Robustness Analysis with Parameter-Varying Integral Quadratic Constraints

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***Abstract*— The paper considers the analysis of the worst-case input/output gain of an interconnection of a known linear parameter varying system and a perturbation. The input/output behavior of the perturbation is described by an integral quadratic constraint (IQC). Recent results have shown that under certain technical conditions IQCs can be formulated as a finite horizon time domain constraint. The worst-case input/output gain of the interconnection can then be bounded using a dissipation inequality that incorporates the IQCs. Unlike the classical frequency domain approach to IQCs, this time domain interpretation opens up a new class of IQCs, where the IQC itself is parameter-varying. Various examples for parameter-varying IQCs for different classes of perturbations are given. A simple numerical example shows that the introduction of parameter-varying IQCs can lead to less conservative bounds on the worst-case gain.**

I. Introduction

Integral quadratic constrains (IQC) [1] provide a general framework for robustness analysis. In the classical IQC framework, an interconnection of a known linear time invariant system and a perturbation is considered. The input/output behavior of the perturbation is bounded by an integral quadratic constraint in the frequency domain. The IQC itself is defined by a multiplier which, in this approach, is a linear time invariant system. Based on the results of [2], it has been recently shown that an equivalent time domain approach can be taken based on dissipation inequalities [3], [4]. These results are summarized in Section II.

Using the time-domain viewpoint the IQC framework can be extended to consider the interconnection of a known linear parameter varying (LPV) system and a perturbation described by a time-domain IQC [5], [6], [7], [8]. The work in [5], [6], [7] consider LPV systems whose state matrices have rational (linear fractional) dependence on the scheduling parameters. The work in [8], on the other hand, provides analysis conditions for LPV systems with arbitrary (possibly non-rational) parameter dependence. This paper also focuses on systems with arbitrary parameter dependence but most results can be specialized to systems with rational parameter dependence. A brief review of the related existing technical results is presented in Section II.

The contribution of this paper is the introduction of a new class of parameter-varying IQCs based on the dissipation inequality framework. Unlike the classical frequency domain approach to IQCs [1], the time domain interpretations allows for the IQC multiplier to be time varying and/or nonlinear. In this work specifically parameter-varying IQCs are studied for the analysis of an LPV system under a perturbation. In Section III the notion of these parameter-varying IQCs is defined and examples for different operators that can be bounded by this new class of IQCs are given. Allowing the IQC to be parameter varying gives additional flexibility which can lead to less conservative results as is shown in the numerical example given in Section IV.

Note that results for parameter-dependent multipliers exist in the literature. For example, the results in [9], [10] consider a known LTI system G interconnected with real parameter uncertainty Δ specified as the LFT $F_u(G,\Delta)$. Analysis conditions are developed based on a parameter-dependent multiplier. The multiplier for Δ depends on Δ itself. This paper considers different class of parameter-dependent multiplier. Specifically, the problem formulation considers the interconnection $F_u(G_\rho,\Delta)$ of an LPV system G_ρ and an uncertainty Δ. The objective is to develop a multiplier for Δ that depends on the parameter ρ that appears in the known part of the model. This allows less conservative results for example if Δ itself depends on ρ as shown in [11].

II. Background

A. Integral Quadratic Constraints

This section describes IQCs for a bounded, causal operator Δ with the input/output behavior described by $w=\Delta(v)$. The input/output signals of Δ can be bounded by an IQC. A precise definition is given below for an IQC in the time domain.

Definition 1. Let M be a symmetric matrix, i.e. $M = M^T \in \mathbb{R}^{n_z\times n_z}$ and Ψ a stable linear system, i.e. $\Psi \in \mathbb{RH}_\infty^{n_z\times(n_v+n_w)}$. A bounded, causal operator $\Delta : L_{2e}^{n_v} \to L_{2e}^{n_w}$ satisfies an IQC defined by (Ψ, M) if the following inequality holds $\forall v \in L_2^{n_v}[0,\infty)$, $w=\Delta(v)$ and $T\geq 0$:

$$\int_0^T z(t)^T M z(t)\, dt \geq 0 \tag{1}$$

where z is the output of the linear system Ψ:

$$\begin{aligned} \dot{x}_\psi(t) &= A_\psi x_\psi(t) + B_{\psi 1} v(t) + B_{\psi 2} w(t),\ x_\psi(0)=0 \\ z(t) &= C_\psi x_\psi(t) + D_{\psi 1} v(t) + D_{\psi 2} w(t) \end{aligned} \tag{2}$$

The notation $\Delta \in IQC(\Psi, M)$ is used if Δ satisfies the IQC defined by (Ψ, M).

Fig. 1 provides a graphic interpretation of the IQC. The input and output signals of Δ are filtered through Ψ. If $\Delta \in IQC(\Psi, M)$ then the output signal z satisfies the constraint in Equation (1) for any finite-horizon $T \geq 0$.

