Robustness Analysis of Linear Parameter Varying Systems Using Integral Quadratic Constraints

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***Abstract*— A general approach is presented to analyze the worst case input/output gain for an interconnection of a linear parameter varying (LPV) system and an uncertain or nonlinear element. The input/output behavior of the nonlinear/uncertain block is described by an integral quadratic constraint (IQC). A dissipation inequality is proposed to compute an upper bound for this gain. This worst-case gain condition can be formulated as a semidefinite program and the result can be interpreted as a Bounded Real Lemma for uncertain LPV systems. The paper shows that this new condition is a generalization of the well known Bounded Real Lemma for LPV systems. The effectiveness of the proposed method is demonstrated on a simple numerical example.**

I. Introduction

This paper presents a method to analyze the robustness of a linear parameter varying (LPV) system with respect to nonlinearities and/or uncertainties. LPV systems are a class of linear systems where the state matrices depend on (measurable) time-varying parameters. The existing analysis and synthesis results for LPV systems provide a rigorous framework for design of gain-scheduled controllers. These results can roughly be categorized based on how the state matrices depend on the scheduling parameters. One approach is to assume the state matrices of the LPV system have a rational dependence on the parameters. In this case finite dimensional semidefinite programs (SDPs) can be formulated to synthesize LPV controllers [1], [2], [3]. An alternative approach is to assume the state matrices have an arbitrary dependence on the parameters. The controller synthesis problem leads to an infinite collection of parameter-dependent linear matrix inequalities (LMIs) [4], [5]. A brief review of this technical result is provided in Section II. The computational solution of such parameter-dependent LMIs requires some finite-dimensional approximation and is typically more involved. The benefit is that arbitrary parameter dependence can be considered which appears in many applications, e.g. aeroelastic vehicles [6] and wind turbines [7], [8], by linearization of nonlinear models.

Integral quadratic constraints (IQCs) are used in this paper to model the uncertain and/or nonlinear components. IQCs, introduced in [9], provide a general framework for robustness analysis. In [9] the system is separated into a feedback connection of a known linear time-invariant (LTI) system and a perturbation whose input-output behavior is described by an IQC. An IQC stability theorem was formulated in [9] with frequency domain conditions and was proved using a homotopy method.

The contribution of this paper is to provide robust performance conditions for a feedback interconnection of an LPV system and a perturbation whose input-output behavior is described by an IQC. The frequency-domain stability condition in [9] does not apply for this case because the LPV system is time-varying. Instead, this paper uses a time-domain interpretation for IQCs, described in Section II, that builds upon the work in [10], [11]. The time-domain viewpoint is used in Section III to derive dissipation-inequality conditions that bound the worst-case induced gain of the system. The conditions are derived for LPV systems whose state matrices have arbitrary dependence on the scheduling parameters. The robust performance analysis conditions can thus be viewed as generalizations of those given for nominal (not-uncertain) LPV systems in [4], [5]. The results in this paper also complement the recent robust performance results obtained for LPV systems whose state matrices have rational dependence on the scheduling parameters [12], [13], [14], [15]. In contrast to [12], [13], [14], [15], an arbitrary dependence on the parameters is assumed in this work and a finite-dimensional approximation based approach is taken. As noted above, this will enable applications to systems, e.g. aeroelastic vehicles or wind turbines, for which arbitrary dependence on scheduling variables is a natural modeling framework.

II. Background

A. Notation

Standard notation is used except for a few cases which are specifically mentioned in this section. $\mathbb{S}^n$ denotes the set of $n \times n$ symmetric matrices. $\mathbb{R}^+$ describes the set of nonnegative real numbers.

