INTERNATIONAL JOURNAL OF ROBUST AND NONLINEAR CONTROL
Int. J. Robust. Nonlinear Control 0000; **00**:1–22
Published online in Wiley InterScience (www.interscience.wiley.com). DOI: 10.1002/rnc

Robustness Analysis of Uncertain Discrete-Time Systems with Dissipation Inequalities and Integral Quadratic Constraints

Bin Hu*, Márcio J. Lacerda, and Peter Seiler

Aerospace Engineering and Mechanics Department, University of Minnesota

SUMMARY

This paper presents a connection between dissipation inequalities and integral quadratic constraints (IQCs) for robustness analysis of uncertain discrete-time systems. Traditional IQC results derived from homotopy methods emphasize an operator-theoretic input-output viewpoint. In contrast, the dissipativity-based IQC approach explicitly incorporates the internal states of the uncertain system, thus providing a more direct procedure to analyze uniform stability with non-zero initial states. The standard dissipation inequality requires a non-negative definite storage function and "hard" IQCs. The term "hard" means that the IQCs must hold over all finite time horizons. This paper presents a modified dissipation inequality that requires neither non-negative definite storage functions nor hard IQCs. This approach leads to linear matrix inequality conditions that can provide less conservative results in terms of robustness analysis. The proof relies on a key J-spectral factorization lemma for IQC multipliers. A simple numerical example is provided to demonstrate the utility of the modified dissipation inequality.

Received ...

1. INTRODUCTION

This paper considers robustness analysis for an uncertain discrete-time system. The uncertain system is given by an interconnection of a discrete-time linear system and a "troublesome" perturbation. The input/output behavior of the perturbation is described by integral quadratic constraints (IQCs) [1]. In particular, the input/output signals of the perturbation are assumed to satisfy a frequency domain constraint specified by a multiplier $\Pi(z)$. The analysis objective is to assess the worst-case gain, robust asymptotic stability, and robust uniform stability of the uncertain system. Such analysis problems are standard in robust control and arise in many applications [2, 3]. More recently, optimization algorithms have been studied as uncertain linear systems whose perturbations are described by discrete-time IQCs [4]. The IQC framework builds on a long history of classical multiplier results, e.g. Zames-Falb multipliers [5, 6]. The original IQC results in [1] were given for continuous-time systems using homotopy arguments. This yields frequency domain analysis conditions which can be adapted with minor changes for discrete-time, e.g. as in [7].

This paper focuses on an alternative time-domain derivation for discrete-time IQC analysis using dissipativity theory [8, 9]. The time-domain approach involves "hard" IQCs that are specified by a stable filter $\Psi(z)$ and a symmetric matrix M. The pair (Ψ, M) defines a time domain integral constraint on the uncertainty that holds over all finite time intervals. The main result (Theorem 2 in Section 4) is a modified dissipation inequality that differs in two respects from a standard

*Correspondence to: Bin Hu, University of Minnesota, 107 Akerman Hall, 110 Union St SE, Minneapolis, Email:huxxx221@umn.edu

dissipation/IQC result. In particular, it allows for non-negative storage functions and so-called "soft" IQCs where the time-domain constraint only holds on infinite time intervals. This can lead to less conservative analysis results as demonstrated by an example in Section 5. The modified dissipation inequality relies on a technical J-spectral factorization [10, 11] result (Lemma 6 in Section 3). It is possible to factorize frequency domain multipliers Π as $\Psi^{\sim} M \Psi$. This factorization is not unique and several important properties of J-spectral factorizations are proved using game-theoretic interpretations. The J-spectral factorization result (Lemma 6) is an important technical result on its own and has other potential applications, e.g. formulating topological separation theorems [12].

This paper complements several existing results in the literature. First, this paper provides a discrete-time counterpart to the continuous-time results in [13–16]. Moreover, Section 3 contains intermediate results regarding discrete-time IQC factorizations and a related open-loop linear quadratic (LQ) difference game. These parallel existing continuous-time results for J-spectral factorizations [17] and open loop LQ differential games [18,19]. The generalization to discrete-time is not immediate since descriptor systems are needed to handle non-proper multipliers that appear in some proofs. Similar discrete-time technical results on factorizations and LQ games are provided in [20] and [11] using operator theoretic methods. This paper provides alternative linear algebra proofs for completeness. In particular, the minimax theorems in [20] were used to construct hard IQCs for both discrete-time and continuous-time systems. This paper extends the game theoretic results to not only construct hard IQCs but also to show that the non-negative constraint can be dropped on the storage function in the dissipation inequality framework.

The benefit of the time-domain dissipation theory is that it enables generalization to cases where the known system in the feedback connection is not necessarily linear time-invariant (LTI). For example, the approach enables the analysis of uncertain linear parameter varying (LPV) systems or uncertain nonlinear systems. The current paper will present discrete-time derivations assuming the nominal system is LTI. However the extension to uncertain LPV and uncertain nonlinear systems follows along the lines of the continuous-time results in [13, 16, 21]. The standard IQC homotopy theory developed for both continuous and discrete-time systems [1, 22–25] can also be generalized for systems which do not have frequency domain interpretations [26]. The homotopy approach emphasizes input-output properties while internal states are incorporated more transparently in the dissipativity approach. In some cases, directly handling internal states can be potentially beneficial, e.g. in the convergence rate analysis of optimization methods [4]. In other cases, it is useful to mask the internal states and focus on input-output stability, e.g. in consensus and synchronization problems [27]. In general, the two approaches are complementary and both are useful.

The rest of the paper is organized as follows. Section 2 formulates the discrete-time analysis problem and summarizes the standard dissipation inequality approach using IQCs. Section 4 presents the discrete-time modified dissipation inequality result. The proof of this main result relies on one main technical J-spectral factorization result presented in Section 3. Several lemmas regarding discrete-time IQC factorizations and a related open-loop linear quadratic difference game are also collected in Section 3 to support the proof of the main J-spectral factorization result. A simple numerical example is given in Section 5.