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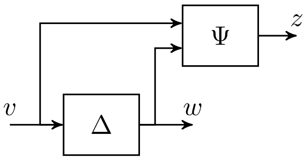

Fig. 1. Graphic interpretation of the IQC

The proposed approach applies to the more general IQC framework introduced in [1] but with some technical restrictions. In particular, [1] provides a library of IQC multipliers that are satisfied by many important system components, e.g. saturation, time delay, and norm bounded uncertainty. The IQCs in [1] are expressed in the frequency domain as an integral constraint defined using a multiplier Π. The multiplier Π can be factorized as $\Pi = \Psi^* M \Psi$ and this connects the frequency domain formulation to the time-domain formulation used in this paper. One technical point is that, in general, the time domain IQC constraint only holds over infinite horizons ($T = \infty$). The work in [1], [2] draws a distinction between hard/complete IQCs for which the integral constraint is valid over all finite time intervals and soft/conditional IQCs for which the integral constraint need not hold over finite time intervals. The formulation of an IQC in this paper as a finite-horizon (time-domain) inequality is thus valid for any frequency-domain IQC that admits a hard/complete factorization (Ψ, M). While this is somewhat restrictive, it has recently been shown in [2] and [12] that a wide class of IQCs have a hard factorization. The remainder of the paper will simply treat, without further comment, (Ψ, M) as the starting point for the definition of the finite-horizon IQC.

B. Robustness Analysis of LPV Systems

Linear Parameter Varying (LPV) systems are a class of linear systems whose state space matrices depend on a time-varying parameter vector $\rho : \mathbb{R} \to \mathbb{R}^{n_\rho}$. The parameter is assumed to be a continuously differentiable function of time and admissible trajectories are restricted, based on physical considerations, to a known compact subset $\mathcal{P} \subset \mathbb{R}^{n_\rho}$. The state-space matrices of an LPV system are continuous functions of the parameter, e.g. $A_G : \mathcal{P} \to \mathbb{R}^{n_x \times n_x}$. Define the LPV system G_ρ with inputs (w, d) and outputs (v, e) as:

$$\begin{aligned} \dot{x}_G(t) &= A_G(\rho(t))x_G(t) + B_G(\rho(t)) \begin{bmatrix} w(t) \\ d(t) \end{bmatrix} \\ \begin{bmatrix} v(t) \\ e(t) \end{bmatrix} &= C_G(\rho(t))x_G(t) + D_G(\rho(t)) \begin{bmatrix} w(t) \\ d(t) \end{bmatrix} \end{aligned} \tag{3}$$

The state matrices at time t depend on the parameter vector at time t. Hence, LPV systems represent a special class of time-varying systems. Throughout this section the explicit dependence on t is suppressed to shorten the notation.

An uncertain LPV system is described by the interconnection of an LPV system G_ρ and an uncertainty Δ. This interconnection represents an upper linear fractional transformation (LFT), which is denoted $F_u(G_\rho, \Delta)$. The uncertainty Δ is assumed to satisfy an IQC described by (Ψ, M). In the basic interconnection $F_u(G_\rho, \Delta)$ the filter Ψ is included as shown in Fig. 2. For fixed Δ, $\|F_u(G_\rho, \Delta)\|$ will denote the largest L_2 gain over all allowable parameter trajectories:

$$\|F_u(G_\rho, \Delta)\| = \sup_{\rho \in \mathcal{P}} \sup_{0 \neq d \in L_2^{n_d}[0,\infty),\ x_G(0)=0} \frac{\|e\|}{\|d\|} \tag{4}$$

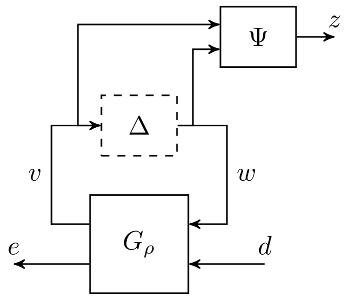

Fig. 2. Analysis Interconnection

The dynamics of the interconnection in Fig. 2 depends on an extended LPV system of the form:

$$\begin{aligned} \dot{x} &= A(\rho)x + B_1(\rho)w + B_2(\rho)d \\ z &= C_1(\rho)x + D_{11}(\rho)w + D_{12}(\rho)d \\ e &= C_2(\rho)x + D_{21}(\rho)w + D_{22}(\rho)d, \end{aligned} \tag{5}$$

where the state vector is $x := \begin{bmatrix} x_G \\ x_\psi \end{bmatrix} \in \mathbb{R}^{n_G + n_\psi}$ with x_G and x_ψ denoting the state vectors of the LPV system G_ρ (3) and the filter Ψ (2), respectively. A dissipation inequality can be formulated to upper bound the worst-case L_2 gain of $\mathcal{F}_u(G_\rho, \Delta)$ (over all uncertainties) using the system (5) and the time domain IQC (1). This dissipation inequality is concretely expressed as a linear matrix inequality in the following theorem.