B. Analysis of LPV Systems

Linear parameter varying (LPV) systems are a class of systems whose state space matrices depend on a time-varying parameter vector $\rho : \mathbb{R}^+ \to \mathbb{R}^{n_\rho}$. The parameter is assumed to be a continuously differentiable function of time and admissible trajectories are restricted, based on physical considerations, to a known compact subset $\mathcal{P} \subset \mathbb{R}^{n_\rho}$. In addition, the parameter rates of variation $\dot{\rho} : \mathbb{R}^+ \to \dot{\mathcal{P}}$ are assumed to lie within a hyperrectangle $\dot{\mathcal{P}}$ defined by

$$\dot{\mathcal{P}} := \{q \in \mathbb{R}^{n_\rho} \,|\, \underline{\nu}_i \leq q_i \leq \bar{\nu}_i,\ i = 1, \ldots, n_\rho\}. \qquad (1)$$

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The set of admissible trajectories is defined as $\mathcal{A} := \{\rho : \mathbb{R}^+ \to \mathbb{R}^{n_\rho} \ : \ \rho(t) \in \mathcal{P}, \ \dot{\rho}(t) \in \dot{\mathcal{P}} \ \forall t \geq 0\}$. The parameter trajectory is said to be rate unbounded if $\dot{\mathcal{P}} = \mathbb{R}^{n_\rho}$.

The state-space matrices of an LPV system are continuous functions of the parameter: $A : \mathcal{P} \to \mathbb{R}^{n_x \times n_x}$, $B : \mathcal{P} \to \mathbb{R}^{n_x \times n_d}$, $C : \mathcal{P} \to \mathbb{R}^{n_e \times n_x}$ and $D : \mathcal{P} \to \mathbb{R}^{n_e \times n_d}$. An n_x^{th} order LPV system, G_ρ, is defined by

$$\begin{aligned} \dot{x}(t) &= A(\rho(t))x(t) + B(\rho(t))d(t) \\ e(t) &= C(\rho(t))x(t) + D(\rho(t))d(t) \end{aligned} \tag{2}$$

The state matrices at time t depend on the parameter vector at time t. Hence, LPV systems represent a special class of time-varying systems. Throughout the remainder of the paper the explicit dependence on t is occasionally suppressed to shorten the notation.

The performance of an LPV system G_ρ can be specified in terms of its induced L_2 gain from input d to output e. The induced L_2-norm is defined by

$$\|G_\rho\| = \sup_{d \neq 0, d \in L_2, \rho \in \mathcal{A}} \frac{\|e\|}{\|d\|}, \tag{3}$$

where $\|.\|$ represents the signal L_2-norm, i.e. $\|e\| = (\int_0^\infty e(t)^T e(t) \ dt)^{\frac{1}{2}}$. The initial condition is assumed to be $x(0) = 0$.

The notation $\rho \in \mathcal{A}$ refers to the entire (admissible) trajectory as a function of time. The analysis below leads to conditions that involve the parameter and rate at a single point in time, i.e. $(\rho(t), \dot{\rho}(t))$. The parametric description $(p, q) \in \mathcal{P} \times \dot{\mathcal{P}}$ is introduced to emphasize that such conditions only depend on the (finite-dimensional) sets $\mathcal{P}$ and $\dot{\mathcal{P}}$.

In [5] a generalization of the LTI Bounded Real Lemma is stated, which provides a sufficient condition to bound the induced L_2 gain of an LPV system. The sufficient condition uses a quadratic storage function that is defined using a parameter-dependent matrix $P : \mathcal{P} \to \mathbb{S}^{n_x}$. It is assumed that P is a continuously differentiable function of the parameter ρ. In order to shorten the notation, a differential operator $\partial P : \mathcal{P} \times \dot{\mathcal{P}} \to \mathbb{S}^{n_x}$ is introduced as in [16]. ∂P is defined as:

$$\partial P(p, q) = \sum_{i=1}^{n_\rho} \frac{\partial P(p)}{\partial p_i} q_i \tag{4}$$

The next theorem states the condition provided in [5] to bound the L_2 gain of an LPV system.

