An Optimal Time-Invariant Approximation for Wind Turbine Dynamics Using the Multi-Blade Coordinate Transformation

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***Abstract*— Wind turbines are subject to periodic loads that result in a time-varying "trim" condition. Linearizing the nonlinear turbine dynamics around this trim condition yields a periodic, linear time-varying (PLTV) system. A linear time-invariant (LTI) approximation is typically obtained in two steps. First, the multi-blade coordinate transformation is applied to the PLTV system to obtain a weakly periodic system. Second, the state matrices of the weakly periodic system are averaged over one period to obtain an LTI approximation. This paper presents an alternative approach to construct optimal LTI approximations using convex optimization. It is also shown that the multi-blade coordinate transformation followed by averaging is equivalent to a special case of the proposed convex optimization procedure. The proposed approach is demonstrated on a linearized model of a utility-scale turbine.**

I. Introduction

Wind turbine models vary widely in complexity from one-state, rotor inertia models up to high fidelity, finite element codes that model the details of the fluid/blade interactions. The Fatigue, Aerodynamics, Structures and Turbulence (FAST) code [9] developed by the National Renewable Energy Laboratory (NREL) lies between these extremes in terms of model complexity. FAST is a widely-used nonlinear aeroelastic turbine simulation. FAST can linearize models through numerical perturbation of the nonlinear dynamics. The nonlinear system is first simulated under steady wind conditions until the turbine reaches a trim operating trajectory. The turbine loads are time-varying even in constant wind conditions due to periodic effects such as tower shadow, gravity and shaft tilt. As a result, the wind turbine trim trajectory is periodic with period equal to one rotation of the rotor. Linearizations can be computed at each rotor position to yield a periodic, linear time-varying (PLTV) system.

It is often desirable to transform the PLTV system to a related linear time invariant (LTI) system in order to apply the well established analysis tools for LTI systems. There are various methods to perform this approximation. The simplest approaches are to evaluate the PLTV system at one rotor position or to average the state matrices over one rotor period. These approaches ignore the periodic modal characteristics of the turbine and typically do not provide an LTI model of sufficient accuracy. Floquet theory [8], [13] gives a time-varying coordinate transformation that transforms a PLTV system into one with a constant state "A" matrix. The Floquet transformation retains the periodic modal characteristics but physical intuition about the system states is lost in the transformed system. The most common approach for wind turbine models is to use the multi-blade coordinate (MBC) transformation [3], [7], [14], [10], [1], [2], [11]. The MBC transformation was originally developed in the helicopter literature [8], [5]. It transforms quantities from rotating blade coordinates into a non-rotating, inertial coordinate frame. The MBC transformation ideally converts the PLTV system into an LTI system. In practice, applying the MBC to the PLTV models generated by FAST yields a system that is still "weakly" periodic, i.e. the transformed system is periodic but with significantly less time variation compared to the original PLTV system. An LTI approximation is obtained by averaging the state matrices of the "weakly" periodic system over one rotor period. This LTI approximation is of sufficient fidelity in many cases [14]. However, the averaging step is ad-hoc and does not rely on a quantifiable error criterion.

The main contribution of this paper is an alternative method to construct optimal LTI approximations for PLTV turbine models. Let $S_{\mathcal{M}}$ denote the set of linear systems that are transformed to an LTI system by the MBC. It is shown that $S_{\mathcal{M}}$ is in one-to-one correspondence with a convex set. The proposed approximation approach is summarized in Figure 1. The upper path in the diagram shows the existing two-step approach. As noted above, a PLTV turbine model G^r generated by FAST is typically not in the set $S_{\mathcal{M}}$, i.e. transforming G^r via the MBC yields a model $\bar{G}^A$ that is not time-invariant. As a result, averaging of the state-space matrices is required to obtain an LTI approximation $\hat{G}^A$. The proposed approach essentially flips the sequence of operations. First, the PLTV model G^r is optimally approximated by a PLTV system $\bar{G}^B \in S_{\mathcal{M}}$. This step can be formulated as a convex optimization. Next, the MBC is applied to $\bar{G}^B$ and, by definition of $S_{\mathcal{M}}$, this leads to an LTI approximation $\hat{G}^B$. The benefit of the proposed approach is that is relies on a quantifiable approximation error and the transformed states retain the physical intuition built around the MBC transformation.

