Model Reduction for Linear Parameter Varying Systems Using Scaled Diagonal Dominance

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***Abstract*— A model-reduction method for linear, parameter-varying (LPV) systems based on parameter-varying balanced realizations is proposed. In general, this requires the solution of a large set of linear matrix inequalities, leading to numerical issues and high computational cost. It has been recognized recently that semidefinite optimization problems (SDP) can be cast into second order cone programs (SOCP) by replacing the positive definiteness constraints with stronger, scaled diagonal dominance conditions. Since the scalability of SOCP solvers is much better than that of the SDPs, the new formulation allows solving large scale model reduction problems more efficiently. A numerical example is provided to demonstrate the efficiency of the approach.**

I. Introduction

This paper considers the model reduction of large scale linear parameter-varying (LPV) systems. LPV systems are a class of linear systems where the state matrices depend on (measurable) time-varying parameters. By large scale, both a high number of states and/or a high number scheduling parameters is meant. Such large scale systems arise naturally in many applications, for instance aeroelastic vehicles [1] or wind turbines [2].

Several LPV model-reduction techniques have been introduced based on solving semidefinite programs [3], [4], [5], see Section III. They guarantee stability of the reduced order model, as well as provide bounds for the approximation error. The main limitation of these approaches is that they do not scale well with the problem size. Recently, an approach to approximate large scale SDPs by solving linear (LP) or second order cone programs (SOCP) has been proposed in [6], [7], [8]. The key idea behind the approximation is that any semidefinite program (SDP) can be converted into an LP (or SOCP) by replacing every positive definiteness constraint by a more restrictive diagonally dominant (or scaled diagonally dominant) condition. By converting the problem into an LP or SOCP specialized solvers can be used which scale much better with the problem dimension than SDP solvers, see Section IV.

In addition to the scaled diagonally dominant (DD) relaxation, another relaxation scheme is proposed in Section V based on exploiting the structure in the LMI conditions. This approach uses the algorithm presented in [9], [10], [11]. A chordal decomposition of the SDP is performed which reduces the semidefinite constraints into an equivalent set of smaller semidefinite constraints. This approach can drastically reduce the computational demand if the problem structure is sparse. In [12], it was used to analyze the stability of large scale LTI systems. In addition, in [12], it was shown that the approach does not introduce conservatism if the system has specific structure. In this paper, we extend the ideas of [12] to solve for sparse Gramians.

In literature, various approaches have been proposed to overcome the scalability issues of the method in [4]. Broadly speaking two different approaches are taken. Heuristic preprocessing can be used until the system is of low enough order to apply the techniques proposed in [4], see e.g. [1]. Alternatively, the LPV system can be reduced "locally" on a grid of frozen parameter values by using standard LTI reduction methods. Afterwards an interpolation rule is sought after that allows obtaining an LPV model based on the reduced order LTI models. This approach is taken, e.g., in [13], [14], [15]. All these approaches have in common that they are based on heuristics. In contrast, the scaled DD based relaxation proposed in this paper still retains the properties of the original balancing technique, namely a stability guarantee and error bound for the reduced order model.

II. Notation

The notations used in the paper are fairly standard. The set of $n\times k$ dimensional real matrices is denoted by $\mathbb{R}^{n\times k}$. If $M\in\mathbb{R}^{n\times k}$ then we refer to its entries and sub-blocks by m_{ij} and M_{ij}, respectively. The notation $M=[m_{ij}]=[M_{ij}]$ is used to emphasize that M is built up from entries m_{ij} and sub-matrices M_{ij}. The sets of symmetric positive definite and semi-definite matrices are denoted by $\mathcal{PD}$ and $\mathcal{PSD}$, respectively. If the dimension is also important then it is indicated in the top-right index, e.g. $\mathcal{PD}^n$.