Theorem 1. ([5]): An LPV system G_ρ is exponentially stable and $\|G_\rho\| \leq \gamma$ if there exists a continuously differentiable $P : \mathcal{P} \to \mathbb{S}^{n_x}$, such that $\forall (p, q) \in \mathcal{P} \times \dot{\mathcal{P}}$

$$P(p) > 0, \tag{5}$$

$$\begin{aligned} &\begin{bmatrix} P(p)A(p) + A(p)^T P(p) + \partial P(p,q) & P(p)B(p) \\ B^T(p)P(p) & -I \end{bmatrix} \\ &\qquad + \frac{1}{\gamma^2} \begin{bmatrix} C(p)^T \\ D(p)^T \end{bmatrix} \begin{bmatrix} C(p) & D(p) \end{bmatrix} < 0. \end{aligned} \tag{6}$$

The conditions (5) and (6) are parameter-dependent LMIs that must be satisfied for all possible $(p, q) \in \mathcal{P} \times \dot{\mathcal{P}}$. Thus (5) and (6) represent an infinite collection of LMI constraints. Since q enters only affinely into the LMI and the set $\dot{\mathcal{P}}$ is a polytope, it is sufficient to check the LMI on the vertices of $\dot{\mathcal{P}}$. On the other hand, p can enter (6) nonlinearly and the set $\mathcal{P}$ does not have to be convex. A remedy to this problem, which works in many practical examples, is to approximate the set $\mathcal{P}$ by a finite set $\mathcal{P}_{grid} \subset \mathcal{P}$ that represents a gridding over $\mathcal{P}$.

In order to avoid the functional dependence of the decision variable, $P(p)$ has to be restricted to a finite dimensional subspace. A common practice [5], [17] is to restrict the storage function variable $P(p)$ to be a linear combination of basis functions,

$$P(p) = \sum_{i=1}^{N_b} g_i(p) P_i \tag{7}$$

where $g_i : \mathbb{R}^{n_\rho} \to \mathbb{R}$ are basis functions ($i = 1, \ldots, N_b$) and the matrix coefficients $P_i \in \mathcal{S}^{n_x}$ are the decision variables in the optimization.

C. Integral Quadratic Constraints

An IQC is defined by a symmetric matrix $M = M^T \in \mathbb{R}^{n_z \times n_z}$ and a stable linear system $\Psi \in \mathbb{RH}_\infty^{n_z \times (m_1 + m_2)}$. Ψ is denoted as

$$\Psi(j\omega) := C_\psi (j\omega I - A_\psi)^{-1} [B_{\psi 1} \ B_{\psi 2}] + [D_{\psi 1} \ D_{\psi 2}] \tag{8}$$

A bounded, causal operator $\Delta : L_{2e}^{m_1} \to L_{2e}^{m_2}$ satisfies an IQC defined by (Ψ, M) if the following inequality holds for all $v \in L_2^{m_1}[0, \infty)$, $w = \Delta(v)$ and $T \geq 0$:

$$\int_0^T z(t)^T M z(t) \, dt \geq 0 \tag{9}$$

where z is the output of the linear system Ψ:

$$\dot{x}_\psi(t) = A_\psi x_\psi(t) + B_{\psi 1} v(t) + B_{\psi 2} w(t), \ x_\psi(0) = 0 \tag{10}$$
$$z(t) = C_\psi x_\psi(t) + D_{\psi 1} v(t) + D_{\psi 2} w(t) \tag{11}$$

The notation $\Delta \in IQC(\Psi, M)$ is used if Δ satisfies the IQC defined by (Ψ, M). Fig. 1 provides a graphic interpretation of the IQC. The input and output signals of Δ are filtered through Ψ. If $\Delta \in IQC(\Psi, M)$ then the output signal z satisfies the (time-domain) constraint in (9) for any finite-horizon $T \geq 0$. Two simple examples are provided below to connect this terminology to standard results used in robust control.

