Stability Analysis with Dissipation Inequalities and Integral Quadratic Constraints

Peter Seiler

Abstract

This paper considers the stability of a feedback connection of a known linear, time-invariant system and a perturbation. The input/output behavior of the perturbation is described by an integral quadratic constraint (IQC). IQC stability theorems can be formulated in the frequency domain or with a time-domain dissipation inequality. The two approaches are connected by a non-unique factorization of the frequency domain IQC multiplier. The factorization must satisfy two properties for the dissipation inequality to be valid. First, the factorization must ensure the time-domain IQC holds for all finite times. Second, the factorization must ensure that a related matrix inequality, when feasible, has a positive semidefinite solution. This paper shows that a class of frequency domain IQC multipliers has a factorization satisfying these two properties. Thus the dissipation inequality test, with an appropriate factorization, can be used with no additional conservatism.

I. Introduction

Integral quadratic constraints (IQCs) [14] provide a framework for robustness analysis building on work by Yakubovich [26]. The system is separated into a feedback connection of a known linear time-invariant (LTI) system and a perturbation. The input-output behavior of the perturbation is assumed to satisfy a frequency-domain IQC defined by a multiplier Π. The IQC stability theorem in [14] involves frequency domain inequalities. The main condition in this theorem is equivalent (by the KYP lemma [17], [23]) to the existence of a matrix $P = P^T$ satisfying a related linear matrix inequality (LMI).

A related stability theorem can be formulated using dissipation theory [1], [24], [25] and a time-domain IQC. There are two issues. First, the frequency domain IQC can be equivalently expressed in the time-domain as an infinite-horizon integral constraint. This step requires a (non-unique) factorization of the multiplier as $\Pi = \Psi^{\sim} M \Psi$. The dissipation theory requires the IQC to be "hard" in the sense that the integral constraint holds over all finite times. Second, the dissipation inequality is equivalent to

P. Seiler is with the Aerospace and Engineering Mechanics Department, University of Minnesota, seiler@aem.umn.edu

existence of a matrix $P \geq 0$ satisfying the KYP LMI. The constraint $P \geq 0$ ensures that P defines a valid storage function. Note that the frequency domain approach does not require $P \geq 0$.

To summarize, the two approaches are related by a non-unique factorization of the frequency domain multiplier as $\Pi = \Psi^\sim M \Psi$. This factorization must be "hard" and must ensure that the KYP LMI, if feasible, has a solution $P \geq 0$. The main contribution of this paper is to show that a class of frequency domain IQC multipliers has a (J-spectral) factorization satisfying these two properties. Thus with an appropriate factorization, the dissipation inequality approach can be used with no additional conservatism. The benefit is that the dissipation theory enables generalization to cases where the known system in the feedback connection is nonlinear and/or time-varying, e.g. Theorem 2 in [15].

Previous work [3], [11], [12], [18], [21] relates IQC analysis to dissipation theory for the special case of hard IQCs. The important point in this paper is that the constraint $P \geq 0$ on the KYP LMI solution must also be considered. Another closely related prior work is [22]. This work extends the systems with additional input/output signals to create an equivalent loop. A dissipation inequality based on IQCs then follows after performing a loop transformation. The work here complements [22] by focusing on the non-unique IQC factorization. This provides additional insight into the use of IQCs. The approach here also avoids extending the input/output dimensions of the loop systems. Hence the dissipation inequality given here may have numerical advantages.

The work here is also related to existing IQC factorizations. Related work on J-spectral factorizations of IQCs appears in [9]. An upper triangular factorization in [19], [20] guarantees that all solutions of the KYP LMI satisfy $P \geq 0$. A lower triangular factorization in [21] is "hard". It was incorrectly stated in [21] that this lower-triangular factorization also yields $P \geq 0$. Finally, a hard factorization theorem is also given in [13] using a minimum phase condition on one block of the factorization. The terms "complete" and "conditional" IQCs in [13] are generalizations of hard and soft IQCs. The hard/soft terminology will be used here. The factorization described in this paper is both hard and ensures $P \geq 0$. This result essentially corrects the flaw in [21].

II. Background

A. Notation

Most notation is from [28]. $ARE(A, B, Q, R, S)$ denotes the following Algebraic Riccati Equation (ARE)

$$A^T X + XA - (XB + S)R^{-1}(XB + S)^T + Q = 0 \tag{1}$$

The stabilizing solution $X = X^T$, if it exists, is such that $A - BR^{-1}(XB + S)^T$ is Hurwitz. For $u \in L_2[0, \infty)$, $(u)_T$ is the truncated function: $(u)_T(t) = u(t)$ for $t \leq T$ and $(u)_T(t) = 0$ otherwise. The para-Hermitian conjugate of $G \in \mathbb{RL}_\infty^{m \times n}$, denoted as $G^\sim$, is defined by $G^\sim(s) := G(-s)^T$. Finally, given a differentiable function $V : \mathbb{R}^n \to \mathbb{R}$ the notation ∇V denotes the gradient of V.

B. Problem Formulation

Consider the feedback interconnection shown in Figure 1. This interconnection is specified by the following equations:

$$v = Gu + f, \qquad u = \Delta(v) + r \tag{2}$$

where $r \in L_{2e}^m[0, \infty)$ and $f \in L_{2e}^n[0, \infty)$ are exogenous inputs. $\Delta : L_{2e}^n[0, \infty) \to L_{2e}^m[0, \infty)$ is a causal operator with bounded gain. G is a linear time-invariant system:

$$\dot{x}_G = Ax_G + Bu, \qquad y = Cx_G + Du \tag{3}$$

where $x_G \in \mathbb{R}^{n_G}$ is the state of G.