2. PRELIMINARIES

2.1. Notation

The notation is standard. $\mathbb{RL}_\infty$ denotes the set of rational functions with real coefficients that have no poles on the unit circle. $\mathbb{RH}_\infty$ is the subset of functions in $\mathbb{RL}_\infty$ that are analytic outside the unit disk of the complex plane. The para-Hermitian conjugate of $\Pi \in \mathbb{RL}_\infty^{m\times n}$, denoted as $\Pi^{\sim}$, is defined by $\Pi^{\sim}(z) := \Pi^T(z^{-1})$. Hence $\Pi^{\sim}(e^{j\omega}) = \Pi^*(e^{j\omega})$ holds on the unit circle. For discrete-time systems $\mathbb{RL}_\infty$ contains improper functions, e.g. polynomials in z, while $\mathbb{RH}_\infty$ contains only proper functions. Thus functions in $\mathbb{RH}_\infty$ have a standard, state space representation but descriptor systems are required, in general, to represent functions in $\mathbb{RL}_\infty$ [28]. The use of descriptor systems is limited to one technical result (Lemma 7 in the appendix).

Consider a (real) sequence $u := (u(0), u(1), \ldots)$ where $u(k) \in \mathbb{R}^n$ for all k. This sequence is said to be in ℓ_2^n if $\sum_{k=0}^{\infty} \|u(k)\|^2 < \infty$ where $\|u(k)\|$ denotes the standard (vector) 2-norm of $u(k)$. In addition, the 2-norm for $u \in \ell_2^n$ is defined as $\|u\|^2 := \sum_{k=0}^{\infty} \|u(k)\|^2$. Then $\|v\|$ denotes the ℓ_2 norm of the signal and $\|v(k)\|$ denotes the Euclidean norm of the vector evaluated at time k.

An inner product on ℓ_2^n is defined as $\langle u_1, u_2 \rangle = \sum_{k=0}^{\infty} u_1^T(k) u_2(k)$ for any $u_1, u_2 \in \ell_2^n$. The truncation operator P_N maps a sequence u to $P_N u$, where

$$(P_N u)(k) := \begin{cases} u(k) & \text{for } k \leq N \\ 0 & \text{for } k > N \end{cases} \tag{1}$$

For simplicity, $P_N u$ is occasionally abbreviated as $(u)_N$. The extended space, denoted ℓ_{2e}^n, is the set of sequences u such that $P_N u \in \ell_2^n$ for all $N \geq 0$.[†] Finally, $DARE(A, B, Q, R, S)$ denotes the following discrete-time Algebraic Riccati Equation (DARE)

$$A^T X A - X - (A^T X B + S)(R + B^T X B)^{-1}(A^T X B + S)^T + Q = 0 \tag{2}$$

The stabilizing solution $X = X^T$, if it exists, is such that $(R + B^T X B)$ is nonsingular. In addition, $A - BK$ is a Schur stable matrix where $K := (R + B^T X B)^{-1}(A^T X B + S)^T$ is the stabilizing DARE gain.

2.2. *Problem Statement*

This paper considers the robustness of uncertain discrete-time systems. Consider the interconnection in Figure 1, denoted by $F_u(G, \Delta)$. This uncertain system is described by the interconnection of a nominal discrete-time LTI system G and an uncertain perturbation Δ. The LTI system G is described by the following state-space model:

$$\begin{aligned} x_G(k+1) &= A_G\, x_G(k) + B_{G1}\, w(k) + B_{G2}\, d(k) \\ v(k) &= C_{G1}\, x_G(k) + D_{G11}\, w(k) + D_{G12}\, d(k) \\ e(k) &= C_{G2}\, x_G(k) + D_{G21}\, w(k) + D_{G22}\, d(k) \end{aligned} \tag{3}$$

where $x_G \in \mathbb{R}^{n_G}$ is the state. The inputs are $w \in \mathbb{R}^{n_w}$ and $d \in \mathbb{R}^{n_d}$ while $v \in \mathbb{R}^{n_v}$ and $e \in \mathbb{R}^{n_e}$ are outputs. The state-space matrices of G have dimensions compatible with these signals, e.g. $A_G \in \mathbb{R}^{n_G \times n_G}$.

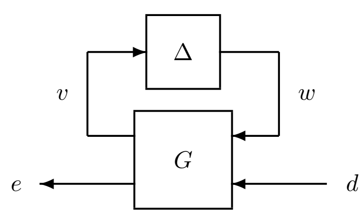

Figure 1. Interconnection for an Uncertain Discrete-time System

The robustness analysis is formulated by specifying a set $\mathbf{\Delta}$ of uncertainties. Each perturbation $\Delta : \ell_{2e}^{n_v} \to \ell_{2e}^{n_w}$ in $\mathbf{\Delta}$ is a bounded, causal operator which maps zero input to zero output. The input/output behavior of the perturbation is specified using quadratic constraints as described further below. At this point it is sufficient to state that Δ can have block-structure as is standard in robust control modeling [2]. The operator Δ can include blocks that are hard nonlinearities (e.g. saturations) and infinite dimensional operators (e.g. time delays) in addition to true system

[†]Note that a sequence having a finite escape time in the 2-norm will have finite escape time in any other p-norm. Hence ℓ is commonly used to denote the extended space in discrete-time. Here the notation ℓ_{2e} is adopted to denote this extended space as this parallels its continuous-time counterpart.

uncertainties. The term "uncertainty" is used for simplicity when referring to the perturbation Δ. For any $\Delta \in \mathbf{\Delta}$, well-posedness of the interconnection $F_u(G, \Delta)$ is defined as follows.

Definition 1. $F_u(G, \Delta)$ is well-posed if for all $x_G(0) \in \mathbb{R}^{n_G}$ and $d \in \ell_{2e}^{n_d}$ there exists a unique solution $x_G \in \ell_{2e}^{n_G}$, $v \in \ell_{2e}^{n_v}$, $e \in \ell_{2e}^{n_e}$, and $w \in \ell_{2e}^{n_w}$ with a causal dependence on d and satisfying Equation (3) and $w = \Delta(v)$.