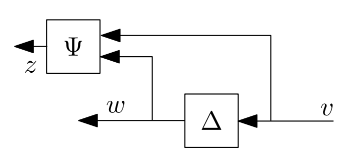

Fig. 1. Graphic interpretation of the IQC

Example 1. Consider a causal (SISO) operator Δ that satisfies the bound $\|\Delta\| \leq b$. The norm bound on Δ implies that $\|w\| \leq b\|v\|$ for any input/output pair $v \in L_2$ and $w = \Delta(v)$. This constraint on (v, w) can be expressed as the following infinite-horizon inequality:

$$\int_0^\infty \begin{bmatrix} v(t) \\ w(t) \end{bmatrix}^T \begin{bmatrix} b^2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} v(t) \\ w(t) \end{bmatrix} dt \geq 0 \qquad (12)$$

Next, the causality of Δ is used to demonstrate that, in fact, the inequality involving (v, w) holds over all finite horizons. Give any $T \geq 0$, define a new input $\tilde{v}$ by $\tilde{v}(t) = v(t)$ for $t \leq T$ and $\tilde{v}(t) = 0$ otherwise. This truncated signal $\tilde{v}$ generates an output $\tilde{w} = \Delta(\tilde{v})$. This new input/output pair $(\tilde{v}, \tilde{w})$ also satisfies $\|\tilde{w}\| \leq b\|\tilde{v}\|$. This implies

$$\begin{aligned} 0 &\leq \int_0^\infty \begin{bmatrix} \tilde{v}(t) \\ \tilde{w}(t) \end{bmatrix}^T \begin{bmatrix} b^2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \tilde{v}(t) \\ \tilde{w}(t) \end{bmatrix} dt \\ &\overset{(a)}{\leq} \int_0^T \begin{bmatrix} \tilde{v}(t) \\ \tilde{w}(t) \end{bmatrix}^T \begin{bmatrix} b^2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \tilde{v}(t) \\ \tilde{w}(t) \end{bmatrix} dt \\ &\overset{(b)}{\leq} \int_0^T \begin{bmatrix} v(t) \\ w(t) \end{bmatrix}^T \begin{bmatrix} b^2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} v(t) \\ w(t) \end{bmatrix} dt \end{aligned}$$

Inequality (a) follows because $\tilde{v}(t) = 0$ for $t \geq T$. Inequality (b) follows from the causality of Δ. Specifically, $\tilde{v}(t) = v(t)$ for $t \leq T$ and hence $\tilde{w}(t) = w(t)$ for $t \leq T$. The final conclusion is that Δ satisfies the IQC defined by (Ψ, M) with $\Psi = I_2$ and $M = \left[\begin{smallmatrix} b^2 & 0 \\ 0 & -1 \end{smallmatrix}\right]$. In this example Ψ contains no dynamics and hence $z = [v; w]$.

Example 2. Next consider an LTI (SISO) system Δ that satisfies the bound $\|\Delta\| \leq b$. Since Δ is LTI it commutes with any stable, minimum phase system $D(s)$, i.e. $\Delta D = D\Delta$. Thus the frequency-scaled system $\bar{\Delta} := D\Delta D^{-1}$ is also norm-bounded by b. Let $(\bar{v}, \bar{w})$ be any input-output pair

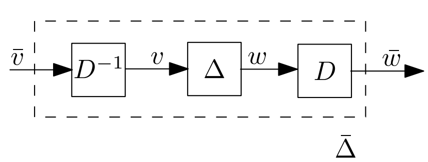

Fig. 2. Scaling of the LTI system Δ

for the scaled system $\bar{\Delta}$, see Fig. 2. The first example implies that

$$\int_0^T \begin{bmatrix} \bar{v}(t) \\ \bar{w}(t) \end{bmatrix}^T \begin{bmatrix} b^2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \bar{v}(t) \\ \bar{w}(t) \end{bmatrix} dt \geq 0 \qquad (13)$$

The associated input/output pair for the original system $w = \Delta(v)$ is related to the input/output pair for the scaled system by $\bar{w} = Dw$ $\bar{v} = Dv$. Thus the inequality in (13) can be equivalently written as

$$\int_0^T z(t)^T M z(t) \geq 0 \qquad (14)$$

where $M = \left[\begin{smallmatrix} b^2 & 0 \\ 0 & -1 \end{smallmatrix}\right]$ and z is the output of $\Psi := \left[\begin{smallmatrix} D & 0 \\ 0 & D \end{smallmatrix}\right]$ generated by the signals $\left[\begin{smallmatrix} v \\ w \end{smallmatrix}\right]$, see Fig. 1. Hence Δ satisfies the IQC defined by (Ψ, M). Note that the use of $D(s)$ directly corresponds to the multipliers used in classical robustness analysis, e.g. the structured singular value μ [18], [19], [20], [21].