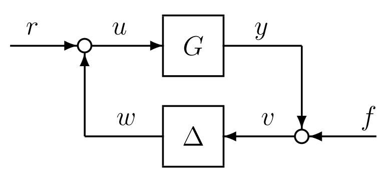

Fig. 1. Feedback interconnection

Definition 1: The interconnection of G and Δ is well-posed if for each $r \in L_{2e}^m[0, \infty)$ and $f \in L_{2e}^n[0, \infty)$ there exist unique $u \in L_{2e}^m[0, \infty)$ and $v \in L_{2e}^n[0, \infty)$ such that the mapping from (r, f) to (u, v) is causal.

Definition 2: The interconnection of G and Δ is stable if it is well-posed and if the mapping from (r, f) to (u, v) has finite L_2 gain for all solutions starting from $x_G(0) = 0$.

C. Frequency Domain IQC Stability Condition

Let $\Pi : j\mathbb{R} \to \mathbb{C}^{(n+m)\times(n+m)}$ be a measurable Hermitian-valued function. Two signals $v \in L_2^n[0, \infty)$ and $w \in L_2^m[0, \infty)$ satisfy the IQC defined by the multiplier Π if

$$\int_{-\infty}^{\infty} \begin{bmatrix} \hat{v}(j\omega) \\ \hat{w}(j\omega) \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} \hat{v}(j\omega) \\ \hat{w}(j\omega) \end{bmatrix} d\omega \geq 0 \tag{4}$$

where $\hat{v}(j\omega)$ and $\hat{w}(j\omega)$ are Fourier transforms of v and w. A bounded, causal operator $\Delta : L_{2e}^{n}[0,\infty) \to L_{2e}^{m}[0,\infty)$ satisfies the IQC defined by Π if Equation 4 holds for all $v \in L_2^n[0,\infty)$ and $w = \Delta(v)$. The next theorem provides a stability condition for the interconnection of G and Δ.

Theorem 1 ([14]): Let $G \in \mathbb{RH}_\infty^{n\times m}$ and $\Delta : L_{2e}^n \to L_{2e}^m$ be a bounded causal operator. Assume for all $\tau \in [0,1]$:

1) the interconnection of G and $\tau\Delta$ is well-posed.
2) $\tau\Delta$ satisfies the IQC defined by Π.
3) $\exists\epsilon > 0$ such that

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} \le -\epsilon I \quad \forall\omega \in \mathbb{R}. \tag{5}$$

Then the feedback interconnection of G and Δ is stable.

For rational multipliers, Condition 3 is equivalent to an LMI. Specifically, any $\Pi \in \mathbb{RL}_\infty^{(n+m)\times(n+m)}$ can be factorized as $\Pi = \Psi^{\sim} M \Psi$ where $M = M^T \in \mathbb{R}^{n_z\times n_z}$ and $\Psi \in \mathbb{RH}_\infty^{n_z\times(n+m)}$. Such factorizations are not unique but can be computed with state-space methods [18]. Denote a state-space realization of Ψ by $(A_\psi, [B_{\psi 1}, B_{\psi 2}], C_\psi, [D_{\psi 1}, D_{\psi 2}])$ where the B_ψ/D_ψ matrices are partitioned compatibly with $\left[\begin{smallmatrix} v \\ w \end{smallmatrix}\right]$. A state-space realization for the system $\Psi \left[\begin{smallmatrix} G \\ I \end{smallmatrix}\right]$ is:

$$\begin{aligned}(\hat{A},\hat{B},\hat{C},\hat{D}) := \Big(&\begin{bmatrix} A & 0 \\ B_{\psi 1}C & A_\psi \end{bmatrix}, \begin{bmatrix} B \\ B_{\psi 2}+B_{\psi 1}D \end{bmatrix}, \\ &\begin{bmatrix} D_{\psi 1}C & C_\psi \end{bmatrix}, D_{\psi 2} + D_{\psi 1}D \Big)\end{aligned} \tag{6}$$

Finally, the KYP Lemma [17], [23] can be applied to demonstrate the equivalence of Condition 3 in Theorem 1 to an LMI condition. This result is stated formally below.

Theorem 2: $\exists\epsilon > 0$ such that Equation 5 holds if and only if there exists a matrix $P = P^T$ such that

$$\begin{bmatrix} \hat{A}^T P + P\hat{A} & P\hat{B} \\ \hat{B}^T P & 0 \end{bmatrix} + \begin{bmatrix} \hat{C}^T \\ \hat{D}^T \end{bmatrix} M \begin{bmatrix} \hat{C} & \hat{D} \end{bmatrix} < 0 \tag{7}$$

D. Time Domain Dissipation Inequality Stability Condition

An alternative time-domain stability condition can be constructed using IQCs and dissipation theory. Let (Ψ, M) be a factorization of Π. Let signals (v,w) satisfy the IQC in Equation 4 and define $\hat{z}(j\omega) := \Psi(j\omega) \begin{bmatrix} \hat{v}(j\omega) \\ \hat{w}(j\omega) \end{bmatrix}$. Then the IQC can be written as $\int_{-\infty}^{\infty} \hat{z}(j\omega)^* M \hat{z}(j\omega) d\omega \ge 0$. By Parseval's theorem [28], this frequency-domain inequality can be equivalently expressed in the time-domain as:

$$\int_0^\infty z(t)^T M z(t)\, dt \ge 0 \tag{8}$$

where z is the output of the LTI system Ψ:

$$\dot{\psi}(t) = A_\psi \psi(t) + B_{\psi 1} v(t) + B_{\psi 2} w(t), \quad \psi(0) = 0 \tag{9}$$

$$z(t) = C_\psi \psi(t) + D_{\psi 1} v(t) + D_{\psi 2} w(t) \tag{10}$$

Thus Δ satisfies the IQC defined by $\Pi = \Psi^\sim M \Psi$ if and only if the filtered signal $z = \Psi \left[\begin{smallmatrix} v \\ w \end{smallmatrix}\right]$ satisfies the time domain constraint (Equation 8) for all $v \in L_2^n[0, \infty)$ and $w = \Delta(v)$.

