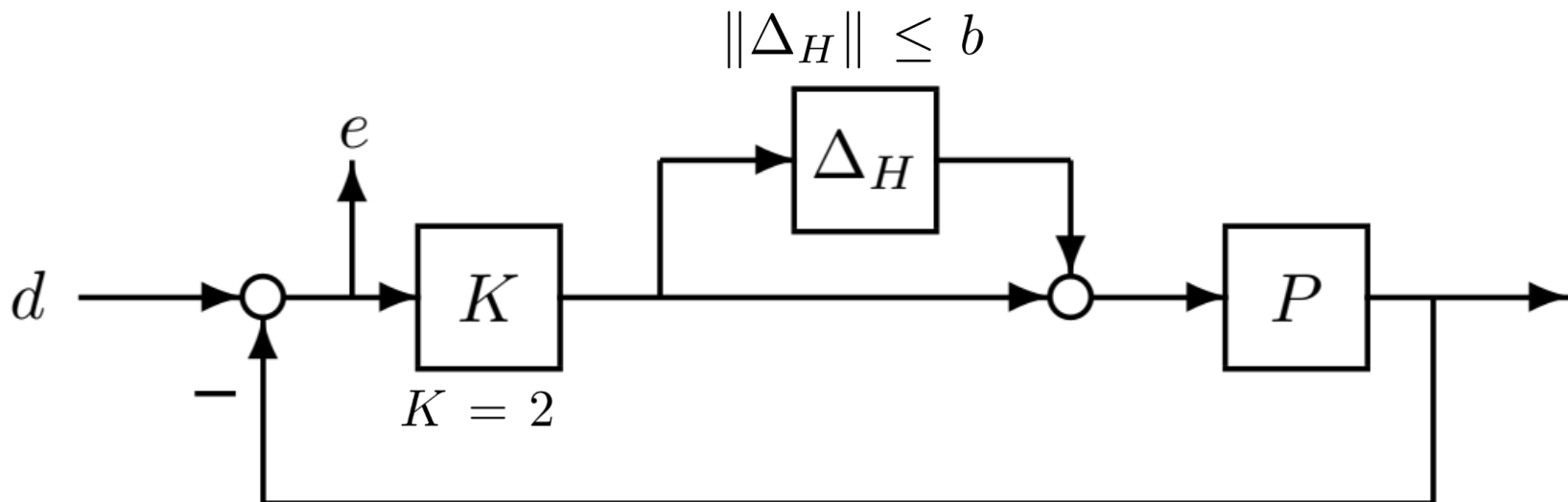
# Example: Robust Analysis

$$\dot{x}_P(t) = \begin{bmatrix} 0 & 1 \\ -4 & -2\zeta(t) \end{bmatrix} x_P(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