The two examples above are simple instances of IQCs. The proposed approach applies to the more general IQC framework introduced in [9] but with some technical restrictions. In particular, [9] provides a library of IQC multipliers that are satisfied by many important system components, e.g. saturation, time delay, and norm bounded uncertainty. The IQCs in [9] are expressed in the frequency domain as an integral constraint defined using a multiplier Π. The multiplier Π can be factorized as $\Pi = \Psi^* M \Psi$ and this connects the frequency domain formulation to the time-domain formulation used in this paper. One technical point is that, in general, the time domain IQC constraint only holds over infinite horizons ($T = \infty$). The work in [9], [10] draws a distinction between hard/complete IQCs for which the integral constraint is valid over all finite time intervals and soft/conditional IQCs for which the integral constraint need not hold over finite time intervals. The formulation of an IQC in this paper as a finite-horizon (time-domain) inequality is thus valid for any frequency-domain IQC that admits a hard/complete factorization (Ψ, M). While this is somewhat restrictive, it has recently been shown in [10] and [11] that a wide class of IQCs have a hard factorization. The remainder of the paper will simply treat, without further comment, (Ψ, M) as the starting point for the definition of the finite-horizon IQC.

Integral quadratic constraints were introduced in [9] to provide a general framework for robustness analysis. In this framework the system is separated into a feedback connection of a known linear time-invariant (LTI) system and a perturbation whose input-output behavior is described by an IQC. The IQC stability theorem in [9] was formulated with frequency domain conditions and was proved using a homotopy method. A contribution of this paper is to use the time-domain view of IQCs in order to derive stability conditions for the interconnection of a known linear parameter-varying (LPV) system and a perturbation whose input-output behavior is described by an IQC. The LPV system is time varying and hence a frequency-domain stability condition is not suitable for such interconnections. A dissipation inequality condition is derived to determine stability of the LPV system in the presences of perturbations.

III. LPV Robustness Analysis

An uncertain LPV system is described by the interconnection of an LPV system G_ρ and an uncertainty Δ, as depicted in Fig. 3. This interconnection represents an upper linear fractional transformation (LFT), which is denoted $\mathcal{F}_u(G_\rho, \Delta)$. The uncertainty Δ is assumed to satisfy an IQC described by (Ψ, M). Note that the perturbation Δ can include hard nonlinearities (e.g. saturations) and infinite dimensional operators (e.g. time delays) in addition to true system uncertainties. The term ”uncertainty” is used for simplicity when referring to the the perturbation Δ.

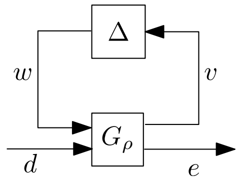

Fig. 3. Uncertain LPV System

The robust performance of $\mathcal{F}_u(G_\rho, \Delta)$ is measured in terms of the worst case induced L_2 gain from the input d to the output e. The worst-case gain is defined as

$$\sup_{\Delta \in IQC(\Psi, M)} \|\mathcal{F}_u(G_\rho, \Delta)\|. \tag{15}$$

This gain is worst-case over all uncertainties Δ that satisfy the IQC defined by (Ψ, M) and admissible trajectories ρ.

A. Bounded Real Lemma including IQCs

As proposed in [11], the basic LFT interconnection in Fig. 3 is considered where Δ satisfies the IQC defined by (Ψ, M). In this basic interconnection the filter Ψ is included as shown in Fig. 4.