III. Model Reduction for LPV Systems

Linear Parameter Varying (LPV) systems are a class of linear systems whose state space matrices depend on a time-varying parameter vector $\rho:\mathbb{R}\to\mathbb{R}^{n_\rho}$. The parameter is assumed to be a continuously differentiable function of time and admissible trajectories are restricted, based on physical considerations, to a known compact subset $\mathcal{P}\subset\mathbb{R}^{n_\rho}$. The state-space matrices of an LPV system are continuous functions of the parameter, e.g. $A:\mathcal{P}\to\mathbb{R}^{n_x\times n_x}$. Define the LPV system G_ρ with input u and outputs y as:

$$\begin{aligned}\dot{x}(t)&=A(\rho(t))x(t)+B(\rho(t))u(t)\\ y(t)&=C(\rho(t))x(t)+D(\rho(t))u(t)\end{aligned}\tag{1}$$

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The state matrices at time t depend on the parameter vector at time t. Hence, LPV systems represent a special class of time-varying systems. Throughout this paper the explicit dependence on t is suppressed to shorten the notation.

A. Truncation and Residualization

For model reduction, the system Equation (1) is partitioned as

$$\begin{aligned} \begin{bmatrix}\dot{x}_1 \\ \dot{x}_2\end{bmatrix} &= \begin{bmatrix}A_{11}(\rho) & A_{12}(\rho) \\ A_{21}(\rho) & A_{22}(\rho)\end{bmatrix} \begin{bmatrix}x_1 \\ x_2\end{bmatrix} + \begin{bmatrix}B_1(\rho) \\ B_2(\rho)\end{bmatrix} u \\ y &= \begin{bmatrix}C_1(\rho) & C_2(\rho)\end{bmatrix} \begin{bmatrix}x_1 \\ x_2\end{bmatrix} + D(\rho)u, \end{aligned} \tag{2}$$

where x_1 is the vector of r states to preserve and x_2 contains $n_x - r$ states to remove. Two different approaches can be used to remove the states x_2, namely truncation and residualization. Truncation is simply the elimination of the vector x_2, i.e. the reduced order model G_r has the form

$$\begin{aligned} \dot{x}_1 &= A_{11}(\rho)x_1 + B_1(\rho)u \\ y &= C_1(\rho)x_1 + D(\rho)u \qquad \forall \rho \in \mathcal{P}. \end{aligned} \tag{3}$$

For a constant ρ, G_r is equal to G at infinite frequency. Hence, the approach is preferred when accuracy at high frequencies is required. Similarly the system can be residualized if accuracy at low frequencies is more important. Residualization yields a reduced order model G_r of the following form:

$$\begin{aligned} \dot{x}_1 &= (A_{11} - A_{12}A_{22}^{-1}A_{21})x_1 + (B_1 - A_{12}A_{22}^{-1}B_2)u \\ y &= (C_1 - C_2A_{22}^{-1}A_{21})x_1 + (D - C_2A_{22}^{-1}B_2)u \end{aligned} \tag{4}$$

B. Balanced Realization

Balanced model reduction is based on the measure of the controllability and observability of the state space model. These measures are given by the controllability and observability Gramians which are formally defined by the following two theorems which are taken from [4].

Theorem 1. Let G_ρ be an LPV system of the form Equation (1) with an initial state $x(0) = x_0$. If there exists a constant matrix $P = P^T > 0$ such that $\forall \rho \in \mathcal{P}$

$$A(\rho)P + PA(\rho)^T + B(\rho)B(\rho)^T < 0 \tag{5}$$

- G_ρ is quadratically stable;
- the input energy required to drive the system G_ρ from $x(-\infty) = 0$ to x_0 is bounded from below according to $\|u\|_2^2 \geq x_0^T P^{-1} x_0$.

Theorem 2. Let G_ρ be an LPV system of the form Equation (1) with an initial state $x(0) = x_0$. If there exists a constant matrix $Q = Q^T > 0$ such that $\forall \rho \in \mathcal{P}$

$$A(\rho)^T Q + QA(\rho) + C(\rho)^T C(\rho) < 0 \tag{6}$$

- G_ρ is quadratically stable;
- the output energy is bounded from above for $u(t) = 0\ \forall t \geq 0$ according to $\|y\|_2^2 \leq x_0^T Q x_0$.