Assume the interconnection is well-posed for a given $\Delta \in \mathbf{\Delta}$. Then the induced ℓ_2 gain from d to e is defined as:

$$\|F_u(G, \Delta)\|_{2\to 2} := \sup_{\substack{0 \neq d \in \ell_2^{n_d} \\ x_G(0)=0}} \frac{\|e\|}{\|d\|} \tag{4}$$

It is emphasized that the initial conditions $x_G(0)$ is assumed to be zero for the induced ℓ_2 gain computation. The objective of this paper is to assess the robustness of the uncertain discrete-time system $F_u(G, \Delta)$. Three types of robustness properties are considered.

1. The *worst-case induced ℓ_2 gain* from input d to the output e is defined as

$$\sup_{\Delta \in \mathbf{\Delta}} \|F_u(G, \Delta)\|_{2\to 2}. \tag{5}$$

   This worst-case gain is defined over all perturbations $\Delta \in \mathbf{\Delta}$.

2. The system has *robust asymptotic stability* if $x_G(k) \to 0$ for any initial condition $x_G(0) \in \mathbb{R}^{n_G}$, disturbance $d \in \ell_2^{n_d}$ and perturbation $\Delta \in \mathbf{\Delta}$.

3. The system has *robust uniform stability*[‡] if $\exists\, c \geq 0$ such that $\|x_G(k)\| \leq c\|x_G(0)\|$ for any initial condition $x_G(0) \in \mathbb{R}^{n_G}$, $k \geq 0$, $d = 0$, and perturbation $\Delta \in \mathbf{\Delta}$. A key requirement in this definition is that the constant c cannot depend on $x_G(0)$. This form of robust stability is important since it can be used to analyze the convergence rate of uncertain linear systems [30].

The worst-case gain (Property 1) requires $x_G(0) = 0$ as is standard. However, in the robust asymptotic and uniform stability concepts (Properties 2 and 3), different initial conditions are allowed for G but not for Δ. More specifically, Δ is treated as an operator mapping v to w. If Δ has an internal state then the initial condition for Δ must be fixed in the analysis. Therefore, the analysis in this paper is most useful for the case where Δ is memoryless, e.g. the static nonlinearity considered in the optimization algorithm analysis [4].

Remark 1. The stability definitions for the interconnection $F_u(G, \Delta)$ focus on the relation from input d and initial condition $x_G(0)$ to the state x_G and output e. An alternative, commonly-used formulation also considers two additional exogenous inputs and two output signals [31, Figure 1.11]. These additional signals are injected and measured at the input and output of Δ. In that setup, the notion of robust stability is defined in terms of the causality and boundedness of the mapping from these extra inputs to outputs [31, Definition 1]. This alternative formulation can easily be accommodated in this paper by two possible approaches. First, one can use the definitions given here but require the interconnection to have the additional inputs and outputs signals absorbed into d and e, respectively. Second, one can explicitly use the definitions given in [31]. This would require a slight modification of the dissipativity proofs given below. This second approach has already been performed for the continuous time case, e.g. Theorem 3 and Lemma 1 in [15]. The discrete-time counterpart can similarly be established with additional notation but is not pursued.

[‡]The notion of stability given here is a special case of the so-called global uniform stability [29, Lemma 4.5] when the required class $\mathcal{K}$ function is a linear function.

2.3. Integral Quadratic Constraints

This section briefly introduces the concept of a discrete-time integral quadratic constraint (IQC). This summary is similar to the discrete-time formulation in [7,20] and parallels the continuous-time formulation in [1]. The formal definition for a frequency domain IQC is given first.

Definition 2. Let $\Pi = \Pi^\sim \in \mathbb{RL}_\infty^{(n_v+n_w)\times(n_v+n_w)}$ be given. A bounded, causal operator $\Delta : \ell_{2e}^{n_v} \to \ell_{2e}^{n_w}$ satisfies the <u>frequency domain IQC</u> defined by the multiplier Π, if the following inequality holds for all $v \in \ell_2^{n_v}$ and $w = \Delta(v)$

$$\int_0^{2\pi} \begin{bmatrix} V(e^{j\omega}) \\ W(e^{j\omega}) \end{bmatrix}^* \Pi(e^{j\omega}) \begin{bmatrix} V(e^{j\omega}) \\ W(e^{j\omega}) \end{bmatrix} d\omega \geq 0 \tag{6}$$

where $V(e^{j\omega})$ and $W(e^{j\omega})$ are discrete-time Fourier transforms of v and w [§].

IQCs can also be defined in the time domain based on the graphical interpretation as shown in Figure 2. Let the input and output signals of Δ be filtered through an LTI system Ψ with zero initial conditions. The time domain IQC is an inequality enforced on the filter output r over infinite (soft IQC) or finite (hard IQC) horizons. The formal definition for a time domain IQC are provided below.

Definition 3. Let $\Psi \in \mathbb{RH}_\infty^{n_r\times(n_v+n_w)}$ and $M = M^T \in \mathbb{R}^{n_r\times n_r}$ be given.

(a) A bounded, causal operator $\Delta : \ell_{2e}^{n_v} \to \ell_{2e}^{n_w}$ satisfies the <u>time domain soft IQC</u> defined by (Ψ, M) if the following inequality holds for all $v \in \ell_2^{n_v}$ and $w = \Delta(v)$

$$\sum_{k=0}^{\infty} r(k)^T M r(k) \geq 0 \tag{7}$$

where r is the output of Ψ driven by inputs (v, w) with zero initial conditions.

(b) A bounded, causal operator $\Delta : \ell_{2e}^{n_v} \to \ell_{2e}^{n_w}$ satisfies the <u>time domain hard IQC</u> defined by (Ψ, M) if the following inequality holds for all $v \in \ell_{2e}^{n_v}$, $w = \Delta(v)$ and for all $N \geq 0$

$$\sum_{k=0}^{N} r(k)^T M r(k) \geq 0 \tag{8}$$

where r is the output of Ψ driven by inputs (v, w) with zero initial conditions.