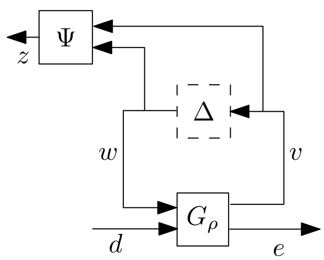

Fig. 4. Analysis Interconnection Structure

The dynamics of the interconnection in Fig. 4 are described by $w = \Delta(v)$ and

$$\begin{aligned} \dot{x} &= A(\rho)x + B_1(\rho)w + B_2(\rho)d \\ z &= C_1(\rho)x + D_{11}(\rho)w + D_{12}(\rho)d \\ e &= C_2(\rho)x + D_{21}(\rho)w + D_{22}(\rho)d, \end{aligned} \tag{16}$$

where the state vector is $x = [x_G; x_\psi]$ with x_G and x_ψ being the state vectors of the LPV system G_ρ and the filter Ψ respectively. The uncertainty Δ is shown in the dashed box in Fig. 4 to signify that it is removed for the analysis. The signal w is treated as an external signal subject to the constraint

$$\int_0^T z(t)^T M z(t) \, dt \geq 0. \tag{17}$$

This effectively replaces the precise relation $w = \Delta(v)$ with the quadratic constraint on z.

A dissipation inequality can be formulated to upper bound the worst-case L_2 gain of $\mathcal{F}_u(G_\rho, \Delta)$ using the system (16) and the time domain IQC (17). The following theorem is based on the dissipation inequality framework given in [11].

Theorem 2. Assume $\mathcal{F}_u(G_\rho, \Delta)$ is well posed for all $\Delta \in$ IQC(Ψ, M). Then the worst-case gain is $\leq \gamma$ if there exists a continuously differentiable $P : \mathcal{P} \to \mathbb{S}^{n_x}$ and a scalar $\lambda > 0$ such that $\forall (p, q) \in \mathcal{P} \times \dot{\mathcal{P}}$

$$P(p) > 0, \tag{18}$$

$$\begin{bmatrix} P(p)A + A^T P(p) + \partial P(p, q) & P(p)B_1 & P(p)B_2 \\ B_1^T P(p) & 0 & 0 \\ B_2^T P(p) & 0 & -I \end{bmatrix} + \\ +\lambda \begin{bmatrix} C_1^T \\ D_{11}^T \\ D_{12}^T \end{bmatrix} M \begin{bmatrix} C_1 & D_{11} & D_{12} \end{bmatrix} \\ + \frac{1}{\gamma^2} \begin{bmatrix} C_2^T \\ D_{21}^T \\ D_{22}^T \end{bmatrix} \begin{bmatrix} C_2 & D_{21} & D_{22} \end{bmatrix} < 0 \tag{19}$$

In (19) the dependency of the state space matrices on p has been omitted to shorten the notation.

The same considerations in regard to gridding and basis functions for $P(p)$ as in Section II-B have to be taken into account, in order to turn Theorem 2 into a computational tractable optimization problem.

In many practical examples, the IQCs include frequency-dependent weightings which do not necessarily have to be rational functions. For instance, the IQC factorization for an LTI norm-bounded operator uses a scaling $D(s)$. In the classical μ-framework, a frequency gridding is applied, so that $D(j\omega)$ can be solved for at each frequency individually. This freedom is lost in the presented approach, as a single $D \in \mathbb{RH}_\infty$ has to be specified. To overcome this shortcoming a basis function approach is used, i.e.

$$D(s) = \sum_{i=1}^{N_D} \alpha_i D_i(s). \tag{20}$$

α_i can be treated as free decision variable by using an independent factorization (Ψ_i, M_i) for each $D_i(s)$ and allowing different λ_i in condition (19) for each factorization. This is analogous to the use of many multipliers in the original IQC analysis [9].

IV. Numerical Example

A simple example is used to demonstrate the applicability of the proposed method. The example is a feedback interconnection of a first-order LPV system with a gain-scheduled proportional-integral controller as shown in Fig. 5.