II. Notation

The notation is standard. $\mathbb{R}$ denotes the set of real numbers. $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote the sets of $n \times 1$ real vectors and $n \times m$ real matrices, respectively. I_n and 0_n are the $n \times n$ identity and zero matrices, respectively. A matrix $M \in \mathbb{R}^{n\times n}$ is orthogonal if $M^T M = I_n$. The trace of a matrix $M \in \mathbb{R}^{n\times n}$ is defined as $\sum_{i=1}^n M_{i,i}$ and is denoted $tr(M)$. The transpose of a matrix $M \in \mathbb{R}^{n\times n}$ is denoted M^T. Finally, a function $M : \mathbb{R} \to \mathbb{R}^{n\times m}$ is said to be *periodic with period* T if $M(t + T) = M(t)$ for all $t \in \mathbb{R}$.

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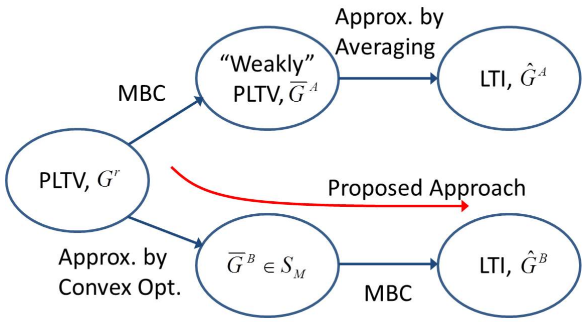

Fig. 1. Two approaches for approximating a PLTV turbine model with an LTI system.

III. Multi-Blade Coordinate Transformation

This section briefly reviews the use of the MBC transformation for wind turbine models. The discussion focuses on linearized models obtained from NREL's FAST simulation package. FAST uses the assumed modes method to model the flexible structural dynamics of the turbine. Blade element momentum theory is used to calculate the aerodynamic loads using AeroDYN [6]. FAST can model on-shore wind turbines with a total of 22-24 degrees of freedom (DOF) including tower fore-aft and side-to-side bending modes as well as blade flapwise and edgewise modes. Off-shore turbines can also be modeled with additional DOFs depending on the platform structure.

For simplicity consider a five DOF (on-shore) turbine model that includes rotor position, first tower fore-aft bending mode, and first flapwise bending mode for each blade. For constant wind conditions, this five DOF model can be specified by a nonlinear dynamical equation of the form:

$$\begin{aligned} \ddot{q} &= f(\dot{q}, q, u) \\ y &= g(\dot{q}, q, u) \end{aligned} \tag{1}$$

where $q \in \mathbb{R}^5$ is defined as:

$$q := \begin{bmatrix} \text{Tower } 1^{st} \text{ Fore-Aft Tip Displacement (m)} \\ \text{Rotor position, } \psi \text{ (rad)} \\ \text{Blade 1 } 1^{st} \text{ Flapwise Tip Displacement (m)} \\ \text{Blade 2 } 1^{st} \text{ Flapwise Tip Displacement (m)} \\ \text{Blade 3 } 1^{st} \text{ Flapwise Tip Displacement (m)} \end{bmatrix} \tag{2}$$

The rotor position, denoted as ψ, is defined to be zero when blade 1 is in the upward position. The input and output vectors, $u \in \mathbb{R}^4$ and $y \in \mathbb{R}^4$, are defined as:

$$u := \begin{bmatrix} \text{Blade 1 Pitch Angle (rad)} \\ \text{Blade 2 Pitch Angle (rad)} \\ \text{Blade 3 Pitch Angle (rad)} \\ \text{Generator Torque (N m)} \end{bmatrix} \tag{3}$$

$$y := \begin{bmatrix} \text{Rotor Speed (rpm)} \\ \text{Blade 1 Root Bending Moment (kN m)} \\ \text{Blade 2 Root Bending Moment (kN m)} \\ \text{Blade 3 Root Bending Moment (kN m)} \end{bmatrix} \tag{4}$$

FAST can produce linear turbine models through numerical perturbation of the nonlinear system in Equation 1. The nonlinear system is first simulated under steady wind conditions until the turbine reaches a trim operating condition $(\bar{q}(t), \dot{\bar{q}}(t), \bar{u}(t), \bar{y}(t))$. The trim condition is, in general, periodic with period T equal to the time for one complete rotation of the rotor. A linear time-varying model is obtained by linearizing the nonlinear system around this periodic trim condition. The resulting linearized model has the form

$$\begin{aligned} \dot{\delta}_x &= A\left(\bar{\psi}(t)\right)\delta_x + B\left(\bar{\psi}(t)\right)\delta_u \\ \delta_y &= C\left(\bar{\psi}(t)\right)\delta_x + D\left(\bar{\psi}(t)\right)\delta_u \end{aligned} \tag{5}$$