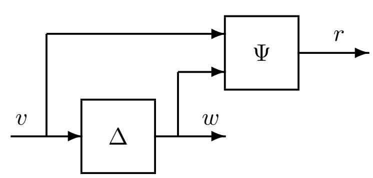

Figure 2. Graphical interpretation for time domain IQCs

The notation $\Delta \in \text{IQC}(\Pi)$, $\Delta \in \text{SoftIQC}(\Psi, M)$ and $\Delta \in \text{HardIQC}(\Psi, M)$ will be used when Δ satisfies the corresponding frequency domain, time domain soft, or time domain hard IQC, respectively. A library of frequency domain IQCs is provided for the continuous-time case in [1]. Many of these continuous-time IQCs have discrete-time counterparts [32, 33]. The dissipation inequality approach developed below requires time domain hard IQCs. Some hard IQCs can be directly derived in the time domain [1]. Many IQCs are more conveniently derived in the frequency

[§]The transform $V(e^{j\omega})$ is unrelated to the storage function V appearing in the dissipation inequalities later in the paper.

domain. Thus it is useful to connect frequency and time domain IQCs so that the full library of known IQCs can be used within the dissipation inequality framework. This connection relies on factorizing a frequency domain multiplier as $\Pi = \Psi^\sim M \Psi$. Such a factorization is always possible as stated in the next lemma although it is not unique.

Lemma 1. If $\Pi = \Pi^\sim \in \mathbb{RL}_\infty^{(n_v+n_w)\times(n_v+n_w)}$ then there exists real matrices A_ψ, B_ψ, Q, S, and R of compatible dimensions with A_ψ Schur, $Q = Q^T$, and $R = R^T$ such that

$$\Pi(z) := \begin{bmatrix} (zI - A_\psi)^{-1}B_\psi \\ I \end{bmatrix}^\sim \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} (zI - A_\psi)^{-1}B_\psi \\ I \end{bmatrix} \tag{9}$$

Proof
The proof given here is a modification of the continuous-time result presented in [34, Section 7.3]. Separate $\Pi = G_S + G_U$ where G_S and G_U are uniformly bounded outside and inside the closed unit disk, respectively. In addition, without loss of generality, one can choose a specific G_S satisfying $G_S(\infty) = 0$. Let $(A_S, B_S, C_S, 0)$ denote a realization for this G_S, i.e. $G_S(z) = C_S(zI - A_S)^{-1}B_S$. Hence A_S is a Schur matrix since G_S is bounded outside the closed unit disk. The assumption $\Pi = \Pi^\sim$ implies that $G_S + G_U = G_S^\sim + G_U^\sim$. This can be rewritten as $G_S - G_U^\sim = G_S^\sim - G_U$ where the left and right sides are analytic outside and inside the (closed) unit disk, respectively. Hence both sides must be analytic in the entire complex plane. By Liouville's theorem, one can conclude that $G_S^\sim - G_U$ is a constant, i.e. there exists matrix R such that $G_U(z) = G_S^\sim(z) + R$ for all $z \in \mathbb{C}$. This implies $\Pi = G_S + G_S^\sim + R$, and $R = R^T$ follows immediately from $\Pi = \Pi^\sim$. Thus Π can be written as in Equation (9) with $A_\psi = A_S, B_\psi = B_S, Q = 0, S = C_S^T$ and the constant matrix R. $\square$

Existing numerical algorithms can be used to construct the factorization presented in Lemma 1. If Π is proper then the Matlab function `stabsep` can be used to separate $\Pi = G_S + G_U$ where G_S is stable and causal. However, $\Pi \in \mathbb{RL}_\infty^{m\times m}$ may be a non-proper (polynomial) function of z, e.g. the multipliers used in [4,25]. A descriptor system representation of Π is required in such cases. This is a key distinction from the continuous-time case where Π is proper if it is bounded on the imaginary axis. In discrete-time, if Π is a non-proper (descriptor) system then the algorithm in [35–37] can be used to separate out the stable part. The stable part G_S in this construction is strictly proper and hence it has a standard state-space description. Finally, the matrix R can be explicitly computed by evaluating $R = \Pi(z_0) - G_S(z_0) - G_S(z_0^{-1})^T$ for some $z_0 \in \mathbb{C}$. For example, evaluating at $z_0 = 1$ is useful as both Π and G_S are bounded on the unit circle.

Frequency and time domain IQCs are connected by these (non-unique) factorizations $\Pi = \Psi^\sim M \Psi$. This is formalized in the next lemma.

Lemma 2. Let $\Pi = \Psi^\sim M \Psi$ with $\Psi \in \mathbb{RH}_\infty^{n_r\times(n_v+n_w)}$ and $M = M^T \in \mathbb{R}^{n_r\times n_r}$. Let $\Delta : \ell_{2e}^{n_v} \to \ell_{2e}^{n_w}$ be a bounded, causal operator. Then

1. $\Delta \in \text{IQC}(\Pi)$ if and only if $\Delta \in \text{SoftIQC}(\Psi, M)$.
2. $\Delta \in \text{IQC}(\Pi)$ if $\Delta \in \text{HardIQC}(\Psi, M)$.

This lemma is an application of Parseval's theorem [2] and hence the proof is omitted. Statement 2 of Lemma 2 states that a time domain hard IQC always leads to a frequency domain IQC. The reverse implication does not hold in general. It is important to emphasize that factorizations of Π are not unique. Some factorizations of Π may yield time domain hard IQCs while others do not. Thus the hard/soft property is not inherent to the multiplier Π but depends on the factorization (Ψ, M). The factorization introduced by Lemma 1 does not, in general, yield a valid time domain hard IQC. A specific hard factorization, presented in Section 3.3, will play a key role in the modified dissipation inequality result.