The constraint in Equation 8 holds, in general, only over infinite time. The term <u>hard IQC</u> in [14] refers to the more restrictive property: $\int_0^T z(t)^T M z(t)\ dt \geq 0$ holds $\forall T \geq 0$. In contrast, IQCs for which the time domain constraint need not hold for all finite times are called <u>soft IQCs</u>. This distinction is important because the dissipation theorem below requires the use of hard IQCs. One issue is that the factorization of Π is not unique. Thus the hard/soft property is not inherent to the multiplier Π but instead depends on the factorization (Ψ, M). A more precise definition is now given.

Definition 3: Let $\Pi \in \mathbb{RL}_\infty^{(n+m)\times(n+m)}$ be factorized as $\Psi^\sim M \Psi$ where $M = M^T \in \mathbb{R}^{n_z \times n_z}$ and $\Psi \in \mathbb{RH}_\infty^{n_z \times (n+m)}$. Then (Ψ, M) is a <u>hard IQC factorization</u> of Π if for any bounded, causal operator Δ satisfying the IQC defined by Π the following inequality holds

$$\int_0^T z(t)^T M z(t)\ dt \geq 0 \tag{11}$$

for all $T \geq 0$, $v \in L_{2e}^n[0, \infty)$, $w = \Delta(v)$, and $z = \Psi \left[\begin{smallmatrix} v \\ w \end{smallmatrix}\right]$.

The stability of the feedback system can be analyzed using Figure 2. This feedback interconnection including Ψ is described by $w = \Delta(v)$ and the following extended linear dynamics (omitting the dependence of all signals on time t):

$$\dot{x} = \hat{A} x + \hat{B} w + \hat{B}_2 \left[\begin{smallmatrix} f \\ r \end{smallmatrix}\right] := F(x, w, f, r) \tag{12}$$

$$\left[\begin{smallmatrix} v \\ u \end{smallmatrix}\right] = \hat{C}_1 x + \hat{D}_{11} w + \hat{D}_{12} \left[\begin{smallmatrix} f \\ r \end{smallmatrix}\right] \tag{13}$$

$$z = \hat{C} x + \hat{D} w + \hat{D}_{22} \left[\begin{smallmatrix} f \\ r \end{smallmatrix}\right] \tag{14}$$

where $x := [x_G^T,\ \psi^T]^T \in \mathbb{R}^{n_G + n_\psi}$ is the extended state. $\hat{A}$, $\hat{B}$, $\hat{C}$, and $\hat{D}$ are defined in Equation 6. The remaining state matrices are defined as:

$$\hat{B}_2 := \left[\begin{smallmatrix} 0 & B \\ B_{\psi 1} & B_{\psi 1} D \end{smallmatrix}\right],\ \hat{C}_1 := \left[\begin{smallmatrix} C & 0 \\ 0 & 0 \end{smallmatrix}\right] \tag{15}$$

$$\hat{D}_{11} := \left[\begin{smallmatrix} D \\ I \end{smallmatrix}\right],\ \hat{D}_{12} := \left[\begin{smallmatrix} I & D \\ 0 & I \end{smallmatrix}\right],\ \hat{D}_{22} := \left[\begin{smallmatrix} D_{\psi 1} & D_{\psi 1} D \end{smallmatrix}\right] \tag{16}$$

The next theorem provides a stability condition using IQCs and a standard dissipation argument.

Theorem 3: Let $G \in \mathbb{RH}_\infty^{n \times m}$ and $\Delta : L_{2e}^n \to L_{2e}^m$ be a bounded causal operator. Assume that:

1) the interconnection of G and Δ is well-posed.

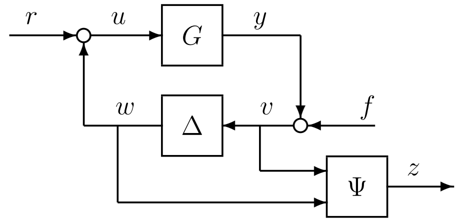

Fig. 2. Analysis Interconnection Structure

2) Δ satisfies the IQC defined by Π and (Ψ, M) is a hard factorization of Π.
3) there exists $P \geq 0$ and a scalar $\gamma > 0$ such that $V(x) := x^T P x$ satisfies

$$z^T M z + \nabla V \cdot F(x, w, f, r) < \gamma \begin{bmatrix} r \\ f \end{bmatrix}^T \begin{bmatrix} r \\ f \end{bmatrix} - \frac{1}{\gamma} \begin{bmatrix} u \\ v \end{bmatrix}^T \begin{bmatrix} u \\ v \end{bmatrix} \tag{17}$$

for all nontrivial $(x, w, r, f) \in \mathbb{R}^{n_G + n_\psi} \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^n$ where u, v, z are defined by Equations 13 and 14.

Then the feedback interconnection of G and Δ is stable.