In Theorem 1 and Theorem 2, P and Q are the controllability and observability Gramian of G_ρ respectively. Note that the conditions Equation (5) and Equation (6) are parameter-dependent LMIs that must be satisfied for all possible $\rho \in \mathcal{P}$. Thus Equation (5) and Equation (6) represent an infinite collection of LMI constraints. A remedy to this problem, which works in many practical examples, is to approximate the set $\mathcal{P}$ by a finite set $\mathcal{P}_{grid} \subset \mathcal{P}$. Specifically, the finite set $\mathcal{P}_{grid} := \{\rho^{(k)}\}_{k=1}^N$ represents a grid of points over the set $\mathcal{P}$. The conditions Equation (5) and Equation (6) are enforced only on these grid points leading to a finite dimensional LMI. Note that the gridding approach is only an approximation for the parameter-dependent LMI conditions in Theorem 1 and Theorem 2.

In order to reduce the system, it is necessary to transform it in a balanced realization. A balanced realization of system Equation (1) is a realization where both Gramians are equal and diagonal, i.e. $\hat{P} = \hat{Q} = \Sigma$. The balancing state transformation T is constant and chosen such that $\hat{P} = TPT^T$ and $\hat{Q} = T^{-T}QT^{-1}$. Note also that $\hat{P}\hat{Q} = TPQT^{-1}$. Therefore, the transformation T leads to an eigenvector decomposition $PQ = T^{-1}\Lambda T$. The realization obtained by the transformation matrix T reflects the combined controllability and observability of the individual states. Small singular values of controllability and observability Gramians indicate that a finite amount of energy in a given input does not result in significant energy in the output; hence, those states can be deleted while retaining the important input/output characteristics of the system.

IV. Balanced Reduction based on Scaled Diagonally Dominance

Applying Theorem 1 and Theorem 2 requires solving a set of LMIs at N grid points. A significant problem with this approach is the growth of the LMI problem with the dynamic order and the number of scheduling parameters of an LPV model. Hence, it is limited to either small systems or requires a heuristic pre-processing, see [1]. Instead of using a heuristic approach, a relaxation based scaled diagonally dominant matrices is proposed.

First, formal definitions of diagonally dominant and scaled diagonally dominant matrices are provided. In addition, an approach to convert an SDP into an SOCP as proposed in [6] is presented. The following definitions were taken from [6] and [16]. The notion of diagonally dominance is defined in the following way:

Definition 1. A matrix $M \in \mathbb{R}^{n\times n}$ is called diagonally dominant, denoted by $M \in \mathcal{DD}$, if $m_{ii} > \sum_{j,j\neq i}^n |m_{ij}|$ holds[1] for all $i = 1, \dots, n$.

Based on the previous definition, scaled diagonally dominance can be defined.

Definition 2. A matrix M is scaled diagonally dominant, denoted by $M \in \mathcal{SDD}$, if there exists a diagonal scaling matrix $S = \mathrm{diag}(s_{ii}, \dots, s_{nn})$, such that $s_{ii} > 0$, $\forall i$ and $MS \in \mathcal{DD}$.

[1]In this paper we define the diagonal dominance by strict inequality, because we would like to exclude the semidefinite solutions in the forthcoming problems.

It is clear from the definition that $\mathcal{DD}^n \subseteq \mathcal{SDD}^n$. Specifically if a matrix $M \in \mathcal{DD}^n$, it is trivial to show that $M \in \mathcal{SDD}^n$ by setting $S = I$ in Definition 2. By using the theorems on eigenvalue localization and Gershgorin disks, it can also be proved that $\mathcal{SDD}^n \subseteq \mathcal{PD}^n$, i.e. any scaled diagonal dominant matrix is also positive definite. To simplify the derivations we introduce the following notation: for a matrix M, the fact that $-M \in \Omega$ with $\Omega \in \{\mathcal{DD}, \mathcal{SDD}, \mathcal{PD}, \mathcal{PSD}\}$ will be denoted by $M \in -\Omega$.