2.4. Standard Dissipation Inequality Approach

This section presents a standard dissipation inequality approach for uncertainty analysis [8,9,29,38]. The results in this section are a discrete-time counterpart of the dissipation inequality robustness analysis given in [13]. The previous section considered the case where the uncertainty Δ satisfies a single IQC. A less conservative analysis test is obtained if multiple IQCs are used to describe the uncertainty. In particular, the uncertainty Δ is assumed to satisfy multiple time domain hard IQCs defined by $\{(\Psi_i, M_i)\}_{i=1}^{N_I}$. All $\{\Psi_i\}_{i=1}^{N_I}$ are first aggregated into a single filter denoted Ψ with the following state-space realization:

$$\begin{bmatrix} \psi(k+1) \\ r(k) \end{bmatrix} = \begin{bmatrix} A_\psi & B_{\psi 1} & B_{\psi 2} \\ C_\psi & D_{\psi 1} & D_{\psi 2} \end{bmatrix} \begin{bmatrix} \psi(k) \\ v(k) \\ w(k) \end{bmatrix} \tag{10}$$

where $r(k) = [r_1(k)^T, \ldots, r_{N_I}(k)^T]^T \in \mathbb{R}^{n_r}$ and $r_i(k)$ is the output of the filter Ψ_i. In addition, define the block diagonal concatenation $M(\lambda) := diag(\lambda_1 M_1, \ldots, \lambda_{N_I} M_{N_I})$ where $\{\lambda_i\}_{i=1}^{N_I}$ are any non-negative real numbers. The stacked filtered output r satisfies the following inequality:

$$\sum_{k=0}^{N} r(k)^T M(\lambda) r(k) = \sum_{i=1}^{N_I} \lambda_i \left(\sum_{k=0}^{N} r_i(k)^T M_i r_i(k) \right) \geq 0 \tag{11}$$

The sum is non-negative because $\lambda_i \geq 0$ and $\Delta \in \text{HardIQC}(\Psi_i, M_i)$ for each i. In summary, Δ satisfies the time domain hard IQC defined by the combined multiplier $(\Psi, M(\lambda))$. This fact enables many IQCs on Δ to be incorporated into the robustness analysis.

The robustness of $F_u(G, \Delta)$ is analyzed using the interconnection shown in Figure 3. The extended system of G (Equation (3)) and Ψ (Equation (10)) is governed by the following state space model:

$$\begin{bmatrix} x(k+1) \\ r(k) \\ e(k) \end{bmatrix} = \begin{bmatrix} \mathcal{A} & \mathcal{B}_1 & \mathcal{B}_2 \\ \mathcal{C}_1 & \mathcal{D}_{11} & \mathcal{D}_{12} \\ \mathcal{C}_2 & \mathcal{D}_{21} & \mathcal{D}_{22} \end{bmatrix} \begin{bmatrix} x(k) \\ w(k) \\ d(k) \end{bmatrix} \tag{12}$$

where the extended state vector is $x := \left[\begin{smallmatrix} x_G \\ \psi \end{smallmatrix}\right] \in \mathbb{R}^{n_G + n_\psi}$ and the state-space matrices are given by

$$\mathcal{A} := \begin{bmatrix} A_G & 0 \\ B_{\psi 1} C_{G1} & A_\psi \end{bmatrix}, \qquad \mathcal{B}_1 := \begin{bmatrix} B_{G1} \\ B_{\psi 1} D_{G11} + B_{\psi 2} \end{bmatrix}, \qquad \mathcal{B}_2 := \begin{bmatrix} B_{G2} \\ B_{\psi 1} D_{G12} \end{bmatrix} \tag{13}$$

$$\mathcal{C}_1 := \begin{bmatrix} D_{\psi 1} C_{G1} & C_\psi \end{bmatrix}, \qquad \mathcal{D}_{11} := D_{\psi 1} D_{G11} + D_{\psi 2}, \qquad \mathcal{D}_{12} := D_{\psi 1} D_{G12} \tag{14}$$

$$\mathcal{C}_2 := \begin{bmatrix} C_{G2} & 0 \end{bmatrix}, \qquad \mathcal{D}_{21} := D_{G21}, \qquad \mathcal{D}_{22} := D_{G22} \tag{15}$$

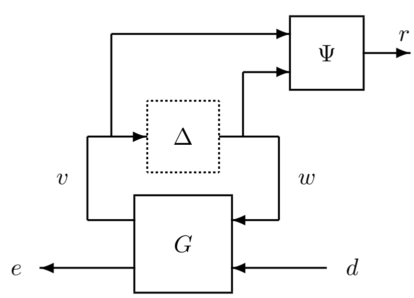

Figure 3. Extended LTI system of G and filter Ψ

Define a linear matrix inequality (LMI) with the extended system of G and $(\Psi, M(\lambda))$:

$LMI_{(G,\Psi)}(P, M(\lambda), \gamma) :=$

$$\begin{bmatrix} \mathcal{A}^T P \mathcal{A} - P & \mathcal{A}^T P \mathcal{B}_1 & \mathcal{A}^T P \mathcal{B}_2 \\ \mathcal{B}_1^T P \mathcal{A} & \mathcal{B}_1^T P \mathcal{B}_1 & \mathcal{B}_1^T P \mathcal{B}_2 \\ \mathcal{B}_2^T P \mathcal{A} & \mathcal{B}_2^T P \mathcal{B}_1 & \mathcal{B}_2^T P \mathcal{B}_2 - \gamma^2 I \end{bmatrix} + \begin{bmatrix} \mathcal{C}_2^T \\ \mathcal{D}_{21}^T \\ \mathcal{D}_{22}^T \end{bmatrix} \begin{bmatrix} \mathcal{C}_2^T \\ \mathcal{D}_{21}^T \\ \mathcal{D}_{22}^T \end{bmatrix}^T + \begin{bmatrix} \mathcal{C}_1^T \\ \mathcal{D}_{11}^T \\ \mathcal{D}_{12}^T \end{bmatrix} M(\lambda) \begin{bmatrix} \mathcal{C}_1^T \\ \mathcal{D}_{11}^T \\ \mathcal{D}_{12}^T \end{bmatrix}^T \tag{16}$$

The next theorem provides an analysis condition formulated with this LMI. The proof uses IQCs and a standard dissipation argument.