where $\delta_u(t) := u(t) - \bar{u}(t)$ and $\delta_y(t) := y(t) - \bar{y}(t)$ are the deviations of the inputs and outputs from their trim values. Similarly define $\delta_q(t) := q(t) - \bar{q}(t)$ as the deviation of the turbine DOF from trim and $\delta_x(t)$ as the deviation of the state from the periodic trim condition:

$$\delta_x(t) := \begin{bmatrix} \delta_q(t) \\ \dot{\delta}_q(t) \end{bmatrix} = \begin{bmatrix} q(t) - \bar{q}(t) \\ \dot{q}(t) - \dot{\bar{q}}(t) \end{bmatrix} \tag{6}$$

The trim rotor position satisfies $\bar{\psi}(t) = \bar{\psi}(t+T)$ for all t since the trim condition is periodic. Hence the state matrices in Equation 5 are also periodic with period T. In other words, the linearized turbine model is PLTV with period T.

The FAST manual [9] contains detailed figures of the coordinate frames used to derive the nonlinear turbine equations of motion (Equation 1). It is important to note that a variety of coordinate frames are used. Specifically, the tower and rotor degrees of freedom are expressed in an earth fixed coordinate frame while quantities associated with individual blades are defined in a frame that rotates with the rotor. For example, the tip displacements of the blade flapwise bending mode are defined with respect to a rotating coordinate frame attached to the blade.

The MBC transformation is used to convert blade quantities back and forth between rotating and non-rotating (inertial) coordinate frames. Define the transformation matrix $M : \mathbb{R} \to \mathbb{R}^{3\times 3}$ as a function of rotor position:

$$M(\psi) := \begin{bmatrix} 1 & \sin(\psi) & \cos(\psi) \\ 1 & \sin(\psi + \frac{2\pi}{3}) & \cos(\psi + \frac{2\pi}{3}) \\ 1 & \sin(\psi + \frac{4\pi}{3}) & \cos(\psi + \frac{4\pi}{3}) \end{bmatrix} \tag{7}$$

For a given rotor position ψ, $M(\psi)$ transforms quantities in the inertial (non-rotating) frame to the rotating frame attached to the rotor. Conversely, the inverse of $M(\psi)$ transforms quantities from a rotating to non-rotating frame. This inverse is explicitly given by

$$M(\psi)^{-1} = \frac{2}{3}\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \sin(\psi) & \sin(\psi + \frac{2\pi}{3}) & \sin(\psi + \frac{4\pi}{3}) \\ \cos(\psi) & \cos(\psi + \frac{2\pi}{3}) & \cos(\psi + \frac{4\pi}{3}) \end{bmatrix} \tag{8}$$

As an example, the last three entries of the output vector are the root bending moments for the three blades measured in the rotating frame. For a given rotor position ψ these quantities are transformed to the non-rotating frame by:

$$\begin{bmatrix} y_{avg}^{nr} \\ y_{yaw}^{nr} \\ y_{tilt}^{nr} \end{bmatrix} = M(\psi)^{-1}\begin{bmatrix} y_2 \\ y_3 \\ y_4 \end{bmatrix} \tag{9}$$

The superscript nr denotes quantities expressed in an inertial non-rotating frame. After the transformation, these quantities have meanings in terms of rotor motion instead of individual blades. y_{avg}^{nr} represents average value of blade root bending moments. The average moment causes the rotor to bend as a cone. y_{tilt}^{nr} and y_{yaw}^{nr} are the blade moments resulting in rotor tilt and yaw, respectively [12]. Similarly, the blade pitch angle inputs and blade flapwise tip displacements can be mapped from rotating to non-rotating coordinates:

$$\begin{bmatrix} u_1^{nr} \\ u_2^{nr} \\ u_3^{nr} \end{bmatrix} = M(\psi)^{-1} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \tag{10}$$

$$\begin{bmatrix} q_3^{nr} \\ q_4^{nr} \\ q_5^{nr} \end{bmatrix} = M(\psi)^{-1} \begin{bmatrix} q_3 \\ q_4 \\ q_5 \end{bmatrix}, \tag{11}$$

q_3^{nr}, q_4^{nr}, and q_5^{nr} are the rotor coning, rotor tip-path-plane fore-aft tilt and rotor tip-path-plane side-side tilt, respectively [1]. Moreover, u_1^{nr} is the collective pitch command while u_2^{nr} and u_3^{nr} are cyclic individual blade pitch commands.