Theorem 1. Let Δ satisfy IQC(Ψ, M) and assume $F_u(G_\rho, \Delta)$ is well posed. Then $\|F_u(G_\rho, \Delta)\| \leq \gamma$ if there exists a scalar $\lambda > 0$ and a matrix $P = P^T \in \mathbb{R}^{n_G + n_\psi}$ such that $P \geq 0$ and for all $\rho \in \mathcal{P}$:

$$\begin{aligned} \begin{bmatrix} PA + A^T P & PB_1 & PB_2 \\ B_1^T P & 0 & 0 \\ B_2^T P & 0 & -I \end{bmatrix} + \lambda \begin{bmatrix} C_1^T \\ D_{11}^T \\ D_{12}^T \end{bmatrix} M \begin{bmatrix} C_1 & D_{11} & D_{12} \end{bmatrix} \\ + \frac{1}{\gamma^2} \begin{bmatrix} C_2^T \\ D_{21}^T \\ D_{22}^T \end{bmatrix} \begin{bmatrix} C_2 & D_{21} & D_{22} \end{bmatrix} < 0 \end{aligned} \tag{6}$$

In Equation (6) the dependency of the state space matrices on ρ has been omitted to shorten the notation.

Proof. The proof is based on defining a storage function $V : R^{n_x + n_\psi} \to \mathbb{R}^+$ by $V(x) = x^T P x$. Left and right multiply Equation (6) by $[x^T, w^T, d^T]$ and $[x^T, w^T, d^T]^T$ to show that V satisfies the dissipation inequality:

$$\lambda z(t)^T M z(t) + \dot{V}(t) \leq \gamma^2 d(t)^T d(t) - e(t)^T e(t) \tag{7}$$

The dissipation inequality Equation (7) can be integrated from $t = 0$ to $t = T$ with the initial condition $x(0) = 0$

to yield:

$$\lambda \int_0^T z(t)^T M z(t)\, dt + V(x(T)) \leq \tag{8}$$
$$\gamma^2 \int_0^T d(t)^T d(t)\, dt - \int_0^T e(t)^T e(t)\, dt$$

It follows from the IQC condition Equation (1), $\lambda \geq 0$, and the non-negativity of the storage function V that

$$\int_0^T e(t)^T e(t)\, dt \leq \gamma^2 \int_0^T d(t)^T d(t)\, dt \tag{9}$$

Hence $\|F_u(G, \Delta)\| \leq \gamma$. □

A detailed proof of Theorem 1 as well as an extension to LPV systems with bounded parameter variation rates using parameter dependent storage functions can be found in [8]. Note that Theorem 1 is a generalization of the Bounded Real Lemma like condition for known LPV systems introduced in [13]. The connection is shown in detail in [8].

III. Parameter-varying IQCs

The dissipation inequality framework opens up the possibility for new classes of IQCs. Unlike the classical frequency domain approach to IQCs, the time domain interpretation allows for Ψ and/or M to be time varying and/or nonlinear. This generalizes the theory in [1] which is restricted to the case where Ψ and M are an LTI system and constant matrix, respectively. As a specific example, Theorem 1 can be extended to consider parameter-varying IQCs where Ψ_ρ and/or M_ρ could depend on ρ. A formal definition of parameter-varying IQCs is now given which extends the one given in Definition 1.

Definition 2. Assume $M_\rho : \mathcal{P} \rightarrow \mathbb{R}^{n_z \times n_z}$, such that $M_\rho(\rho) = M_\rho(\rho)^T$ for all $\rho \in \mathcal{P}$. In addition, let Ψ_ρ be a stable LPV system of the form

$$\begin{aligned} \dot{x}_\psi &= A_\psi(\rho) x_\psi + B_{\psi 1}(\rho) v + B_{\psi 2}(\rho) w,\ x_\psi(0) = 0 \\ z &= C_\psi(\rho) x_\psi + D_{\psi 1}(\rho) v + D_{\psi 2}(\rho) w. \end{aligned} \tag{10}$$

A bounded, causal operator $\Delta : L_{2e}^{n_v} \rightarrow L_{2e}^{n_w}$ satisfies an parameter-varying IQC defined by (Ψ_ρ, M_ρ) if the following inequality holds for all $v \in L_2^{n_v}[0, \infty)$, $w = \Delta(v)$, $\rho \in \mathcal{P}$ and $T \geq 0$:

$$\int_0^T z(t)^T M_\rho(\rho(t)) z(t)\, dt \geq 0 \tag{11}$$

where z is the output of the LPV system Ψ_ρ.