Proof: All solutions of the interconnection satisfy the dynamics in Equations 12-14. From well-posedness, the dissipation inequality (Equation 17) can be integrated from $t = 0$ to $t = T$ with the initial condition $x(0) = 0$. It then follows from the IQC (Equation 11) and $P \geq 0$ that:

$$\frac{1}{\gamma} \int_0^T \begin{bmatrix} u(t) \\ v(t) \end{bmatrix}^T \begin{bmatrix} u(t) \\ v(t) \end{bmatrix} dt \leq \gamma \int_0^T \begin{bmatrix} r(t) \\ f(t) \end{bmatrix}^T \begin{bmatrix} r(t) \\ f(t) \end{bmatrix} dt \tag{18}$$

Hence the feedback interconnection of G and Δ is stable. ∎

Equation 17 is an algebraic inequality on the variables (x, w, f, r). This constraint, when evaluated along solutions of the extended system, represents the differential form for a dissipation inequality satisfied by the extended system. The next lemma shows that the dissipation inequality in Equation 17 is also equivalent to the KYP LMI.

Lemma 1: There exists $P \geq 0$ satisfying the dissipation inequality (Equation 17) for some $\gamma > 0$ if and only if there exists $P \geq 0$ satisfying the KYP LMI (Equation 7).

Proof: The dissipation inequality (Equation 17) can be expressed as a quadratic constraint on (x, w, f, r):

$$\begin{bmatrix} x \\ w \\ f \\ r \end{bmatrix}^T \left[\begin{array}{c|c} Q(P, \gamma) & S(P, \gamma) \\ \hline S(P, \gamma)^T & R(\gamma) \end{array} \right] \begin{bmatrix} x \\ w \\ f \\ r \end{bmatrix} < 0 \tag{19}$$

where

$$Q(P,\gamma) := \left[\begin{smallmatrix} \hat{A}^T P + P\hat{A} & P\hat{B} \\ \hat{B}^T P & 0 \end{smallmatrix}\right] + \left[\begin{smallmatrix} \hat{C}^T \\ \hat{D}^T \end{smallmatrix}\right] M \left[\begin{smallmatrix} \hat{C} & \hat{D} \end{smallmatrix}\right] + \tfrac{1}{\gamma} \left[\begin{smallmatrix} \hat{C}_1^T \\ \hat{D}_{11}^T \end{smallmatrix}\right] \left[\begin{smallmatrix} \hat{C}_1 & \hat{D}_{11} \end{smallmatrix}\right]$$

$$S(P,\gamma) := \left[\begin{smallmatrix} P\hat{B}_2 \\ 0 \end{smallmatrix}\right] + \left[\begin{smallmatrix} \hat{C}^T \\ \hat{D}^T \end{smallmatrix}\right] M \hat{D}_{22} + \frac{1}{\gamma} \left[\begin{smallmatrix} \hat{C}_1^T \\ \hat{D}_{11}^T \end{smallmatrix}\right] \hat{D}_{12}$$

$$R(\gamma) := \frac{1}{\gamma} \hat{D}_{12}^T \hat{D}_{12} + \hat{D}_{22}^T M \hat{D}_{22} - \gamma I$$

By Schur complements [3], P satisfies the KYP LMI if and only if it satisfies Equation 19 for sufficiently large γ. ■

III. Main Result

Section II summarizes two IQC stability theorems. Theorem 1 involves a frequency domain condition with a multiplier Π. Theorem 3 involves a dissipation inequality with a multiplier (Ψ, M). The multipliers are connected by a non-unique factorization $\Pi = \Psi^{\sim} M \Psi$. Theorems 1 and 3 are clearly related as the Condition 3 in each theorem is equivalent to the same KYP LMI. Two important properties are required for the dissipation inequality approach:

1) (Ψ, M) must be a "hard" factorization to ensure the time-domain constraint holds over all finite intervals.
2) The solution to the KYP LMI must satisfy $P \geq 0$. This is not required for the frequency domain test.

The main result is: For a class of multipliers, Π has a factorization (Ψ, M) that is both "hard" and such that any feasible solution of the KYP LMI satisfies $P \geq 0$.

A. Condition for Hard Factorization

Define the following cost functional J on $v \in L_2^n[0,\infty)$, $w \in L_2^m[0,\infty)$, and $\psi_0 \in \mathbb{R}^{n_\psi}$:

$$J(v, w, \psi_0) := \int_0^\infty z(t)^T M z(t)\, dt \tag{20}$$

subject to:

$$\dot{\psi}(t) = A_\psi \psi(t) + B_{\psi 1} v(t) + B_{\psi 2} w(t), \ \psi(0) = \psi_0$$

$$z(t) = C_\psi \psi(t) + D_{\psi 1} v(t) + D_{\psi 2} w(t)$$

Also define the upper value $\bar{J}$ as

$$\bar{J}(\psi_0) := \inf_{v \in L_2^n[0,\infty)} \sup_{w \in L_2^m[0,\infty)} J(v, w, \psi_0) \tag{21}$$

Lemma 2: Let $\Pi \in \mathbb{RL}_\infty^{(n+m)\times(n+m)}$ be a multiplier and (Ψ, M) any factorization of Π with Ψ stable. Assume $\Delta : L_{2e}^n[0,\infty) \to L_{2e}^m[0,\infty)$ is a casual, bounded operator that satisfies the IQC defined by Π. Then for all $T \geq 0$, $v \in L_{2e}^m[0,\infty)$ and $w = \Delta(v)$, the output of Ψ satisfies:

$$\int_0^T z(t)^T M z(t)\, dt \geq -\bar{J}(\psi_T) \tag{22}$$

where ψ_T denotes the state of Ψ at time T when driven by inputs (v, w) with initial condition $\psi(0) = 0$.