The system states, inputs, and outputs are defined in a mixed coordinate system, i.e. they have entries expressed in both rotating and non-rotating (inertial) coordinate frames. The MBC transformation is used to convert all quantities to a non-rotating coordinate frame. Specifically, the transformation M introduced in Equation 7 is used to define a transformation M_q acting on the linearized DOF:

$$M_q(\bar{\psi}(t)) := \begin{bmatrix} I_2 & 0_2 \\ 0_2 & M(\bar{\psi}(t)) \end{bmatrix} \tag{12}$$

The linearized DOFs are transformed as:

$$\delta_q(t) = M_q(\bar{\psi}(t))\ \delta_q^{nr}(t) \tag{13}$$

M_q transforms only those quantities that are specified in the rotating frame. Quantities specified in the inertial frame are left unchanged. Similarly, the state, input, and output transformations for the linearized system are:

$$\delta_x(t) = M_x(\bar{\psi}(t))\ \delta_x^{nr}(t) \tag{14}$$

$$\delta_u(t) = M_u(\bar{\psi}(t))\ \delta_u^{nr}(t) \tag{15}$$

$$\delta_y(t) = M_u(\bar{\psi}(t))\ \delta_y^{nr}(t) \tag{16}$$

where the transformation matrices are given by

$$M_x(\bar{\psi}(t)) = \begin{bmatrix} M_q(\bar{\psi}(t)) & 0_5 \\ \frac{d}{dt} M_q(\bar{\psi}(t)) & M_q(\bar{\psi}(t)) \end{bmatrix} \tag{17}$$

$$M_u(\bar{\psi}(t)) := \begin{bmatrix} M(\bar{\psi}(t)) & 0 \\ 0 & 1 \end{bmatrix} \tag{18}$$

$$M_y(\bar{\psi}(t)) := \begin{bmatrix} 1 & 0 \\ 0 & M(\bar{\psi}(t)) \end{bmatrix} \tag{19}$$

The transformation M_x is derived by applying the chain rule to Equation 13 and using the definition of δ_x in terms of δ_q and $\dot{\delta}_q$. The complete MBC transformation for the linearized system is given by the collection of state, input, and output transformations (M_x, M_u, M_y). Applying these transformations to the PLTV model in Equation 5 reduces the variation due to rotor position but it typically does not lead to an LTI system. Averaging the remaining variations over one rotor period often gives an LTI model of sufficient fidelity [14]. The basic approach reviewed in this section can easily be generalized to turbine models with additional DOFs specified in the inertial and/or rotating frames. Additional details of MBC can be found in [1] and in the manual for the NREL MATLAB utilities that implement the MBC transformations [2].

IV. Optimal LTI Approximations

This section gives a precise formulation of the two approximation methods described in Section I. Section IV-A develops the necessary mathematical background and the Section IV-B formulates the two approximation methods. Finally, Section IV-C shows that the multi-blade coordinate transformation followed by averaging is equivalent to a special case of the proposed convex optimization procedure.

A. Mathematical Background

Let G^r denote a PLTV turbine model obtained from linearization. The superscript r denotes that this model, in general, contains some quantities in the rotating coordinate frame. Assume G^r has n states, m inputs, and p outputs. Moreover, let $M_x : \mathbb{R} \to \mathbb{R}^{n\times n}$, $M_u : \mathbb{R} \to \mathbb{R}^{m\times m}$, and $M_y : \mathbb{R} \to \mathbb{R}^{p\times p}$ be the state, input, and output transformations that comprise the complete MBC transformation for this system. This formulation generalizes the discussion in Section III to handle an arbitrary number of DOFs. The first result provides a useful decomposition of the MBC transformation matrices.

Lemma 1: If trim rotor speed is constant $\ddot{\bar{\psi}}(t) = 0$ then the MBC transformation matrices satisfy:

$$M_x(\psi) = W_x(\psi) N_x \tag{20}$$

$$M_u(\psi) = W_u(\psi) N_u \tag{21}$$

$$M_y(\psi) = W_y(\psi) N_y \tag{22}$$

where $W_x(\psi)$, $W_u(\psi)$, $W_y(\psi)$ are orthogonal $\forall \psi \in [0, 2\pi]$.