The remainder of the section provides various examples of parameter-varying IQCs for different classes of uncertainties/nonlinearities. In general, IQCs can be categorized in the following ways: First, either the described operator is memoryless or not, meaning that Δ may or may not include internal dynamics. In the paper, examples for both of these cases are given. Second, the IQC itself can be memoryless or not. This implies that either Ψ_ρ is a dynamical system or the identity matrix. This paper only considers the latter case where $\Psi_\rho = I_{n_z}$. The consideration of a dynamical, potentially parameter-varying Ψ_ρ will be pursued in future research.

A. Time-varying Parametric Uncertainties

The first example considers repeated time varying real parameters, i.e. $\Delta(t) = \delta(t) I_{n_v}$, where $\delta : \mathbb{R} \rightarrow \mathbb{R}$ with $\|\delta\|_\infty \leq 1$. For Δ of this form a simple IQC in [1] is given by $\Psi := I_{2n_v}$ and $M := \begin{bmatrix} X & Y \\ Y^T & -X \end{bmatrix}$ where $X \geq 0$ and $Y = -Y^T$. This IQC is equivalent to the use of constant D and G-scales in classical robustness analysis, e.g. the structured singular value μ [14], [15], [16]. Consider the case of a parameter-varying system with this time-varying real parameter uncertainty, i.e. $\mathcal{F}_u(G_\rho, \Delta)$. The constant IQC just described for Δ is, in general, conservative in this case. The conservatism can be reduced by allowing the IQC from [1] to depend on ρ. The next Lemma demonstrates that a parameter-dependent IQC is indeed a valid IQC for Δ.

Lemma 1. Let $X : \mathcal{P} \rightarrow \mathbb{R}^{n_v \times n_v}$ and $Y : \mathcal{P} \rightarrow \mathbb{R}^{n_v \times n_v}$ satisfy $X(\rho) \geq 0$ and $Y(\rho) = -Y(\rho)^T$ for all $\rho \in \mathcal{P}$. Then $\Delta = \delta I_{n_v}$ with $\delta : \mathbb{R} \rightarrow \mathbb{R}$, $\|\delta\|_\infty \leq 1$ satisfies the IQC(Ψ_ρ, M_ρ) where $\Psi_\rho := I_{2n_v}$ and

$$M_\rho(\rho(t)) := \begin{bmatrix} X(\rho(t)) & Y(\rho(t)) \\ Y(\rho(t))^T & -X(\rho(t)) \end{bmatrix} \tag{12}$$

Proof. Let $v \in L_2^{n_v}[0, \infty)$. At each point in time, $v(t)$ and $w(t) = \Delta(t) v(t)$ satisfies the quadratic relation:

$$\begin{aligned} &\begin{bmatrix} v \\ w \end{bmatrix}^T \begin{bmatrix} X(\rho) & Y(\rho) \\ Y(\rho)^T & -X(\rho) \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix} = \\ &v^T[(1-\delta^2) X(\rho) + \delta(Y(\rho) + Y(\rho)^T)] v \end{aligned} \tag{13}$$

The assumptions on δ, X, and Y immediately imply that $\left[\begin{smallmatrix} v \\ w \end{smallmatrix}\right]^T M_\rho(\rho) \left[\begin{smallmatrix} v \\ w \end{smallmatrix}\right] \geq 0$ at each point in time. Integration of this quadratic constrain over any finite time interval $[0, T]$ thus implies $\Delta \in$ IQC(Ψ_ρ, M_ρ). □

This parameter-varying IQC can be used within the dissipation inequality framework to develop a less conservative analysis condition for LPV systems under time-varying real parametric uncertainties. Let $\mathcal{F}_u(G_\rho, \Delta)$ denote a uncertain LPV system and consider $\Delta \in$ IQC(Ψ_ρ, M_ρ) as defined by Lemma 1. The analysis interconnection (Fig. 2) simplifies in this case because $\Psi_\rho = I_{2n_v}$ means that $z = \left[\begin{smallmatrix} v \\ w \end{smallmatrix}\right]$. This leads to the following analysis result:

Corollary 1. Assume $\mathcal{F}_u(G_\rho, \Delta)$ is well posed for all $\Delta = \delta I_n$ with $\delta : \mathbb{R} \rightarrow \mathbb{R}$, $\|\delta\|_\infty \leq 1$. Then $\|F_u(G_\rho, \Delta)\| \leq \gamma$ if there exists a scalar $\lambda > 0$ and a matrix $P = P^T \in \mathbb{R}^{n_x + n_\psi}$ such that $P \geq 0$ and for all $\rho \in \mathcal{P}$ Equation (6) holds where $M := M_\rho$ and Ψ_ρ are specified in accordance with Lemma 1.

Allowing X and Y to be a function depending on ρ gives additional flexibility in comparison to restricting them to be constant. This added flexibility comes at the cost of introducing additional unknowns in the problem formulation.