Proof: By assumption, Δ satisfies the "soft" IQC defined by (Ψ, M). For any $T \geq 0$, the (infinite time) integral constraint can be re-arranged as:

$$\int_0^T z(t)^T M z(t)\, dt \geq -\int_T^\infty z(t)^T M z(t)\, dt \tag{23}$$

This provides a simple lower bound on $\int_0^T z(t)^T M z(t)\, dt$. A more useful lower bound can be obtained by invoking the causality of Δ. Specifically, let $\tilde{v} \in L_2^n[0,\infty)$ be any signal that matches v up to time T, i.e. $(\tilde{v})_T = (v)_T$. Define $\tilde{w} := \Delta(\tilde{v})$ and let $\tilde{z} := \Psi\left[\begin{smallmatrix}\tilde{v}\\ \tilde{w}\end{smallmatrix}\right]$ denote the corresponding output of Ψ from $\psi(0) = 0$. By causality of Δ and Ψ, if $(\tilde{v})_T = (v)_T$ then $(\tilde{w})_T = (w)_T$ and $(\tilde{z})_T = (z)_T$. Hence $\int_0^T \tilde{z}(t)^T M \tilde{z}(t)\, dt = \int_0^T z(t)^T M z(t)\, dt$. Moreover, $\int_0^\infty \tilde{z}(t)^T M \tilde{z}(t) \geq 0$ because the (infinite-horizon) IQC holds for all input/output pairs of Δ. Thus any $\tilde{v}$ satisfying $(\tilde{v})_T = (v)_T$ can be used to lower bound the integral $\int_0^T z(t)^T M z(t)\, dt$ for v. Maximizing over feasible $\tilde{v}$ yields the following lower bound on $\int_0^T z(t)^T M z(t)\, dt$:

$$\sup_{\tilde{v} \in L_2^n[0,\infty)} -\int_T^\infty \tilde{z}(t)^T M \tilde{z}(t)\, dt \tag{24}$$

$$\text{subject to: } (\tilde{v})_T = (v)_T,\ \tilde{w} = \Delta(\tilde{v}),\ \tilde{z} = \Psi\left[\begin{smallmatrix}\tilde{v}\\ \tilde{w}\end{smallmatrix}\right]$$

The integral cost in the optimization only depends on the state of Ψ at $t = T$ and the signals $(\tilde{v}, \tilde{w})$ for $t \geq T$. Note that $\tilde{\psi}(T)$ is the same for all feasible $\tilde{v}$. In particular, $((\tilde{v})_T, (\tilde{w})_T) = ((v)_T, (w)_T)$ for any feasible $\tilde{v}$. Hence Ψ evolves from $\tilde{\psi}(0) = 0$ to the state $\tilde{\psi}(T) = \psi_T$ given by the inputs (v, w). Thus the lower bound can be expressed as:

$$\sup_{\tilde{v}_f \in L_2^n[T,\infty)} -\int_T^\infty \tilde{z}(t)^T M \tilde{z}(t)\, dt \tag{25}$$

$$\text{subject to: } \tilde{w} = \Delta(\tilde{v}) \text{ where } \tilde{v}(t) = \begin{cases} v(t) & t \leq T \\ \tilde{v}_f(t) & t > T \end{cases},$$

$$\dot{\tilde{\psi}}(t) = A_\psi \tilde{\psi}(t) + B_{\psi 1}\tilde{v}(t) + B_{\psi 2}\tilde{w}(t), \quad \tilde{\psi}(T) = \psi_T$$

$$\tilde{z}(t) = C_\psi \tilde{\psi}(t) + D_{\psi 1}\tilde{v}(t) + D_{\psi 2}\tilde{w}(t)$$

In this bound, the relation $\tilde{w} = \Delta(\tilde{v})$ is the only constraint that connects the past ($t < T$) to the future ($t > T$). This connection is removed by replacing the true future output of Δ with a minimization over all possible output signals. This leads to the following lower bound on $\int_0^T z(t)^T M z(t)\, dt$:

$$\sup_{\tilde{v} \in L_2^n[T,\infty)} \inf_{\tilde{w} \in L_2^m[T,\infty)} -\int_T^\infty \tilde{z}(t)^T M \tilde{z}(t)\, dt \tag{26}$$

subject to:

$$\dot{\tilde{\psi}}(t) = A_\psi \tilde{\psi}(t) + B_{\psi 1}\tilde{v}(t) + B_{\psi 2}\tilde{w}(t), \quad \tilde{\psi}(T) = \psi_T$$

$$\tilde{z}(t) = C_\psi \tilde{\psi}(t) + D_{\psi 1}\tilde{v}(t) + D_{\psi 2}\tilde{w}(t)$$

This removes the dependence on Δ but introduces some conservatism, i.e. the bound in Equation 26 is no greater than the bound in Equation 25. The time-invariance of Ψ is used to equivalently write Equation 26 as $-\bar{J}(\psi_T)$. ■

B. Condition for Positive Semidefinite KYP Solution

Define the lower value $\underline{J}$ as

$$\underline{J}(\psi_0) := \sup_{w \in L_2^m[0,\infty)} \inf_{v \in L_2^n[0,\infty)} J(v, w, \psi_0) \tag{27}$$

Lemma 3: Let $\Pi \in \mathbb{RL}_\infty^{(n+m)\times(n+m)}$ be a multiplier and (Ψ, M) any factorization of Π with Ψ stable. Given $G \in \mathbb{RH}_\infty^{n\times m}$, assume the corresponding KYP LMI (Equation 7) is feasible with state matrices $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ defined in Equation 6. Let $P = P^T$ denote a solution to the KYP LMI. Then $V(x_0) := x_0^T P x_0 \geq \underline{J}(\psi_0)$ for all $x_0 := \left[\begin{smallmatrix} x_{G,0}^T, \psi_0^T \end{smallmatrix} \right]^T \in \mathbb{R}^{n_G + n_\psi}$.