Proof: Define a scaled version of the basic coordinate transformation defined in Equation 7:

$$W(\psi) := M(\psi) \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{2}{3} & 0 \\ 0 & 0 & \frac{2}{3} \end{bmatrix}^{\frac{1}{2}} \tag{23}$$

It can be shown by direct multiplication and trigonometric identities that $W(\psi)^T W(\psi) = I_3$ for all ψ. The input transformation for the MBC $M_u(\psi)$ is, in general, a block diagonal augmentation of identity matrices and the basic coordinate transform $M(\psi)$. It follows that $M_u(\psi) = W_u(\psi) N_u$ where $W_u(\psi)$ is orthogonal for all ψ and N_u is constant (independent of ψ). Both the transformations M_q and M_y have similar expressions as a product of an orthogonal and constant matrix. The decomposition $M_x(\psi) = W_x(\psi) N_x$ with W_x orthogonal follows because $M_q^{-1}(\bar{\psi}(t)) \dot{M}_q(\bar{\psi}(t))$ is constant if $\ddot{\bar{\psi}}(t) = 0$. ■

The next result concerns the class of systems that become LTI via the MBC transformation. Let S_T denote the set of PLTV systems with period T and state, input, output dimension equal to n, m, p, respectively. The PLTV turbine model G^r is in this set S_T. More generally, any system $G \in S_T$ has the form:

$$\begin{aligned}\dot{x}(t) &= A(t)x(t) + B(t)u(t)\\ y(t) &= C(t)x(t) + D(t)u(t)\end{aligned} \tag{24}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^p$. The state matrices have compatible dimensions and are periodic functions with period T. The MBC is a transformation $\mathcal{M} : S_T \to S_T$ that maps any system $G \in S_T$ to a new periodic system $G^{nr} = \mathcal{M}(G)$. The states of the new periodic system G^{nr} are interpreted as being entirely in non-rotating (inertial) coordinates. In particular, $\mathcal{M}$ transforms the states, inputs, and outputs of G as:

$$x(t) = M_x(t)x^{nr}(t) \tag{25}$$

$$u(t) = M_u(t)u^{nr}(t) \tag{26}$$

$$y(t) = M_y(t)y^{nr}(t) \tag{27}$$

The transformed system G^{nr} is

$$\begin{aligned}\dot{x}^{nr}(t) &= A^{nr}(t)x^{nr}(t) + B^{nr}(t)u^{nr}(t)\\ y^{nr}(t) &= C^{nr}(t)x^{nr}(t) + D^{nr}(t)u^{nr}(t)\end{aligned} \tag{28}$$

where the state matrices are given by

$$A^{nr}(t) = M_x^{-1}(t)A(t)M_x(t) - M_x^{-1}(t)\dot{M}_x(t) \tag{29}$$

$$B^{nr}(t) = M_x^{-1}(t)B(t)M_u(t) \tag{30}$$

$$C^{nr}(t) = M_y^{-1}(t)C(t)M_x(t) \tag{31}$$

$$D^{nr}(t) = M_y^{-1}(t)D(t)M_u(t) \tag{32}$$

The state matrices of G^{nr} are, in general, periodic with period T and hence $G^{nr} \in S_T$. However, some systems become time-invariant under the MBC transformation $\mathcal{M}$. Let $S_{\mathcal{M}}$ denote the set of systems in S_T that are transformed to LTI systems by $\mathcal{M}$. $S_{\mathcal{M}}$ is formally defined as

$$S_{\mathcal{M}} := \{G \in S_T \;:\; \mathcal{M}(G) \in S_0\} \tag{33}$$

$S_0 \subset S_T$ is the set of systems of appropriate dimensions with constant state matrices.[1]

Next, note that state matrices of systems in S_T can be packed into a single periodic, matrix function $g(t) := \left[\begin{smallmatrix} A(t) & B(t)\\ C(t) & D(t)\end{smallmatrix}\right]$. Conversely, periodic functions $g : \mathbb{R} \to \mathbb{R}^{(n+p)\times(n+m)}$ of period T can be properly partitioned and unpacked to yield state space matrices corresponding to a system in S_T. Thus there is a natural one-to-one correspondence between systems $G \in S_T$ and periodic functions. To formalize this connection, let F_T denote the set of periodic functions $g : \mathbb{R} \to \mathbb{R}^{(n+p)\times(n+m)}$ of period T. As noted, there is a natural one-to-one correspondence between periodic systems in S_T and periodic functions in F_T. Similarly, let $F_{\mathcal{M}} \subset F_T$ denote periodic functions corresponding to the packed state matrices of systems in $S_{\mathcal{M}}$. The following result provides a useful characterization of the systems that become LTI via the MBC transformation.