Using the relations above, it was shown in Theorem 3.1 in [17] that by relaxing the positive definite constraints by scaled DD constraints, an SDP can be cast into an SOCP. A feasible solution of the SOCP will still be a feasible solution of the original SDP. This follows from the following Theorem that provides a definition for *symmetric* scaled DD matrices.

Theorem 3 ([17]). Let $M \in \mathbb{R}^{n\times n}$ be a symmetric matrix, i.e. $M = M^T$. M is scaled diagonally dominant if it can be decomposed in the following way:

$$M = \sum_{i=1,j>i}^{i=n,j=n} M^{(ij)}, M_{2\times 2}^{(ij)} := \begin{bmatrix} m_{ii}^{(ij)} & m_{ij}^{(ij)} \\ m_{ji}^{(ij)} & m_{jj}^{(ij)} \end{bmatrix} > 0 \quad (7)$$

where for all (i, j), $M^{(ij)} \in \mathbb{R}^{n\times n}$ is a symmetric matrix with all entries zero, except the elements $m_{ii}^{(ij)}$, $m_{ij}^{(ij)}$, $m_{ji}^{(ij)}$, $m_{jj}^{(ij)}$ and $M_{2\times 2}^{(ij)}$ is a 2×2 matrix that contains the nonzero entries of $M^{(ij)}$.

The following example shall illustrate the decomposition.

Example 1. Let $M \in \mathbb{R}^{3\times 3}$ be a symmetric scaled DD matrix, i.e. $M = M^T$. Then the following decomposition exists:

$$M = \underbrace{\begin{bmatrix} m_{11}^{(12)} & m_{12}^{(12)} & 0 \\ m_{21}^{(12)} & m_{22}^{(12)} & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{M^{(12)}} + \underbrace{\begin{bmatrix} m_{11}^{(13)} & 0 & m_{13}^{(13)} \\ 0 & 0 & 0 \\ m_{31}^{(13)} & 0 & m_{33}^{(13)} \end{bmatrix}}_{M^{(13)}} + \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & m_{22}^{(23)} & m_{23}^{(23)} \\ 0 & m_{32}^{(23)} & m_{33}^{(23)} \end{bmatrix}}_{M^{(23)}} \quad (8)$$

For this decomposition the 2×2 matrices have the form

$$M_{2\times 2}^{(12)} = \begin{bmatrix} m_{11}^{(12)} & m_{12}^{(12)} \\ m_{21}^{(12)} & m_{22}^{(12)} \end{bmatrix} \qquad M_{2\times 2}^{(13)} = \begin{bmatrix} m_{11}^{(13)} & m_{13}^{(13)} \\ m_{31}^{(13)} & m_{33}^{(13)} \end{bmatrix}$$
$$M_{2\times 2}^{(23)} = \begin{bmatrix} m_{22}^{(23)} & m_{23}^{(23)} \\ m_{32}^{(23)} & m_{33}^{(23)} \end{bmatrix}. \quad (9)$$

Theorem 3 states that checking whether a symmetric matrix is scaled DD amounts to checking whether a set of 2×2 matrices is positive definite. The positive definiteness of a 2×2 symmetric matrix is equivalent to a rotated quadratic cone constraint, e.g. in (7), $M_{2\times 2}^{(ij)} > 0$ is equivalent to $m_{ii}^{(ij)} > 0$, $m_{jj}^{(ij)} > 0$, $m_{ii}^{(ij)} m_{jj}^{(ij)} - (m_{ij}^{(ij)})^2 > 0$. Thus the condition $M = M^T \in \mathcal{SDD}^n$ can be checked by a second order cone program (SOCP). If in any optimization problem, every LMI constraint $M \in \mathcal{PD}$ is replaced by the stronger, scaled diagonally dominance condition $M \in \mathcal{SDD}$ then the problem can be solved more efficiently by a SOCP solver. As this reformulation restricts the search space to a subset of positive definite matrices, more conservative solutions can be expected.