Proof: $\underline{J}(\psi_0)$ involves a max over w followed by a min over v. Hence the choice of v may depend on w. Choose v to be the output of G generated by w with some initial condition $x_{G,0}$. This specific choice of v yields a value that is no lower than the infimum over all possible $v \in L_2$. Hence $\underline{J}(\psi_0) \leq V^*(x_0)$ where V^* is defined as:

$$V^*(x_0) := \sup_{w \in L_2^m[0,\infty)} \int_0^\infty z(t)^T M z(t)\, dt \tag{28}$$

subject to:

$$\dot{x} = \hat{A}x + \hat{B}w, \quad x(0) = x_0$$

$$z = \hat{C}x + \hat{D}w$$

The proof is completed by showing $V(x_0) \geq V^*(x_0)$ for all x_0. This follows from Theorems 2 and 3 in [23] and hence the proof is only sketched. Let $x(t)$, $z(t)$ be the resulting solutions of $\Psi \left[\begin{smallmatrix} G \\ I \end{smallmatrix} \right]$ for a given input $w \in L_2^m[0,\infty)$ and initial condition x_0. Multiply the KYP LMI on the left/right by $\left[\begin{smallmatrix} x(t) \\ w(t) \end{smallmatrix}\right]^T$ and $\left[\begin{smallmatrix} x(t) \\ w(t) \end{smallmatrix}\right]$ to show $\dot{V}(x(t)) + z(t)^T M z(t) \leq 0$. Integrate this inequality from $t=0$ to $t=T$ to obtain

$$V(x(T)) + \int_0^T z(t)^T M z(t)\, dt \leq V(x_0) \tag{29}$$

$\lim_{T\to\infty} x(T) = 0$ for any $w \in L_2^m[0,\infty)$ because $\hat{A}$ is Hurwitz. Maximizing the left side of Equation 29 over $w \in L_2^m[0,\infty)$ for $T=\infty$ thus yields $V(x_0) \geq V^*(x_0)$. ■

C. Dissipation Inequalities with J-Spectral Factorizations

By Lemma 2, (Ψ, M) is a hard factorization if $\bar{J}(\psi) \leq 0\ \forall\psi$. By Lemma 3, all KYP LMI solutions satisfy $P \geq 0$ if $\underline{J}(\psi) \geq 0\ \forall\psi$. Moreover, weak duality implies that the lower and upper values satisfy $\underline{J}(\psi) \leq \bar{J}(\psi)$. Hence a factorization $\Pi = \Psi^\sim M \Psi$ that is both "hard" and ensures $P \geq 0$ for all KYP LMI solutions must have $0 \leq \underline{J}(\psi) \leq \bar{J}(\psi) \leq 0$. In other words, for such a factorization the lower and upper values must satisfy $\underline{J}(\psi) = \bar{J}(\psi) = 0$. The following special factorization plays a key role in the main result below.

Definition 4: (Ψ, M) is called a $J_{n,m}$-spectral factor of $\Pi = \Pi^\sim \in \mathbb{RL}_\infty^{(n+m)\times(n+m)}$ if $\Pi = \Psi^\sim M \Psi$, $M = \left[\begin{smallmatrix} I_n & 0 \\ 0 & -I_m \end{smallmatrix}\right]$, and $\Psi, \Psi^{-1} \in \mathbb{RH}_\infty^{(n+m)\times(n+m)}$.

Lemma 4 in the appendix provides sufficient conditions for the existence of a J-spectral factor. The main result can now be stated.

Theorem 4: Let $\Pi = \Pi^\sim \in \mathbb{RL}_\infty^{(n+m)\times(n+m)}$ and partition as $\left[\begin{smallmatrix} \Pi_{11} & \Pi_{21}^\sim \\ \Pi_{21} & \Pi_{22} \end{smallmatrix}\right]$ where $\Pi_{11} \in \mathbb{RL}_\infty^{n\times n}$ and $\Pi_{22} \in \mathbb{RL}_\infty^{m\times m}$. If $\Pi_{11}(j\omega) > 0$ and $\Pi_{22}(j\omega) < 0\ \forall\omega \in \mathbb{R}\cup\{\infty\}$, then

1) Π has a $J_{n,m}$-spectral factorization (Ψ, M).
2) The $J_{n,m}$-spectral factorization (Ψ, M) is a hard factorization of Π.
3) For $G \in \mathbb{RH}_\infty^{n\times m}$, let $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ denote the state-space realization of $\Psi \left[\begin{smallmatrix} G \\ I \end{smallmatrix}\right]$ in Equation 6. All solutions $P = P^T$ to the KYP LMI (Equation 7) satisfy $P \geq 0$.