Theorem 1: The set $F_{\mathcal{M}}$ is convex.

Proof: It follows directly from the transformation relations (Equations 29-32) that a PLTV system G with state matrices $(A(t), B(t), C(t), D(t))$ is in $S_{\mathcal{M}}$ if and only if there exist constant matrices $(A^{nr}, B^{nr}, C^{nr}, D^{nr})$ such that

$$A(t) = M_x(t)\left[A^{nr} + M_x^{-1}(t)\dot{M}_x(t)\right]M_x^{-1}(t) \tag{34}$$

$$B(t) = M_x(t)B^{nr}M_u^{-1}(t) \tag{35}$$

$$C(t) = M_y(t)C^{nr}M_x^{-1}(t) \tag{36}$$

$$D(t) = M_y(t)D^{nr}M_u^{-1}(t) \tag{37}$$

These relations can be used to show that $F_{\mathcal{M}}$ satisfies the basic definition of convexity [4]: $g_1, g_2 \in F_{\mathcal{M}}$ implies that $\lambda g_1 + (1-\lambda)g_2 \in F_{\mathcal{M}}$ for all $\lambda \in [0, 1]$. Alternatively, the result follows by simply noting that $F_{\mathcal{M}}$ is an affine subspace of periodic functions. ■

By Theorem 1, the class of n-state systems that become LTI via the MBC can be viewed as a convex set in the space of their packed state-space matrices.

B. Approximation Methods

Next let $G^r \in S_T$ be a given PLTV turbine model. Consider the following question: Does the MBC $\mathcal{M}$ transform the turbine model G^r to an LTI system? The answer is yes if and only if $G^r \in S_{\mathcal{M}}$ (subject to the technical point concerning unobservable/uncontrollable modes). If $G^r \notin S_{\mathcal{M}}$ then the answer to this question is no and the following approximation problem is of interest: Find an n-state LTI system $\hat{G}$ that is approximately related to G^r through the MBC transformation $\mathcal{M}$. Two specific approaches to this approximation problem will be considered.

Approach A: First apply $\mathcal{M}$ to obtain $\bar{G} = \mathcal{M}(G^r) \in S_T$. Next, compute an n-state LTI system $\hat{G}$ that approximates $\bar{G}$. The following optimization is proposed as a means to perform the approximation step:

$$\hat{g} := \arg\min_{g \in F_0} \|\bar{g} - g\| \tag{38}$$

$\bar{g} \in F_T$ denotes the periodic function representation of $\bar{G}$ as a packed state-space matrix. F_0 are packed state space matrices corresponding to n-state LTI systems, i.e. $g \in F_0$ is basically a constant matrix $\mathbb{R}^{(n+p)\times(n+m)}$. Finally, $\|\cdot\|$ is any norm on the space F_T of periodic matrix functions. The optimization searches for constant state-space matrices $\hat{g}$ that best approximate $\bar{g}$ in the given norm. This yields the corresponding LTI system $\hat{G}$ that is the best approximation.

As a concrete example, the optimization in Approach A can be formulated with the following norm:

$$\|h\|_2 := \int_0^T tr\left(h(\tau)^T h(\tau)\right) d\tau \tag{39}$$

In words, the 2-norm is the sum of the entries of h squared and integrated over one period. The next result shows that

[1] There is one technical point concerning the definition of $S_{\mathcal{M}}$. It is possible for $G^{nr} = \mathcal{M}(G)$ to have unobservable and/or uncontrollable modes such that its state matrices vary with time and yet it has a minimal realization with constant state matrices. Such systems are not contained in $S_{\mathcal{M}}$ as defined in Equation 33.

the optimal solution to Equation 38 has a particularly simple form when the 2-norm is used.

Theorem 2: Let $\bar{G} \in S_T$ be given and let $\bar{g} \in F_T$ denote the corresponding periodic function representation of the packed state matrices. The optimal solution

$$\hat{g} := \arg \min_{g \in F_0} \|\bar{g} - g\|_2 \tag{40}$$

is given by:

$$\hat{g} := \frac{1}{T} \int_0^T \bar{g}(\tau) d\tau \tag{41}$$

Proof: Substitute $g = \hat{g} + \Delta g$ into $\|\bar{g} - g\|_2$, to obtain

$$\int_0^T tr\left((\bar{g}(\tau) - \hat{g})^T(\bar{g}(\tau) - \hat{g})\right) \tag{42}$$
$$- 2tr\left((\bar{g}(\tau) - \hat{g})^T \Delta g\right) + tr\left(\Delta g^T \Delta g\right) d\tau$$

The second term is zero by definition of the constant matrix $\hat{g}$. Hence $\Delta g \neq 0$ increases the cost. ■

Thus averaging of the periodic state matrices is the optimal solution to Equation 38 when the 2-norm is used in the cost. Thus by Theorem 2, the standard LTI approximation method of MBC followed by averaging is equivalent to Approach A in the 2-norm. However, the use of another norm on the space F_T of periodic matrix functions will yield a different optimal approximation. Next, the alternative approach introduced in Section I is formally defined below.