Using the fact that $\mathcal{SDD}^n \subseteq \mathcal{PD}^n$ and Theorem 3, the LMI conditions in Theorem 1 and Theorem 2 can be replaced by scaled DD conditions, which gives the following Corollaries. These allows solving for scaled DD Gramians as a SOCP, which is computationally more efficient than solving the SDPs in Theorem 1 and Theorem 2.

Corollary 1. Let G_ρ be an LPV system of the form Equation (1) with an initial state $x(0) = x_0$. If there exists a constant matrix $P = P^T \in \mathcal{SDD}$ such that $\forall \rho \in \mathcal{P}$

$$A(\rho)P + PA(\rho)^T + B(\rho)B(\rho)^T \in -\mathcal{SDD} \quad (10)$$

- G_ρ is quadratically stable;
- the input energy required to drive the system G_ρ from $x(-\infty) = 0$ to x_0 is bounded from below according to $\|u\|_2^2 \geq x_0^T P^{-1} x_0$.

Corollary 2. Let G_ρ be an LPV system of the form Equation (1) with an initial state $x(0) = x_0$. If there exists a constant matrix $Q = Q^T \in \mathcal{SDD}$ such that $\forall \rho \in \mathcal{P}$

$$A(\rho)^T Q + QA(\rho) + C(\rho)^T C(\rho) \in -\mathcal{SDD} \quad (11)$$

- G_ρ is quadratically stable;
- the output energy is bounded from above for $u(t) = 0 \ \forall t \geq 0$ according to $\|y\|_2^2 \leq x_0^T Q x_0$.

The SOCP is solved iteratively based on an approach proposed in [3]. The iterative approach can be described in the following way. Initialize the algorithm by setting $Q^{(0)} = I$. Then, iteratively solve $\min_{P^{(k)}} \text{trace}(P^{(k)} Q^{(k-1)})$ such that Equation (10) holds $\forall \rho \in \mathcal{P}$ and $\min_{Q^{(k)}} \text{trace}(P^{(k-1)} Q^{(k)})$ such that Equation (11) holds $\forall \rho \in \mathcal{P}$.

It shall be emphasized that the results of the scaled DD approach are not invariant to state transformations. This contrasts the LMI approach. Unfortunately, it is currently not clear how an optimal state transformation can be chosen to reduce the conservatism of the scaled DD approach.

V. Alternative Approach based on Exploiting the Sparsity

In literature an alternative approach to tackle large scale SDPs exists. It is based on exploiting the sparsity pattern of the constraints. Specifically, in [12], a relaxation based on chordal decomposition for solving Lyapunov type LMIs for sparse A matrices is proposed. This idea can be extended to relax the LMIs to compute LPV Gramians. Given sparse matrices A, B and C that describe an LPV system (1), a sparsity pattern for P and Q can be chosen. This implies a sparsity pattern for the LMIs (5) and (6). The Matlab toolbox SparseCoLO [10] can be used to detect chordal sparsity and preprocess the LMIs accordingly. For details of the approach the reader is referred to [9], [10], [11], [12].

A simple iterative algorithm can be used to search for sparse Gramians. The algorithm is a simple extension of the one proposed in [12]. Initialize the algorithm by setting $Q^{(0)} = I$ and restricting $P^{(1)}$ to be diagonal. Then, solve $\min_{P^{(1)}} \text{trace}(P^{(1)} Q^{(0)})$ such that Equation (5) holds $\forall \rho \in$

$\mathcal{P}$. If the optimization does not find a feasible solution, increase the bandwidth of $P^{(1)}$ by one until a feasible solutions is found. If the algorithm terminates successfully, proceed by restricting $Q^{(1)}$ to be diagonal and repeat the procedure to find a feasible $Q^{(1)}$. The feasible solutions $P^{(1)}$ and $Q^{(1)}$ determine the bandwidth of P and Q respectively. Then follow the iterative procedure given in the previous section to obtain a sequence of P and Q that minimize $\mathrm{trace}(PQ)$.