Proof: Statement 1) follows from Lemma 4 in the appendix. Statements 2) and 3) follow from known results on linear quadratic games (Lemma 5 in the appendix). Specifically, let $(A_\psi, B_\psi, C_\psi, D_\psi)$ be a realization for the J-spectral factor Ψ. Define $Q := C_\psi^T M C_\psi$, $R := D_\psi^T M D_\psi$, and $S := C_\psi^T M D_\psi$. Then $X_0 = 0$ is a solution of $ARE(A_\psi, B_\psi, Q, R, S)$ and this solution gives $A_\psi - B_\psi R^{-1}(X_0 B_\psi + S)^T = A_\psi - B_\psi D_\psi^{-1} C_\psi$. The matrix $A_\psi - B_\psi D_\psi^{-1} C_\psi$ is Hurwitz because Ψ^{-1} is stable and hence $X_0 = 0$ is the stabilizing solution of the ARE. Next, $\Pi_{11}(j\omega) > 0$ implies the ARE in Condition 2 of Lemma 5 has a stabilizing solution. This follows from the spectral factorization theorem [27], [28]. Similarly, $\Pi_{22}(j\omega) < 0$ implies that the ARE in Condition 3 of Lemma 5 has a stabilizing solution. Finally, Lemma 5 implies $\underline{J}(\psi) = \bar{J}(\psi) = \psi^T X_0 \psi = 0\ \forall \psi$. Statements 2) and 3) now follow from Lemmas 2 and 3. ∎

Factorization conditions in [4], [8] connect classical passivity multipliers and their IQC counterparts. Theorem 4 provides a connection between classical passivity multipliers and dissipation theory. Specifically, let H be a classical passivity multiplier proving stability for the interconnection of G and a finite-gain system Δ. It follows by a simple perturbation argument, e.g. as in [4], that stability can be demonstrated with the (frequency-domain) IQC test using $\Pi = \begin{bmatrix} \epsilon I & H^* \\ H & \frac{-\epsilon}{\|\Delta\|^2} I \end{bmatrix}$. The conditions in Theorem 4 hold for this multiplier and thus a J-spectral factorization of Π exists. Moreover, there is a dissipation inequality that proves stability of the feedback interconnection. In other words, if stability can be demonstrated by a classical passivity multiplier then it can also be demonstrated via a dissipation inequality.

IV. CONCLUSIONS

This paper explored the connections between frequency domain and time domain IQC stability theorems. The approaches are related by a (non-unique) factorization of the frequency domain multiplier. It was shown that if a J-spectral factorization is used then the approaches are equivalent except for minor differences in technical assumptions. Thus the dissipation theory, with an appropriate IQC factorization, can be used with no additional conservatism.

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VI. J-Spectral Factorizations

Existence conditions for a J-spectral factor of Π are provided by the canonical factorization theorem [2]. The conditions involve the modal subspaces for Π and Π^{-1}. This subspace property is connected to a related Riccati equation. Chapter 7 of [7] summarizes these results. Existence conditions for a J-spectral factor can also be specified using the notion of an equalizing vector as defined in [16].

Specifically, $\hat{u} \in \mathbb{H}_2$ is an equalizing vector of Π if $\hat{u}$ is non-zero and $\Pi\hat{u} \in \mathbb{H}_2^{\perp}$. The next lemma provides an alternative existence condition in terms of definiteness properties on Π.

Lemma 4: Let $\Pi = \Pi^{\sim} \in \mathbb{RL}_{\infty}^{(n+m)\times(n+m)}$ be partitioned as $\begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12}^{\sim} & \Pi_{22} \end{bmatrix}$ where $\Pi_{11} \in \mathbb{RL}_{\infty}^{n\times n}$ and $\Pi_{22} \in \mathbb{RL}_{\infty}^{m\times m}$. If $\Pi_{11}(j\omega) > 0$ and $\Pi_{22}(j\omega) < 0$ $\forall\omega \in \mathbb{R} \cup \{\infty\}$, then

1) There exists real matrices A, B, Q, S, R of compatible dimensions with A Hurwitz, $Q = Q^T$, and $R = R^T$ such that Π can be expressed as

$$\Pi(s) = \begin{bmatrix} B^T(-sI-A^T)^{-1} & I \end{bmatrix} \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} (sI-A)^{-1}B \\ I \end{bmatrix}$$

2) Π has no poles and zeros on the imaginary axis including ∞ and Π has no equalizing vectors.
3) R is nonsingular and there exists a unique stabilizing solution $X = X^T$ to $ARE(A, B, Q, R, S)$.
4) Π has a $J_{n,m}$-spectral factorization (Ψ, M). Moreover, (Ψ, M) is a $J_{n,m}$-spectral factor of Π if and only if Ψ has a state-space realization $\left(A, B, MW^{-*}\left(B^T X + S^T\right), W\right)$ where W is a solution of $R = W^T M W$.

Proof: Conclusion 1) follows from the results in Section 7.3 of [7]. Next, the block-determinant formula yields

$$\det\left(\Pi(j\omega)\right) = \det\left(\Pi_{22}(j\omega)\right) \cdot \qquad (30)$$
$$\det\left(\Pi_{11}(j\omega) - \Pi_{12}(j\omega)\Pi_{22}^{-1}(j\omega)\Pi_{12}^{*}(j\omega)\right)$$

Hence $\Pi_{11}(j\omega) > 0$ and $\Pi_{22}(j\omega) < 0$ implies $\det\left(\Pi(j\omega)\right) \neq 0$ $\forall\omega \in \mathbb{R} \cup \{\infty\}$. Thus Π is nonsingular and has no zeros on the imaginary axis. Π is also bounded on the imaginary axis and hence it has no poles there.