Approach B: First approximate G^r by $\bar{G} \in S_{\mathcal{M}}$. Next apply the MBC transformation $\mathcal{M}$ to obtain the n-state LTI system $\hat{G} = \mathcal{M}(\bar{G})$. The following optimization is proposed as a means to perform the approximation step:

$$\bar{g} := \arg \min_{g \in F_{\mathcal{M}}} \|g^r - g\| \tag{43}$$

$g^r \in F_T$ denotes the periodic function representation of G^r as a packed state-space matrix. $\|\cdot\|$ again denotes any norm on the space F_T of periodic matrix functions.

It follows from Theorem 1 that Equation 43 is a convex optimization. This optimization approximates G^r in the sense of finding a system $\bar{G}$ whose state matrices are near those of G^r in some norm. Unfortunately it is not clear at this point how to formulate an approximation problem using norms on the systems themselves. Similarly Equation 38 is also a convex optimization because F_0 is clearly convex. These optimization are only an approximation in the sense of matching the state matrices.

C. Relations Between Approximation Approaches

The previous section introduced two general LTI approximation methods using the MBC transformation. This section demonstrates an equivalence between these two methods when using the norm $\|.\|_2$ defined in Equation 39.

Theorem 3: Assume the trim rotor speed is constant $\ddot{\psi}(t) = 0$. For any $G^r \in S_T$ the optimal LTI approximation obtained with Approach A using the 2-norm is identical to that given by Approach B also using 2-norm.

Proof: The first step of Approach B is to approximate G^r by $\bar{G} \in S_{\mathcal{M}}$ via a convex optimization

$$\bar{g} := \arg \min_{g \in F_{\mathcal{M}}} \|g^r - g\|_2 \tag{44}$$

As in the proof of Theorem 1, $g \in F_{\mathcal{M}}$ if and only if there exist constant matrices $(A^{nr}, B^{nr}, C^{nr}, D^{nr})$ such that

$$g(t) = \begin{bmatrix} M_x(t) & 0 \\ 0 & M_y(t) \end{bmatrix} \begin{bmatrix} A^{nr} & B^{nr} \\ C^{nr} & D^{nr} \end{bmatrix} \begin{bmatrix} M_x^{-1}(t) & 0 \\ 0 & M_u^{-1}(t) \end{bmatrix}$$
$$+ \begin{bmatrix} \dot{M}_x(t) M_x^{-1}(t) & 0 \\ 0 & 0 \end{bmatrix}$$

By Lemma 1, the MBC transformation matrices can be decomposed as a product of a time-varying orthogonal matrix and a constant matrix (Equations 20-22). Moreover, the 2-norm is unchanged by the introduction of time-varying orthogonal matrices. Thus the optimization in Equation 44 can be equivalently written as:

$$\min_{g \in F_{\mathcal{M}}} \|Q_L(g^r - g)Q_R\|_2 \tag{45}$$

where

$$Q_L(t) := \begin{bmatrix} W_x^T(t) & 0 \\ 0 & W_y^T(t) \end{bmatrix} \tag{46}$$

$$Q_R(t) := \begin{bmatrix} W_x(t) & 0 \\ 0 & W_u(t) \end{bmatrix} \tag{47}$$

These orthogonal matrices separate out the time-varying components of g. In particular,

$$Q_L(t)(g^r(t) - g(t))Q_R(t)$$
$$= Q_L(t) \left(g^r(t) - \begin{bmatrix} \dot{M}_x(t) M_x^{-1}(t) & 0 \\ 0 & 0 \end{bmatrix} \right) Q_R(t)$$
$$- \begin{bmatrix} N_x & 0 \\ 0 & N_y \end{bmatrix} \begin{bmatrix} A^{nr} & B^{nr} \\ C^{nr} & D^{nr} \end{bmatrix} \begin{bmatrix} N_x^{-1} & 0 \\ 0 & N_u^{-1} \end{bmatrix}$$