B. Memoryless Nonlinearities in a Sector

The next example considers repeated monotonic nonlinearities described by an operator $\Phi : \mathbb{R}^{n_v} \rightarrow \mathbb{R}^{n_v}$ that has the form

$$\Phi(v) = [\phi(v_1), \ldots, \phi(v_{n_v})]^T \tag{14}$$

where ϕ is a scalar monotonically nondecreasing nonlinearity belonging to a finite sector $[0, k]$. Note that by an operator ϕ belonging to a sector $[\alpha, \beta]$, it is meant that $(\phi(v)-\alpha v)(\beta v-\phi(v)) \geq 0$ for all $v \in \mathbb{R}$. IQCs that bound the input/output behavior of Φ are proposed in [17] which are extended in this section into the parameter-varying IQC framework. The key idea in [17] is to use diagonally dominant multipliers as an extension of classical diagonal multipliers for repeated nonlinearities.

Lemma 2. Let $Q : \mathcal{P} \to \mathbb{R}^{n_v \times n_v}$ be diagonally dominant for all $\rho \in \mathcal{P}$, i.e. $Q_{ii}(\rho) \geq \sum_{j=1,j\neq i}^{n} |Q_{ij}(\rho)|, \forall i = 1, \ldots, n$. Assume $Q_{ij}(\rho) \leq 0$ for all $i \neq j$ and $\rho \in \mathcal{P}$. Then Φ as defined by Equation (14) with ϕ monotonic nondecreasing and belonging to a finite sector $[0, k]$ satisfies the IQC(Ψ_ρ, M_ρ) where $\Psi_\rho := I_{2n_v}$ and

$$M_\rho(\rho(t)) := \begin{bmatrix} 0 & Q(\rho(t)) \\ Q(\rho(t)) & 0 \end{bmatrix} \tag{15}$$

Proof. Let $v \in L_2^{n_v}[0, \infty)$ and define $w = \Phi(v)$. At each point in time v and w satisfy

$$\begin{bmatrix} v \\ w \end{bmatrix}^T \begin{bmatrix} 0 & Q(\rho) \\ Q(\rho) & 0 \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix} = w^T Q(\rho) v. \tag{16}$$

The remainder of the proof uses the given assumptions to demonstrate that $w^T Q(\rho) v \geq 0$ at each point in time. First, by symmetry of $Q(\rho)$, Equation (16) can be written as

$$\sum_{i=1}^{n} w_i v_i Q_{ii}(\rho) + \frac{1}{2} \sum_{j=1,j\neq i}^{n} Q_{ij}(\rho)(w_i v_j + w_j v_i) \tag{17}$$

Next, the assumptions on ϕ can be used to show (Lemma 5 in the appendix) that the following inequality holds for each $i \neq j$

$$w_j v_j + w_i v_i \geq w_i v_j + w_j v_i \tag{18}$$

Moreover, Equation (18) and the assumption $Q_{ij}(\rho) \leq 0$ for all $i \neq j$ and all $\rho \in \mathcal{P}$ to imply the following inequality:

$$\sum_{j=1,j\neq i}^{n} Q_{ij}(w_i v_j + w_j v_i) \geq \sum_{j=1,j\neq i}^{n} - |Q_{ij}| (w_i v_i + w_i v_i) \tag{19}$$

This relation along with Equation (17) can be used to show that

$$w^T Q(\rho) v \geq \sum_{i=1}^{n} w_i v_i \left(Q_{ii}(\rho) - \sum_{j=1,j\neq i}^{n} |Q_{ij}(\rho)| \right). \tag{20}$$

Since $Q(\rho)$ is diagonally dominant, the right hand side of Equation (20) is ≥ 0. This implies that $w^T Q(\rho) v \geq 0$ at each point in time. Integration over a finite time interval yields the corresponding IQC. □

It is straightforward to obtain an analysis result similar to Corollary 1 for the interconnection of an LPV system and a repeated monotonic nonlinearity. Analogously to the approach in [17], the parameter varying IQC can be extended to consider repeated nonlinearities belonging to a given sector $[\alpha, \beta]$ with $\alpha < \beta$, i.e. $(\phi(v) - \alpha v)(\beta v - \phi(v)) \geq 0$ $\forall v \in \mathbb{R}$.

Lemma 3. Let $Q : \mathcal{P} \to \mathbb{R}^{n_v \times n_v}$ be diagonally dominant for all $\rho \in \mathcal{P}$ and assume $Q_{ij}(\rho) \leq 0$ for all $i \neq j$ and $\rho \in \mathcal{P}$. Then Φ as defined by Equation (14), where ϕ is monotonically nondecreasing and belongs to the sector $[\alpha, \beta]$ with $\alpha < \beta$, satisfies the IQC(Ψ_ρ, M_ρ) where $\Psi_\rho := I_{2n_v}$ and

$$M_\rho(\rho(t)) := \begin{bmatrix} -2\alpha\beta Q(\rho(t)) & (\alpha+\beta) Q(\rho(t)) \\ (\alpha+\beta) Q(\rho(t)) & -2Q(\rho(t)) \end{bmatrix} \tag{21}$$

Proof. The proof is similar to that given for the extension to $[\alpha, \beta]$ in Theorem 1 of [17]. Perform the loop transformation shown in Fig. 3 below.