Theorem 1. Let $G \in \mathbb{RH}_\infty^{(n_v+n_e)\times(n_w+n_d)}$ be a stable LTI system defined by (3) and $\Delta : \ell_{2e}^{n_v} \to \ell_{2e}^{n_w}$ be a bounded, causal operator. Assume $F_u(G,\Delta)$ is well-posed and $\Delta \in \text{HardIQC}(\Psi_i, M_i)$ for $i = 1, \ldots, N_I$. If $\exists$ a matrix $P = P^T \geq 0$ and non-negative scalars γ, $\{\lambda_i\}_{i=1}^{N_I}$ such that $LMI_{(G,\Psi)}(P, M(\lambda), \gamma) < 0$, then

1. *(Worst-case Gain):* $\|F_u(G,\Delta)\|_{2\to2} < \gamma$
2. *(Robust Asymptotic Stability):* $x_G \in \ell_2^{n_G}$ and $\lim_{k\to\infty} x_G(k) = 0$ for any $x_G(0) \in \mathbb{R}^{n_G}$ and $d \in \ell_2^{n_d}$.
3. *(Robust Uniform Stability):* There exists c such that $\|x_G(k)\| \leq c\|x_G(0)\|$ holds for all $x_G(0) \in \mathbb{R}^{n_G}$ and $d = 0$.

Proof

This theorem is a standard dissipation result [8,9,29,38] but the proof is given for clarity. The LMI is strictly feasible by assumption and hence it remains feasible under small perturbations. Specifically, $\epsilon I + LMI_{(G,\Psi)}(P + \epsilon I, M(\lambda), \gamma) < 0$ for some sufficiently small $\epsilon > 0$. $F_u(G,\Delta)$ is assumed to be well-posed and hence there is a unique causal ℓ_{2e} solution (x_G, w, v, e) satisfying Equation (3) and $w = \Delta(v)$ for any given initial condition $x_G(0)$ and input $d \in \ell_2^{n_d}$. Let r be the output of Ψ driven by inputs (v, w) with the initial condition $\psi(0) = 0$, and set $x = \left[\begin{smallmatrix} x_G \\ \psi \end{smallmatrix}\right]$. Clearly both r and x are ℓ_{2e} signals. Define a storage function by $V(x) := x^T(P + \epsilon I)x$. Left and right multiply the perturbed LMI by $[x^T, w^T, d^T]$ and $[x^T, w^T, d^T]^T$ and apply (12) to show that V satisfies:

$$\epsilon x(k)^T x(k) + V(x(k+1)) - V(x(k)) + e(k)^T e(k) + \sum_{i=1}^{N_I} \lambda_i r_i(k)^T M_i r_i(k) \leq (\gamma^2 - \epsilon) d(k)^T d(k) \tag{17}$$

This dissipation inequality can be summed from $k = 0$ to $k = N$ to yield:

$$\begin{aligned} \epsilon \sum_{k=0}^{N} x(k)^T x(k) + V(x(N+1)) - V(x(0)) + \sum_{k=0}^{N} e(k)^T e(k) + \\ \sum_{i=1}^{N_I} \lambda_i \left(\sum_{k=0}^{N} r_i(k)^T M_i r_i(k) \right) \leq (\gamma^2 - \epsilon) \sum_{k=0}^{N} d(k)^T d(k) \end{aligned} \tag{18}$$

Apply the hard IQC conditions and $\lambda_i \geq 0$ to conclude:

$$\epsilon \sum_{k=0}^{N} x(k)^T x(k) + V(x(N+1)) + \sum_{k=0}^{N} e(k)^T e(k) \leq (\gamma^2 - \epsilon) \sum_{k=0}^{N} d(k)^T d(k) + V(x(0)) \tag{19}$$

The three robustness results follow as special cases of this inequality. First, if $x_G(0) = 0$ then $V \geq 0$ implies $e \in \ell_2$ and $\|e\| \leq (\gamma^2 - \epsilon)\|d\|$. Hence $\|F_u(G,\Delta)\|_{2\to2} < \gamma$.

Second, if $x_G(0)$ is non-zero then Equation (19) with $V \geq 0$ implies

$$\epsilon \sum_{k=0}^{N} x(k)^T x(k) \leq (\gamma^2 - \epsilon) \sum_{k=0}^{N} d(k)^T d(k) + V(x(0)) \tag{20}$$

This inequality yields $\|x\| < \infty$. Therefore $x_G \in \ell_2^{n_G}$ and $\lim_{k\to\infty} x_G(k) = 0$.

Third, if $d = 0$ and $x_G(0)$ is non-zero then Equation (19) implies $V(x(N+1)) \leq V(x(0))$, which is equivalent to $V(x(N)) \leq V(x(0))$. The filter has zero initial conditions ($\psi(0) = 0$) so that

$$\|x_G(N)\|^2 \leq \|x(N)\|^2 \leq \text{cond}(P + \epsilon I)\|x(0)\|^2 = \text{cond}(P + \epsilon I)\|x_G(0)\|^2 \tag{21}$$

Here cond denotes the condition number of a matrix. Thus $\|x_G(N)\| \leq \sqrt{\text{cond}(P + \epsilon I)}\|x_G(0)\|$. The condition number is finite since the perturbation ensures $P + \epsilon I > 0$. □

Conclusion 1 (Worst-case gain) of the above theorem will hold in a non-strict sense, i.e. $\|F_u(G,\Delta)\|_{2\to 2} \leq \gamma$, if the strict LMI is replaced by the non-strict condition $LMI_{(G,\Psi)}(P, M(\lambda), \gamma) \leq 0$. Similarly, Conclusion 3 (Robust uniform stability) still holds if $P > 0$ and $LMI_{(G,\Psi)}(P, M(\lambda), \gamma) \leq 0$. In both cases, one can no longer conclude that $\lim_{k\to\infty} x_G(k) = 0$. Also note that there are other related results using similar dissipativity. For example, the induced gain was defined in Equation 4 assuming zero initial conditions ($x_G(0) = 0$). The effect of nonzero initial conditions can be included in the result using the stored initial energy $V(x_G(0))$. These variations are minor and are not discussed further. A more significant variation concerns the condition for robust uniform stability. Specifically, an LMI condition with smaller dimensions can be formulated specifically for robust uniform stability.