Thus the optimization in Equation 45 is converted to one where the objective is to approximate a periodic function with a constant. The optimal solution is

$$\begin{bmatrix} A^{nr} & B^{nr} \\ C^{nr} & D^{nr} \end{bmatrix} := \frac{1}{T} \int_0^T \hat{g}(\tau) d\tau \tag{48}$$

where $\hat{g} \in F_T$ is the periodic function corresponding to the state matrices of $\hat{G} = \mathcal{M}(G^r)$. Hence the optimal solution for Approach B is identical to that of Approach A. ■

It was shown (Theorem 2) that the standard LTI approximation method of MBC followed by averaging is equivalent to Approach A in the 2-norm. Theorem 3 shows that the proposed approach, i.e. Approach B, is also equivalent to this standard LTI method in the 2-norm. However, Approaches A and B need not be equivalent to each other nor to the MBC/Averaging method when another norm is used for approximation. Other norms may yield better LTI approximations from a dynamical systems perspective. Moreover, it is not clear if approximating the system G^r before (Approach B) or after (Approach A) the MBC transformation leads to better results from a dynamical systems perspective. Finally, it is again worth noting that the two approaches approximate systems using norms on the state space matrices.

V. Numerical Results

This section presents initial numerical for the three-bladed Controls Advanced Research Turbine (CART3) located at the National Wind Technology Center. The CART3 is a $40\ m$ diameter turbine with $600\ kW$ rated power. A 5 DOF model, as described in Section III, was linearized at 14 m/sec mean wind speed using FAST. The trim condition is periodic and some key characteristics of the trim condition are provided in Table I. The state matrices for the PLTV model $(A^r(\psi), B^r(\psi), C^r(\psi), D^r(\psi))$ are continuous functions of rotor position. FAST computes these matrices on a finite grid. In this example, the state matrices were computed at 180 equally spaced rotor positions in $[0, 2\pi]$.

TABLE I
TRIM CONDITIONS

Description	Value
Mean wind speed, F	14.0 m/s
Vertical shear factor	0.2
Rotor speed, $\bar{\Omega}$	3.881 rad/s
Collective blade pitch, $\bar{u}$	0.1713 rad
Generator torque (High Speed Side)	3524.0 Nm

The nonlinear optimization solvers in MATLAB were used to compute LTI approximations for the CART3 PLTV model using both Approaches A and B described in Section IV. The decision variables are the elements of the LTI system matrices $(A^{nr}, B^{nr}, C^{nr}, D^{nr})$. The matrices A^{nr} and B^{nr} must have special structure because the first five states are position DOFs whose time derivatives are equal to the second five states. This structure is exploited in the optimization. In addition, the decision variables and the cost of optimization are all scaled to 1 at the optimization starting point. This improves the numerical conditioning of the optimization.

First, the optimizations in both Approaches A and B were performed in MATLAB using the 2-norm. The optimizations were initialized with the LTI approximation obtained using the MBC transformation followed by averaging. In both cases the optimization terminated immediately and reported that cost did not decrease for any change in the decision variables. This result verifies that both Approaches A and B are equivalent to the standard MBC transformation followed by averaging as shown in Section IV.

Next, the optimizations in both Approaches A and B were performed in MATLAB using the ∞-norm:

$$\|h\|_\infty := \max_{0 \le t \le T} \max_{i,j} |h_{i,j}(t)| \tag{49}$$

The LTI approximation using Approach A can be computed analytically. The LTI approximation with Approach B was obtained from numerical optimization in MATLAB. The optimization had approximately 150 decision variables and took less than 2 minutes to solve on a desktop computer. Using the $||.||_\infty$ norm, Approaches A and B yielded different matrices than each other and both LTI approximations were different than that obtained by simple MBC followed by averaging. However, the differences were relatively minor. The state matrices A^{nr} obtained in these cases all had poles whose damping ratios differed by less than 0.5%. Morever, the relative difference between the entries of any of the computed A^{nr} matrices was less than 4%. We performed an identical analysis at a higher wind speed of $18 m/s$ with similar results. Future work will apply the various approximation methods to larger turbines. Larger turbines are more flexible and hence significant differences in the approximations may be observed in this case.

VI. Conclusions

This paper introduced a procedure for computing LTI approximations for turbine models. The approach first approximates the periodic model using convex optimization and then applies the multi-blade coordinate transformation. The common procedure of transforming the periodic model with the multi-blade coordinate transformation followed by averaging of the state matrices is a special instance of the proposed procedure. Future work will investigate the use of different norms in the approximation step.

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