VI. Numerical Example

A simple nonlinear mass-spring-damper taken from [18] is used to illustrate the efficiency of the scaled DD relaxation. The system is a series interconnection of M blocks with mass $m = 1$, as depicted in Fig. 1. Each block is connected to the neighboring blocks and the "ground" by linear dampers and nonlinear springs. All damping coefficients are set to $b = 1$ and all springs have the stiffness $k(q) = 0.5 + q^2$, where q is the displacement. A disturbance force ρ and a input force u are acting on the M^{th}-block. Hence, the dynamics of the mass-spring-damper are governed by the following differential equations:

$$\begin{aligned} m\ddot{q}_1 &= -F_1 - F_{1,2} \\ m\ddot{q}_i &= -F_i - F_{i,i-1} - F_{i,i+1} \qquad i = 2, \ldots, M-1 \\ m\ddot{q}_M &= -F_M - F_{M,M-1} + \rho + u, \end{aligned} \tag{12}$$

where the force $F_i = b\dot{q}_i + k(q_i)q_i$ is due to the connection of the i^{th} block with the "ground" and the force $F_{i,j} = b\big(\dot{q}_i - \dot{q}_j\big) + k(q_i - q_j)\,\big(q_i - q_j\big)$ is the force between the i^{th} and the j^{th} block. Equation (12) can be written in state space form as $\dot{z} = f(z, u, \rho)$ and $y = h(z, u, \rho)$.

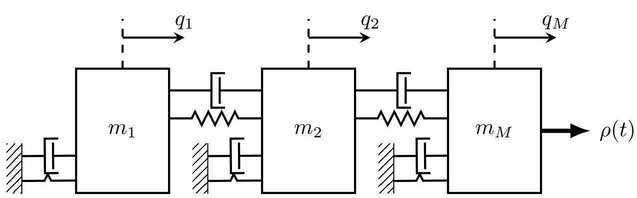

Fig. 1. Mass-Spring-Damper System

The LPV model of the mass-spring-damper is obtained by linearization of Equation (12) along an equilibrium trajectory $\bar{x}(\rho)$ which satisfies $f(\bar{x}(\rho), 0, \rho) = 0$. This yields the LPV system

$$\begin{aligned} \dot{x} &= A(\rho)x + B(\rho)u + \frac{d}{dt}\bar{x}(\rho) \\ y &= C(\rho)x, \end{aligned} \tag{13}$$

where the term $\frac{d}{dt}\bar{x}(\rho)$ results from the time dependence of the equilibrium condition, see [19] for details. For the example fifty blocks are considered, i.e. $M = 50$ which results in a 100 state LPV model. The disturbance ρ lies within the interval $[0, 2]$.

The SOCPs are solved using Mosek [20] and the SDPs using LMILab [21]. The parameter-dependent Lyapunov inequalities are evaluated on the grid $\rho = \{0, 1, 2\}$. In Tab. I, the computational times of both algorithm are summarized. Both approaches take three iterations to converge. The SOCP is solved in about three minutes whereas the LMI takes approximately 2.5 hours. The alternative approach based on the sparsity of the problem is also applied to the problem. The algorithm is initialized with diagonal Gramians and then iteratively off-diagonals are added up to 20 or until a feasible solution is found. The approach did not find a feasible solution, even though the system has a highly sparse structure. Specifically, the A matrix of the system only contains 346 nonzero entries out of its 100000 entries.

TABLE I

Computational Time of the Order Reduction

Method	Computational time [s]	Speedup
LMI Gramians	9535	
Scaled DD Gramians	189	98%

In both approaches, the LPV model is truncated to four states. Fig. 2 shows the comparison of the frequency responses of the full order model and the two reduced order ones at frozen parameter values. Both reduced order models match the full order well at low frequencies. The scaled DD based on Corollary 1 and Corollary 2 is, however, less accurate at high frequencies. Note that the difference is at such a high frequency that is well beyond the bandwidth of the system. Hence, it only has minor impact on the quality of the reduced order model.