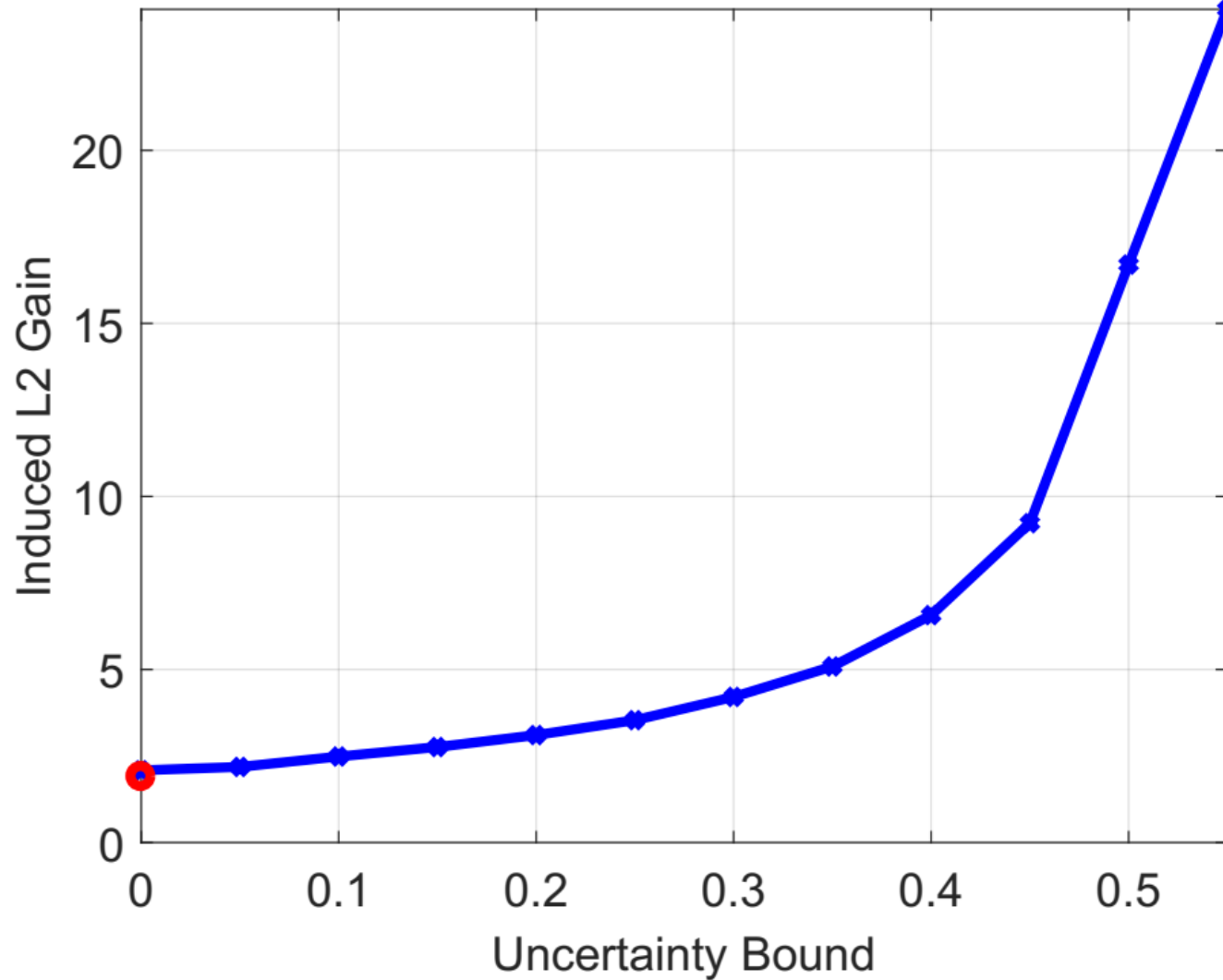
**Plant, P**

$$y(t) = \begin{bmatrix} 4 & 0 \end{bmatrix} x_P(t)$$

$$\zeta(t) := 0.2 + \frac{0.63t}{1+0.9t}$$



# Example: Robust Analysis



# Conclusions

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- **Main Result:** Bounds on finite-horizon robust performance can be computed using differential equations or inequalities.
  - These results complement the use of nonlinear Monte Carlo simulations.
  - It would be useful to construct worst-case inputs / uncertainties analogous to  $\mu$  lower bounds.
  - An LTVTools toolbox is in development with  $\beta$ -code of the proposed methods.
- **References**
  - Moore, Finite Horizon Robustness Analysis Using Integral Quadratic Constraints, MS Thesis, 2015.
  - Seiler, Moore, Meissen, Arcak, Packard, Finite Horizon Robustness Analysis of LTV Systems Using Integral Quadratic Constraints, arXiv + submission to Automatica.
  - Related work by Biertümpfel and Pfifer with application to rocket launchers submitted to the 2018 IEEE Conference on Control Technology and Applications.

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- NASA
  - NRA NNX14AL36A: "Lightweight Adaptive Aeroelastic Wing for Enhanced Performance Across the Flight Envelope," Tech. Monitor: J. Ouelette.
  - NRA NNX12AM55A: “Analytical Validation Tools for Safety Critical Systems Under Loss-of-Control Conditions.” Tech. Monitor: C. Belcastro.
  - SBIR contract #NNX12CA14C: “Adaptive Linear Parameter-Varying Control for Aeroservoelastic Suppression.” Tech. Monitor. M. Brenner.
- Eolos Consortium and Saint Anthony Falls Laboratory
  - <http://www.eolos.umn.edu/> & <http://www.safl.umn.edu/>