Corollary 1. Let $G \in \mathbb{RH}_\infty^{(n_v+n_e)\times(n_w+n_d)}$ be a stable LTI system defined by (3) and $\Delta : \ell_{2e}^{n_v} \to \ell_{2e}^{n_w}$ be a bounded, causal operator. Assume $F_u(G,\Delta)$ is well-posed and $\Delta \in \text{HardIQC}(\Psi_i, M_i)$ for $i = 1, \ldots, N_I$. If $\exists$ a matrix $P = P^T \geq 0$ and non-negative scalars $\{\lambda_i\}_{i=1}^{N_I}$ such that

$$\begin{bmatrix} \mathcal{A}^T P \mathcal{A} - P & \mathcal{A}^T P \mathcal{B}_1 \\ \mathcal{B}_1^T P \mathcal{A} & \mathcal{B}_1^T P \mathcal{B}_1 \end{bmatrix} + \begin{bmatrix} \mathcal{C}_1^T \\ \mathcal{D}_{11}^T \end{bmatrix} M(\lambda) \begin{bmatrix} \mathcal{C}_1^T \\ \mathcal{D}_{11}^T \end{bmatrix}^T < 0 \tag{22}$$

then there exists c such that $\|x_G(k)\| \leq c\|x_G(0)\|$ holds for all $x_G(0) \in \mathbb{R}^{n_G}$ and $d = 0$, i.e. the system has *Robust Uniform Stability*.

Proof
The LMI in Equation (22) is feasible if and only if $LMI_{(G,\Psi)}(P, M(\lambda), \gamma) < 0$ for some sufficiently large γ. This can be shown via a Schur complement argument, e.g. see the proof of Lemma 1 in [15]. Hence the corollary follows from Theorem 1. Alternatively, a dissipation inequality argument, similar to that used in the proof of Theorem 1, can be used to directly prove this result. $\square$

As commented in Remark 1, an alternative commonly-used definition of input/output stability includes additional inputs/outputs on both sides of Δ [31, Definition 1]. It is possible to show that LMI (22) and the dissipativity framework can be used to guarantee input-output stability as in this alternative stability definition. The required steps parallel the continuous-time result in [15, Lemma 1]. It is also noted that related work in [39] incorporates the effect of initial conditions. The proof is non-constructive in the sense that it does not yield an easily computable bound c for the uniform stability constant. The main utility of the results contained here is that the constant c is directly bounded by $\sqrt{\text{cond}(P + \epsilon I)}$.

3. J-SPECTRAL FACTORIZATIONS AND RELATED GAMES

As noted previously, the factorization $\Pi = \Psi^\sim M \Psi$ is not unique. The modified dissipation inequality relies on a specific factorization. $(\hat{\Psi}, \hat{J})$ is called a *J-spectral factorization* of $\Pi = \Pi^\sim$ if: (i) $\Pi = \hat{\Psi}^\sim \hat{J} \hat{\Psi}$, (ii) $\hat{J} = diag(I_{n_v}, -I_{n_w})$ and (iii) $\hat{\Psi}, \hat{\Psi}^{-1} \in \mathbb{RH}_\infty^{(n_v+n_w)\times(n_v+n_w)}$ [10]. In other words, the factorization yields a square, stable filter $\hat{\Psi}$ with a stable inverse and $\hat{J}$ is a signature matrix. A simple condition for the existence of a J-spectral factorization can be stated using the following definition.

Definition 4. Let $\Pi = \Pi^\sim \in \mathbb{RL}_\infty^{(n_v+n_w)\times(n_v+n_w)}$ be partitioned as $\begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12}^\sim & \Pi_{22} \end{bmatrix}$ where $\Pi_{11} \in \mathbb{RL}_\infty^{n_v\times n_v}$ and $\Pi_{22} \in \mathbb{RL}_\infty^{n_w\times n_w}$. Π is a Strict Positive-Negative (PN) multiplier if there exists $\epsilon > 0$ such that

(a) $\Pi_{11}(e^{j\omega}) \geq \epsilon I$ for all $\omega \in [0, 2\pi]$.

(b) $\Pi_{22}(e^{j\omega}) \leq -\epsilon I$ for all $\omega \in [0, 2\pi]$.

Π is simply called a PN multiplier if (a) and (b) hold with $\epsilon = 0$.

The PN terminology refers to the Positive semidefinite and Negative semidefinite properties specified by conditions (a) and (b) with $\epsilon = 0$. Strict-PN multipliers strictly satisfy (a) and (b) over all frequencies. It will be shown (Lemma 6 in Section 3.3) that if Π is a Strict-PN multiplier then it has a J-spectral factorization. This result is a variation of the canonical factorization theorem in [10]. Condition (a) with $\epsilon = 0$ is necessary and sufficient for the zero operator $\Delta \equiv 0$ to satisfy the frequency domain IQC defined by Π. Condition (b) with $\epsilon > 0$ implies that if $\Delta \in \text{IQC}(\Pi)$ then Δ maps zero input to zero output. Bounded gain operators automatically have this zero input-zero output property. Condition (b) with $\epsilon = 0$ further implies that the set of all $\Delta \in \text{IQC}(\Pi)$ is a convex set [40,41]. The class of PN multipliers is quite general and covers the most typical multipliers used in IQC analysis. In fact, all of the IQCs listed in [1] satisfy Conditions (a) and (b) with $\epsilon = 0$ except for the IQCs for certain sector bounded nonlinearities and polytopic uncertainties.