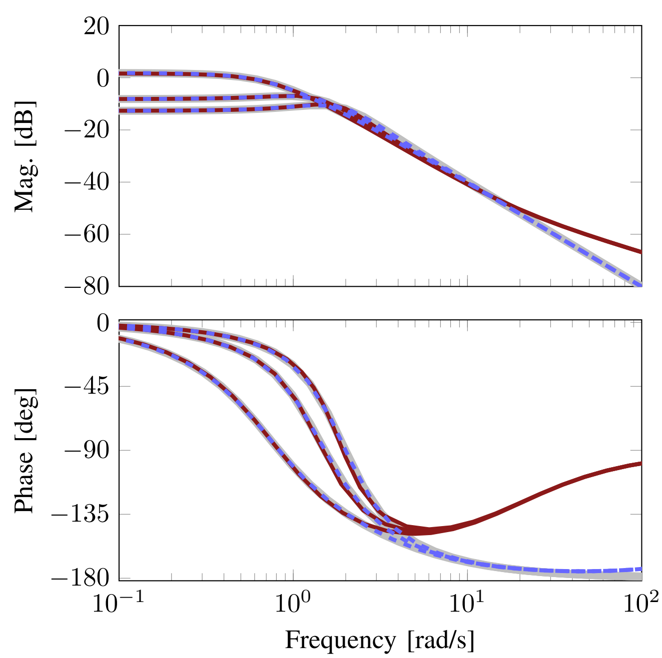

Fig. 2. Bode plots at $\rho = \{0, 1, 2\}$ of the full order model ( ), and reduced order models from balanced truncation based on the LMI ( ) and scaled DD solution ( ).

Finally, the two reduced order models are compared to the original nonlinear model (12). Time histories are compared using as control input a step response with $u(t) = 0.5$ and as external disturbance the trajectory $\rho(t) = 1 + \cos(0.5t)$. The results of the simulation are shown in Fig. 3. Both reduced order models match the nonlinear model very well and there is no noticeable difference in the time response between the scaled DD and the LMI based models.

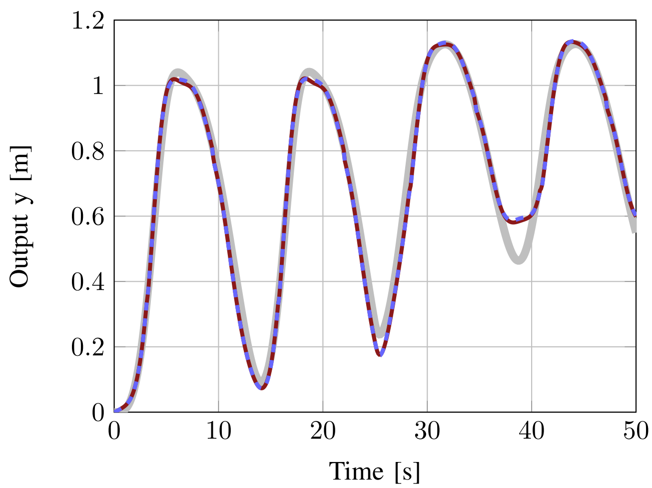

Fig. 3. Output of the full order model (▬), and reduced order models from balanced truncation based on the LMI (- - -) and scaled DD solution (▬).

VII. Conclusion

This paper introduced a relaxation method for the model reduction of large scale LPV systems based on scaled diagonally dominance. This significantly improves the scalability of the semidefinite optimization problem involved in obtain the Gramians of an LPV systems. An alternative approach to this relaxation is given that is trying to exploit the sparsity structure of the problem. The potential of the new approach is demonstrated with a numerical examples. The proposed scaled DD approach is considerably outperforming the LMI approach. Even though the considered example is has a high sparsity, the alternative relaxation yields no results. Future work will explore potential state transformations for LPV systems that can reduce the conservatism of the results, as well as consider more realistic applications like aeroservoelastic systems or wind turbines.

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References

[1] C. Moreno, P. Seiler, and G. Balas, "Model reduction for aeroservoelastic systems," *Journal of Aircraft*, vol. 51, no. 1, pp. 280–290, 2014.

[2] J. Annoni, P. Seiler, K. Johnson, P. Fleming, and P. Gebraad, "Evaluating wake models for wind farm control," in *American Control Conference*, 2014, pp. 2517–2523.