Finally, assume that Π has an equalizing vector, i.e. assume there exists a nonzero $\hat{u} \in \mathbb{H}_2$ such that $\hat{y} := \Pi\hat{u} \in \mathbb{H}_2^{\perp}$. By the spectral factorization theorem [27], [28], $\Pi_{11} > 0$ and $-\Pi_{22} > 0$ have spectral factors denoted by G_1 and G_2, respectively. Define $\bar{u} := \begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix} \hat{u}$ and $\bar{y} := \begin{bmatrix} G_1^{\sim} & 0 \\ 0 & G_2^{\sim} \end{bmatrix}^{-1} \hat{y}$. The spectral factors G_1 and G_2 are stable with stable inverses and hence $\bar{u} \in \mathbb{H}_2$ and $\bar{y} \in \mathbb{H}_2^{\perp}$. With these definitions, the relation $\hat{y} := \Pi\hat{u} \in \mathbb{H}_2^{\perp}$ is transformed to two coupled equations consistent with the partitioning of Π:

$$\bar{y}_1 = \bar{u}_1 + X\bar{u}_2 \qquad (31)$$

$$\bar{y}_2 = X^{\sim}\bar{u}_1 - \bar{u}_2 \qquad (32)$$

where $X := (G_1^{\sim})^{-1}\Pi_{12}G_2^{-1} \in \mathbb{RL}_{\infty}^{n\times m}$. Let P_+ and P_- denote the projection operators to $\mathbb{H}_2$ and $\mathbb{H}_2^{\perp}$, respectively. By Liouville's theorem, Equation 31 implies that $\bar{u}_1 = -P_+(X\bar{u}_2)$. Similarly, Equation 32 implies $\bar{u}_2 = P_+(X^{\sim}\bar{u}_1)$. Thus $\bar{u}_2$ must satisfy the following equation:

$$\bar{u}_2 = -P_+\left(X^{\sim}P_+(X\bar{u}_2)\right) \tag{33}$$

Take the inner product of $\bar{u}_2$ with itself to obtain:

$$0 \leq \langle \bar{u}_2, \bar{u}_2 \rangle = -\langle \bar{u}_2, P_+\left(X^{\sim}P_+(X\bar{u}_2)\right)\rangle \tag{34}$$

The projection operator P_+ is self-adjoint and, moreover, $P_+\bar{u}_2 = \bar{u}_2$ because $\bar{u}_2 \in \mathbb{H}_2$. Hence the inequality in Equation 34 yields $\langle X\bar{u}_2, P_+(X\bar{u}_2)\rangle \leq 0$. Use $P_+ = (P_+)^2$ to express this inequality as:

$$0 \geq \langle X\bar{u}_2, P_+(X\bar{u}_2)\rangle = \langle P_+(X\bar{u}_2), P_+(X\bar{u}_2)\rangle \tag{35}$$

This implies that $P_+(X\bar{u}_2) = 0$ and hence both $\bar{u}_1 = 0$ and $\bar{u}_2 = 0$. This contradicts the assumption that Π has a (non-zero) equalizing vector. Conclusion 2) follows.

Finally conclusion 3) as well as the existence of a J-spectral factor both follow from Theorem 2.4 in [16]. Q and R are not sign definite in general but the stabilizing solution X can still be computed by standard Hamiltonian methods, see Chapter 2 of [5]. The specific conclusion that Π has a $J_{n,m}$-spectral factorization follows from the inertia of the matrix R. In particular, $R = \Pi(j\infty)$ and hence this matrix is symmetric with $R_{11} > 0$ and $R_{22} < 0$. The Courant-Fischer minimax theorem [10] thus implies that R has n positive eigenvalues and m negative eigenvalues. ∎

A. Linear Quadratic Differential Games

This section briefly summarizes one technical result on linear quadratic games related to the cost functional J, upper value $\bar{J}$, and lower value $\underline{J}$ as defined in Equations 20, 21, and 27, respectively. The cost functional J defines a two-player, non-cooperative game with player 1 choosing input v to minimize J and player 2 choosing input w to maximize J. The dynamic game with J includes a quadratic integral cost and LTI dynamics. There is an extensive literature on LQ differential games and the most relevant work is [5], [6].

Lemma 5: Assume A_ψ is Hurwitz. Define $Q := C_\psi^T M C_\psi$, $R := D_\psi^T M D_\psi$, and $S := C_\psi^T M D_\psi$ where $B_\psi = \begin{bmatrix} B_{\psi 1} & B_{\psi 2} \end{bmatrix}$ and $D_\psi = \begin{bmatrix} D_{\psi 1} & D_{\psi 2} \end{bmatrix}$. In addition, partition R and S as $\begin{bmatrix} R_{11} & R_{12} \\ R_{12}^T & R_{22} \end{bmatrix}$ and $\begin{bmatrix} S_1 & S_2 \end{bmatrix}$ compatible with the dimensions of v and w. Finally, assume the following conditions hold:

1) R is nonsingular and there exists a stabilizing solution $X_0 = X_0^T$ to $ARE(A_\psi, B_\psi, Q, R, S)$.
2) $R_{11} := D_{\psi 1}^T M D_{\psi 1} > 0$ and there exists a stabilizing solution $X_1 = X_1^T$ to $ARE(A_\psi, B_{\psi 1}, Q, R_{11}, S_1)$.
3) $R_{22} := D_{\psi 2}^T M D_{\psi 2} < 0$ and there exists a stabilizing solution $X_2 = X_2^T$ to $ARE(A_\psi, B_{\psi 2}, Q, R_{22}, S_2)$.

Then $\underline{J}(\psi) = \bar{J}(\psi) = \psi^T X_0 \psi$ for all ψ.

Proof: This lemma is a generalization of Proposition 7.20 in [5] to include the cross terms $D_{\psi 1}^T M D_{\psi 2}$, $C^T M D_{\psi 1}$, and $C^T M D_{\psi 2}$. It can be proven using existing results for non-zero sum games in [6]. The proof is similar to the proof of Proposition 7.20 in [5] and details are omitted. ■