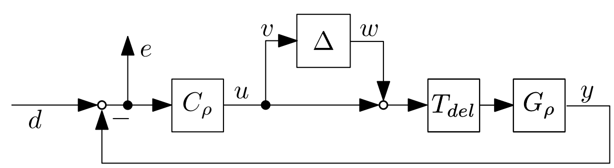

Fig. 5. Closed Loop Interconnection with Dynamic Uncertainty

The system G_ρ, taken from [22], is a first order system with dependence on a single parameter ρ. It can be written as

$$\dot{x}_G = -\frac{1}{\tau(\rho)} x_G + \frac{1}{\tau(\rho)} u_G \\ y = K(\rho) x_G \tag{21}$$

with the time constant $\tau(\rho)$ and output gain $K(\rho)$ depending on the scheduling parameter as follows:

$$\tau(\rho) = \sqrt{133.6 - 16.8\rho} \\ K(\rho) = \sqrt{4.8\rho - 8.6}. \tag{22}$$

The scheduling parameter ρ is restricted to the interval $[2, 7]$. For all the following analysis scenarios a grid of six points is used that span the parameter space equidistantly. A time-delay of 0.5 seconds is included at the control input. The time delay in the system is represented by a second order Pade approximation:

$$T_{del}(s) = \frac{\frac{(T_d s)^2}{12} - \frac{T_d s}{2} + 1}{\frac{(T_d s)^2}{12} + \frac{T_d s}{2} + 1}, \tag{23}$$

where $T_d = 0.5$.

For G_ρ a gain-scheduled PI-controller C_ρ is designed that guarantees a closed loop damping $\zeta_{cl} = 0.7$ and a closed loop frequency $\omega_{cl} = 0.25$ at each frozen value of ρ. The controller gains that satisfy these requirements are given by

$$K_p(\rho) = \frac{2\zeta_{cl}\omega_{cl}\tau(\rho) - 1}{K(\rho)} \\ K_i(\rho) = \frac{\omega_{cl}^2 \tau(\rho)}{K(\rho)}. \tag{24}$$

The controller is realized in the following state space form:

$$\dot{x}_c = K_i(\rho) e \\ u = x_c + K_p(\rho) e \tag{25}$$

As an extension to the original example in [22], a multiplicative norm bounded uncertainty Δ is inserted at the input of the plant. The norm of the uncertainty is assumed to be less than a positive scalar b, i.e. $\|\Delta\| \leq b$. The uncertainty is described by the integral quadratic constraints (Ψ, M) with $\Psi = I$ and $M = \left[\begin{smallmatrix} b^2 & 0 \\ 0 & -1 \end{smallmatrix}\right]$. The gain of the channel from the reference signal d to the control error e is used as performance measurement and γ is an upper bound on the worst case gain as defined by (15).

Established techniques are used to compute two gains for comparison with the proposed method: (I) the worst-case LTI gain in the presence of uncertainty at each frozen value of ρ, and (II) the bound on induced L_2 gain of the nominal LPV system ($\Delta = 0$). The worst-case LTI gain (I) at each frozen value of ρ is computed using the Matlab command `wcgain` [23]. The nominal LPV gain (II) is computed using the standard LMI condition in Theorem 1.

The first analysis is performed assuming unbounded parameter variation rates. Hence, a constant matrix P is assumed. In this case, the nominal gain for the system without uncertainty is 18.7. The results of the robust LPV performance analysis for different uncertainty sizes is shown in the top plot of Fig. 6. The convex optimization could not find feasible solutions for $b > 0.1$, meaning that for uncertainties larger than ten percent no finite gain can be guaranteed. For the case $b = 0.05$ this analysis yields a bound on the worst case gain of 29.1. The following approach is used to estimate the conservativeness of the upper bound. First, the frozen LTI analysis (I) is performed for $b = 0.05$ to obtain the worst values of ρ and Δ. The worst value of Δ returned by `wcgain` is used to construct a (not-uncertain) LPV system $F_u(G_\rho, \Delta)$. The conditions in Theorem 1 are then used to compute the induced L_2 gain bound for this system. The result of this analysis is a gain of 27.5 which his close to the computed worst-case gain of 29.1. This approach does not provide a true lower bound on the worst-case gain due to the conservatism in the Bounded Real Lemma for LPV systems. However, it does provide an indication of the conservatism in the proposed robustness bound.