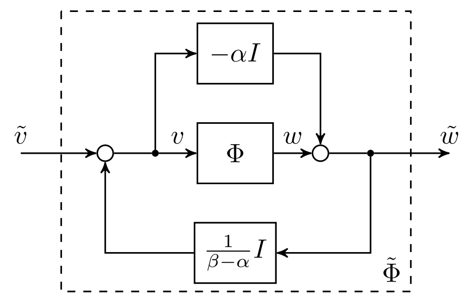

Fig. 3. Loop transformation on Φ

As shown in Claim A.3 of [17], $\tilde{\Phi}$ is monotonically nondecreasing and belongs to a finite sector $[0, k]$ for some $k < \infty$. By Lemma 2, the input/output signals of $\tilde{\Phi}$ satisfy a quadratic inequality at each time

$$\begin{bmatrix} \tilde{v} \\ \tilde{w} \end{bmatrix}^T \begin{bmatrix} 0 & Q(\rho) \\ Q(\rho) & 0 \end{bmatrix} \begin{bmatrix} \tilde{v} \\ \tilde{w} \end{bmatrix} \tag{22}$$

Substitute for $(\tilde{v}, \tilde{w})$ in terms of (v, w) to conclude that Φ satisfies IQC(Ψ_ρ, M_ρ) with $\Psi_\rho := I_{2n_v}$ and M_ρ as defined in Equation (21). □

C. Time Delays

A final example is given for the analysis of time-delayed LPV systems. Unlike the previous examples, in the case of the time-delay Δ has an internal dynamics. A constant delay $w = \mathcal{D}_\tau(v)$ is defined by $w(t) = v(t - \tau)$ where τ specifies the constant delay. To be precise the constant delay is defined as $\mathcal{D}_\tau : L_2^n[0, \infty) \to L_2^{n_v}[0, \infty)$ such that $w = \mathcal{D}_\tau(v)$ satisfies $w(t) = 0$ for $t \in [0, \tau)$ and $w(t) = v(t - \tau)$ for $t \geq \tau$. The next lemma provides a parameter-varying IQC for $\mathcal{D}_\tau$. Note that this IQC is delay independent i.e. it does not depend on the amount of delay. The basic idea for the proposed IQC is taken from [18] which develops stability conditions for a delayed LPV system using the Lyapunov-Krasovskii framework.

Lemma 4. Let $Q : \mathcal{P} \to \mathbb{R}^{n_v \times n_v}$ satisfy $Q(\rho) \geq 0$ for all $\rho \in \mathcal{P}$. Then $\mathcal{D}_\tau \in IQC(\Psi_\rho, M_\rho)$ where $\Psi_\rho := I_{2n_v}$ and $M_\rho := \left[\begin{smallmatrix} Q(\rho(t)) & 0 \\ 0 & -Q(\rho(t-\tau)) \end{smallmatrix}\right]$

Proof. Let $v \in L_2^{n_v}[0,\infty)$ and define $w = \mathcal{D}_\tau(v)$. The assumption $Q(\rho) > 0$ implies that the following inequality holds for all $v \in L_2^{n_v}[0,\infty)$ and for all $T \geq 0$:

$$\int_{T-\tau}^{T} v^T(t) Q(\rho(t)) v(t)\ dt \geq 0 \tag{23}$$

This integral term appears as one term of the Lyapunov-Krasovskii function of Theorem 4.1 in [18]. With some algebra this expression can be re-written in the following IQC form:

$$\int_0^T \begin{bmatrix} v(t) \\ w(t) \end{bmatrix}^T \begin{bmatrix} Q(\rho(t)) & 0 \\ 0 & -Q(\rho(t-\tau)) \end{bmatrix} \begin{bmatrix} v(t) \\ w(t) \end{bmatrix}\ dt \geq 0 \tag{24}$$

Thus $\mathcal{D}_\tau \in IQC(\Psi_\rho, M_\rho)$. □

This IQC can be also be obtained via a scaling argument. Specifically, the norm bound $\|\mathcal{D}_\tau\| \leq 1$ leads to the simple (constant) IQC $\mathcal{D}_\tau \in IQC(\Psi, M)$ where $\Psi = I_{2n_v}$ and $M = \begin{bmatrix} I_{n_v} & 0 \\ 0 & -I_{n_v} \end{bmatrix}$. The delay $\mathcal{D}_\tau$ also satisfies the following swapping relation: $\mathcal{D}_\tau X(\rho(t)) = X(\rho(t-\tau))\mathcal{D}_\tau$, see Fig. 4. Thus the matrix-scaled system $\bar{\Delta} = X(\rho(t-\tau))\mathcal{D}_\tau X(\rho(t))^{-1}$ is also norm bounded by 1. The associated input/output pair for the original system $w = \Delta(v)$ is related to the input/output pair for the scaled system by $\bar{w} = Xw$ and $\bar{v} = Xv$. Using this relation and the condition $\|\bar{\Delta}\| \leq 1$ leads to the conclusion of Lemma 4: $D_\tau \in$ IQC(Ψ_ρ, M_ρ) where $Q(\rho) = X^T(\rho)X(\rho) > 0$.