[3] G. D. Wood, "Control of parameter-dependent mechanical systems," Ph.D. dissertation, Cambridge, 1995.

[4] G. D. Wood, P. J. Goddard, and K. Glover, "Approximation of linear parameter-varying systems," in *Proceedings of the 35th IEEE Conference on Decision and Control*, Kobe, Japan, 1996, pp. Vol. 4, pp. 406–411.

[5] S. Lall and C. Beck, "Error-bounds for balanced model-reduction of linear time-varying systems," *Automatic Control, IEEE Transactions on*, vol. 48, no. 6, pp. 946–956, 2003.

[6] A. Majumdar, A. A. Ahmadi, and R. Tedrake, "Control and verification of high-dimensional systems with DSOS and SDSOS programming," in *Submitted to the IEEE Conference on Decision and Control*, 2014.

[7] A. A. Ahmadi and P. A. Parrilo, "Towards scalable algorithms with formal guarantees for lyapunov analysis of control systems via algebraic optimization," in *IEEE Conference on Decision and Control*, 2014.

[8] T. Peni and H. Pfifer, "Analysis of large scale parameter-varying systems by using scaled diagonal dominance," in *European Control Conference*, 2015.

[9] H. Waki, S. Kim, M. Kojima, M. Muramatsu, and H. Sugimoto, "Algorithm 883: Sparsepop—a sparse semidefinite programming relaxation of polynomial optimization problems," *ACM Transactions on Mathematical Software (TOMS)*, vol. 35, no. 2, p. 15, 2008.

[10] K. Fujisawa, S. Kim, M. Kojima, Y. Okamoto, and M. Yamashita, "Users manual for sparsecolo: Conversion methods for sparse conic-form linear optimization problems," Research Report B-453, Dept. of Math. and Comp. Sci., Tokyo Institute of Technology, Tokyo 152-8552, Japan, Tech. Rep., 2009.

[11] S. Kim, M. Kojima, M. Mevissen, and M. Yamashita, "Exploiting sparsity in linear and nonlinear matrix inequalities via positive semidefinite matrix completion," *Mathematical programming*, vol. 129, no. 1, pp. 33–68, 2011.

[12] R. P. Mason and A. Papachristodoulou, "Chordal sparsity, decomposing sdps and the lyapunov equation," in *American Control Conference (ACC), 2014*. IEEE, 2014, pp. 531–537.

[13] D. Amsallem and C. Farhat, "An online method for interpolating linear parametric reduced-order models," *SIAM Journal on Scientific Computing*, vol. 33, no. 5, pp. 2169–2198, 2011.

[14] C. Poussot-Vassal and C. Roos, "Generation of a reduced-order lpv/lft model from a set of large-scale mimo lti flexible aircraft models," *Control Engineering Practice*, vol. 20, no. 9, pp. 919–930, 2012.

[15] J. Theis, B. Takarics, H. Pfifer, G. Balas, and H. Werner, "Modal matching for lpv model reduction of aeroservoelastic vehicles," in *AIAA Science and Technology Forum*, 2015.

[16] R. A. Horn and C. R. Johnson, *Matrix analysis*. Cambridge University Press, 1985.

[17] A. A. Ahmadi and A. Majumdar, "DSOS and SDSOS optimization: more tractable alternatives to sos optimization," *In preparation. (http://aaa.lids.mit.edu/publications)*, 2014.

[18] J. Annoni and P. Seiler, "A method to construct reduced-order parameter varying models," *submitted to the International Journal of Robust and Nonlinear Control*, 2015.

[19] B. Takarics and P. Seiler, "Gain scheduling for nonlinear systems via integral quadratic constraints," in *American Control Conference*, 2015.

[20] The MOSEK optimization toolbox for MATLAB. (http://docs.mosek.com/7.0/toolbox/index.html).

[21] P. Gahinet, A. Nemirovski, A. J. Laub, and M. Chilali, *LMI Control Toolbox – For use with MATLAB*. The MathWorks, 1995.