In the next step, the parameter variation rate is bounded by $|\dot{\rho}| \leq 2$. Affine and quadratic parameter dependences are considered for P, i.e. $P(\rho) = P_0 + \rho P_1$ and $P(\rho) = P_0 + \rho P_1 + \rho^2 P_2$ respectively. The results are given in the middle plot of Fig. 6. Allowing a higher order $P(\rho)$ reduces the worst case gains and also allows finding finite gains up to $b = 0.3$. In comparison, the affine storage function could only guarantee finite gains for $b \leq 0.25$.

Finally, to compare the proposed method with the LTI gains the rate is bounded by $|\dot{\rho}| \leq 0.1$. Both affine and quadratic dependences of $P(\rho)$ are presented in relation to the results of the Matlab function `wcgain` in the lower plot of Fig. 6. The following two facts mostly contribute to the difference between the results from `wcgain` and the proposed method: First, the parameter dependence of the storage function is restricted in the latter, while the first can compute a individual P at each grid point. Second, `wcgain` assumes that Δ is an LTI operator, whereas it can be any norm-bounded operator in the latter case.

V. Conclusions

In this paper the analysis frameworks for LPV systems and uncertainties described by IQCs have been combined. This leads to computationally efficient conditions to assess the robust performance of an LPV system interconnected with uncertainties and nonlinearities. The proposed robust LPV analysis framework is a generalization of the well known nominal LPV Bounded Real Lemma. A simple numerical example was presented to show the potential of the proposed approach. Future work will consider the synthesis of robust controllers for uncertain LPV systems.

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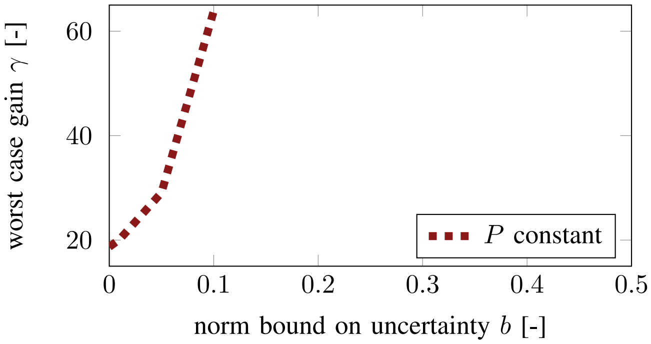
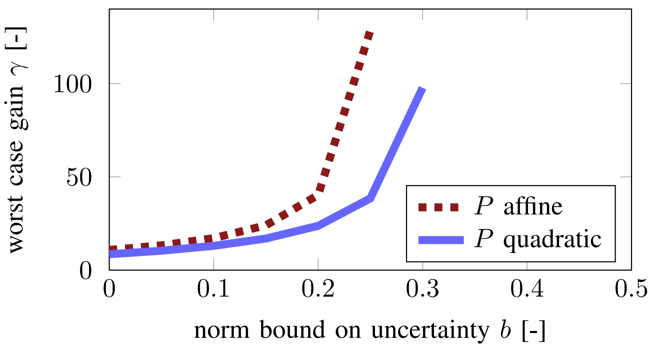
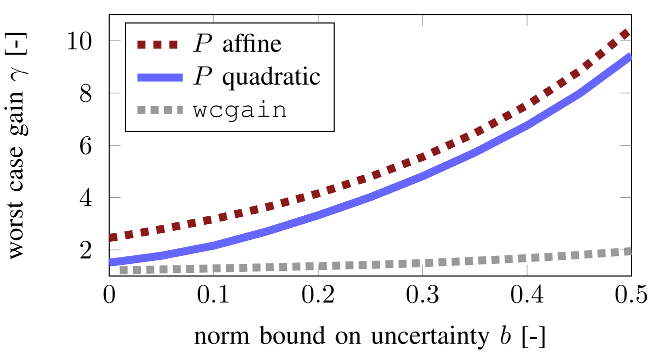


(a) Unbounded Parameter Variation Rate



(b) Parameter Variation bounded by $|\dot{\rho}| \leq 2$



(c) Parameter Variation bounded by $|\dot{\rho}| \leq 0.1$

Fig. 6. Worst Case Gain vs Norm Bound on Uncertainty

conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of NASA or NSF.

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