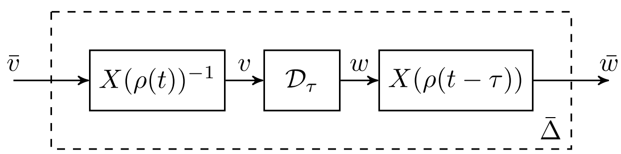

Fig. 4. Scaling of the time delay $\mathcal{D}_\tau$

Due to the appearance of both terms $Q(\rho(t))$ and $Q(\rho(t-\tau)$ in Lemma 4, Theorem 1 can not be applied directly to obtain a bound on the worst case gain of the interconnection $\mathcal{F}_u(G_\rho, \mathcal{D}_\tau)$. Instead the following slightly modified theorem needs to be used.

Theorem 2. Assume $\mathcal{F}_u(G_\rho, \mathcal{D}_\tau)$ is well posed for the constant delay $\tau > 0$. Then $\|\mathcal{F}_u(G_\rho, \mathcal{D}_\tau)\| < \gamma$ if there exists a matrix $P = P^T > 0$ and a function $Q : \mathcal{P} \to \mathbb{R}^{n_v \times n_v}$ such that $\forall \rho_1, \rho_2 \in \mathcal{P}\ Q(\rho_1) > 0$ and

$$\begin{bmatrix} PA(\rho_1) + A(\rho_1)^T P & PB_1(\rho_1) & PB_2(\rho_1) \\ B_1(\rho_1)^T P & 0 & 0 \\ B_2(\rho_1)^T P & 0 & -I \end{bmatrix} + \begin{bmatrix} C_1(\rho_1)^T \\ D_{11}(\rho_1)^T \\ D_{12}(\rho_1)^T \end{bmatrix} M_\rho(\rho_1, \rho_2) \begin{bmatrix} C_1(\rho_1) & D_{11}(\rho_1) & D_{12}(\rho_1) \end{bmatrix} + \frac{1}{\gamma^2} \begin{bmatrix} C_2(\rho_1)^T \\ D_{21}(\rho_1)^T \\ D_{22}(\rho_1)^T \end{bmatrix} \begin{bmatrix} C_2(\rho_1) & D_{21}(\rho_1) & D_{22}(\rho_1) \end{bmatrix} < 0 \tag{25}$$

where $M_\rho(\rho_1, \rho_2) := \begin{bmatrix} Q(\rho_1) & 0 \\ 0 & -Q(\rho_2) \end{bmatrix}$.

Proof. The proof uses the IQC for $\mathcal{D}_\tau$ defined in Lemma 4 and is similar to that given for Theorem 1. Details are omitted. □

IV. Numerical Example

A simple example is used to demonstrate the applicability of the proposed method. The example is a feedback interconnection of a second-order LPV system with an input saturation and an anti-windup controller as shown in Fig. 5. The LPV system G_ρ is a second order system depending on a single scheduling parameter ρ that can vary arbitrary fast in the interval $[0,1]$. It has the form

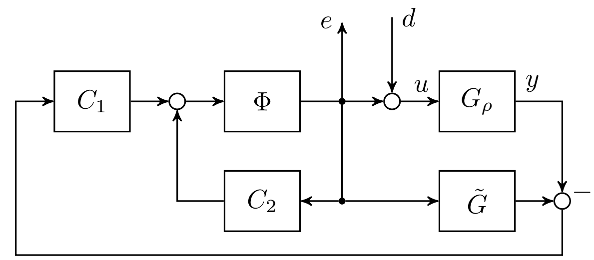

Fig. 5. Closed Loop Interconnection with Input Saturation

$$\dot{x}_G = A(\rho)x_G + Bu \qquad y = C(\rho)x_G \qquad \text{with} \tag{26}$$

$$\begin{aligned} A(\rho) &:= \begin{bmatrix} -0.01 & 0 \\ 0 & -0.01 \end{bmatrix} - \frac{\rho}{9} \begin{bmatrix} 0.01 & 0 \\ 0 & 0.01 \end{bmatrix}, \\ C(\rho) &:= \begin{bmatrix} 0.4 & -0.5 \\ -0.3 & 0.4 \end{bmatrix} + \frac{\rho}{9} \begin{bmatrix} 0.4 & -0.5 \\ -0.3 & 0.4 \end{bmatrix}. \end{aligned} \tag{27}$$

The controller elements C_1, C_2 and $\tilde{G}$ represent an anti-windup controller which has been designed in [19] for the linear system $G_\rho(\rho = 0)$. The controller elements have the following transfer matrices:

$$\begin{aligned} C_1 &:= \frac{2.5(s+1)}{(20s+1)(0.1s+1)} \begin{bmatrix} 1.6s+1 & 2s \\ 1.2s & 1.6s+1 \end{bmatrix}, \\ C_2 &:= \frac{1}{100s+1} \begin{bmatrix} 99 & \frac{-125(s+1)}{0.1s+1} \\ \frac{-75(s+1)}{0.1s+1} & 99 \end{bmatrix}, \\ \tilde{G} &:= \frac{10}{100s+1} \begin{bmatrix} 4 & \frac{-5}{0.1s+1} \\ \frac{-3}{0.1s+1} & 4 \end{bmatrix}. \end{aligned} \tag{28}$$

The saturation Φ is a 2-by-2 MIMO operator. It is bounded by the IQC(Ψ_ρ, M_ρ) as defined in Lemma 3 with the sector bound set to $[0,1]$, i.e. $M_\rho(\rho) = \begin{bmatrix} 0 & Q(\rho) \\ Q(\rho) & -2Q(\rho) \end{bmatrix}$.

The results of the study are summarized in Tab. I. First, a linear analysis on the vertices of G_ρ is performed to gain some insight in the achievable worst case gain. The worst-case gain is estimated at the frozen grid points $\rho = 0$ and $\rho = 1$ using an LTI version of Theorem 1 with Q set to a constant matrix (i.e. Theorem 2 in [17]). Next, Theorem 1 is invoked using both a constant matrix Q and a parameter varying $Q(\rho) = Q_0 + \rho Q_1$. In general, Q can have an arbitrary dependence on ρ. An affine dependence is sufficient in the presented example, as the system itself only depends affinely on ρ. Since the system G_ρ depends affinely on ρ, the LMI constraints only need to be enforced at $\rho = 0$ and $\rho = 1$. This example shows the benefit of parameter varying IQCs. Using a constant Q yields a significantly higher upper bound on the worst-case gain than $Q(\rho)$.

System	Method	Worst Case Gain
LTI: $G_\rho(\rho = 0)$	constant Q	15.18
LTI: $G_\rho(\rho = 1)$	constant Q	24.43
LPV: G_ρ	constant Q	80.09
LPV: G_ρ	parameter varying $Q(\rho)$	43.38

TABLE I

SUMMARY OF ROBUST PERFORMANCE OF THE LPV SYSTEM UNDER SATURATION

V. CONCLUSION

This paper introduced a new class of IQCs based on the time-domain viewpoint. In this time-domain framework the IQC multiplier is no longer restricted to linear time invariant systems. Allowing the multiplier itself to depend on the scheduling parameter adds additional flexibility in the robustness analysis of LPV systems. For different operators parameter-varying IQCs are given which are extension of known classical time invariant results. The potential advantage of using parameter-varying IQCs was demonstrated with a simple numerical example. Future work will explore the usage of LPV filters in the IQC description, as well as extending the operators that can be described by parameter-varying IQCs.

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APPENDIX

The following Lemma is a simplified version of Lemma A.1. from [17]. It is used to prove Lemma 2.

Lemma 5. Let $\phi : \mathbb{R} \to \mathbb{R}$ be monotonically nondecreasing and belonging to some finite sector $[0, k]$ with $k < \infty$. Then the following inequality holds for all $x, y \in \mathbb{R}$:

$$\phi(x)x + \phi(y)y \geq \phi(x)y + \phi(y)x \tag{29}$$

Proof. The monotonicity of ϕ implies that the following inequality holds for any α, $\beta \in \mathbb{R}$: $(\beta - \alpha)\phi(\beta) \geq \int_\alpha^\beta \phi(v)dv$. Setting $\alpha = x$ and $\beta = y$ yields

$$\phi(y)y - \phi(y)x \geq \int_x^y \phi(v)dv. \tag{30}$$

Similarly $\alpha = y$ and $\beta = x$ gives

$$\phi(x)x - \phi(x)y \geq \int_y^x \phi(v)dv = -\int_x^y \phi(v)dv. \tag{31}$$

Equations (30) and (31) can be used to bound the integral $\int_x^y \phi(v)dv$ from above and below respectively.

$$\phi(y)y - \phi(y)x \geq \int_x^y \phi(v)dv \geq \phi(x)y - \phi(x)x \tag{32}$$

Using these bounds and rearranging the terms finally yields

$$\phi(x)x + \phi(y)y \geq \phi(x)y + \phi(y)x \tag{33}$$

which concludes the proof. □