The remainder of this section presents the J-spectral factorization lemma (Lemma 6) which is required to prove the modified dissipation inequality theorem. This J-spectral factorization lemma is first preceded by several intermediate game theory results. The reader only interested in the main modified dissipation inequality theorem (Theorem 2) can skip ahead to Section 4.

3.1. Open-loop Dynamic Games and IQC Factorizations

Suppose $\Pi = \Psi^\sim M \Psi$ is an arbitrary (not necessarily hard) factorization of the frequency domain IQC multiplier Π. This section connects properties of the factorization (Ψ, M) to the upper and lower values of an open-loop linear quadratic discrete-time game. There is a large body of literature on linear quadratic discrete-time games [42,43]. The results here build on previous results connecting discrete-time IQCs to min/max games [20]. Consider a two-player, zero-sum, linear quadratic difference game based on Ψ (with state space representation in Equation (10)) and matrix $M = M^T$:

$$J_{\Psi,M}(v, w, \psi_0) := \sum_{k=0}^{\infty} r(k)^T M r(k) \tag{23}$$

$$\begin{aligned} &\text{subject to:} \\ &\psi(k+1) = A_\psi \psi(k) + B_{\psi 1} v(k) + B_{\psi 2} w(k), \ \psi(0) = \psi_0 \\ &r(k) = C_\psi \psi(k) + D_{\psi 1} v(k) + D_{\psi 2} w(k) \end{aligned}$$

The infinite horizon cost function $J_{\Psi,M}$ is defined on $v \in \ell_2^{n_v}$, $w \in \ell_2^{n_w}$, and $\psi_0 \in \mathbb{R}^{n_\psi}$. Player 1 uses the "control variable" v to minimize $J_{\Psi,M}$ while Player 2 uses w to maximize $J_{\Psi,M}$. The game has an open-loop information structure and neither player can adapt their action during the game. The upper value of the game is defined as:

$$\bar{J}_{\Psi,M}(\psi_0) := \inf_{v \in \ell_2^{n_v}} \sup_{w \in \ell_2^{n_w}} J_{\Psi,M}(v, w, \psi_0) \tag{24}$$

The lower value of the game is defined as

$$\underline{J}_{\Psi,M}(\psi_0) := \sup_{w \in \ell_2^{n_w}} \inf_{v \in \ell_2^{n_v}} J_{\Psi,M}(v, w, \psi_0) \tag{25}$$

The next two lemmas relate the upper and lower values of this open-loop game to the properties of the IQC factorization (Ψ, M). The proofs are omitted as they are similar to those used in the continuous-time counterparts [15, Lemma 2, Lemma 3].

Lemma 3. Let $\Pi = \Psi^\sim M \Psi \in \mathbb{RL}_\infty^{(n_v+n_w)\times(n_v+n_w)}$ be any factorization with $\Psi \in \mathbb{RH}_\infty^{n_r \times (n_v+n_w)}$. Let Δ be a bounded, casual operator with $\Delta \in \text{IQC}(\Pi)$. Then the following inequality holds for all $v \in \ell_{2e}^{n_v}$, $w = \Delta(v)$ and $N \geq 0$:

$$\sum_{k=0}^{N} r(k)^T M r(k) \geq -\bar{J}_{\Psi,M}(\psi(N+1)) \tag{26}$$

where r and ψ are the output and state of Ψ, respectively, driven by inputs (v, w) with initial condition $\psi(0) = 0$. Moreover, if $\bar{J}_{\Psi,M}(\psi) \leq 0 \; \forall \psi \in \mathbb{R}^{n_\psi}$ then $\Delta \in \text{HardIQC}(\Psi, M)$.

Lemma 4. Let $\Pi = \Psi^\sim M \Psi \in \mathbb{RL}_\infty^{(n_v+n_w)\times(n_v+n_w)}$ be any factorization with $\Psi \in \mathbb{RH}_\infty^{n_r\times(n_v+n_w)}$. Let $G \in \mathbb{RH}_\infty^{(n_v+n_e)\times(n_w+n_d)}$ be given. If $P = P^T$ satisfies $LMI_{(G,\Psi)}(P, M, \gamma) < 0$ for any $\gamma > 0$ then

$$V(x_0) := x_0^T P x_0 \geq \underline{J}_{\Psi,M}(\psi_0) \;\; \forall x_0 := \begin{bmatrix} x_{G0} \\ \psi_0 \end{bmatrix} \in \mathbb{R}^{n_G+n_\psi} \tag{27}$$

Moreover, if $\underline{J}_{\Psi,M}(\psi_0) \geq 0 \; \forall \psi_0 \in \mathbb{R}^{n_\psi}$ then $P \geq 0$.

By Lemma 3, $\bar{J}_{\Psi,M}(\psi) \leq 0$ ensures the factorization (Ψ, M) is a hard IQC. By Lemma 4, $\underline{J}_{\Psi,M}(\psi_0) \geq 0$ ensures the storage matrix satisfies $P \geq 0$. It is easily shown that the two costs satisfy $\underline{J}_{\Psi,M}(\psi_0) \leq \bar{J}_{\Psi,M}(\psi)$ [42,43]. Hence the two conditions in Lemmas 3 and 4 can only be satisfied if $\bar{J}_{\Psi,M}(\psi_0) = \underline{J}_{\Psi,M}(\psi_0) = 0$ for all $\psi_0 \in \mathbb{R}^{n_\psi}$. It will be shown in Section 3.3 that the lower and upper values of the game are both equal to zero if (Ψ, M) is a J-spectral factorization. Hence this factorization plays an important role in the proof of the modified dissipation inequality.

3.2. Nash Equilibrium for the Two-Player Game

This section provides explicit values for $\bar{J}_{\Psi,M}(\psi_0)$ and $\underline{J}_{\Psi,M}(\psi_0)$ using the stabilizing solution of a related discrete-time algebraic Riccati equation. It is known that the upper and lower values can be effectively computed if a Nash equilibrium for the game exists [18, Theorem 3.26]. The basic intuition is provided before formally stating the result. Let $\Pi = \Psi^\sim M \Psi$ be the frequency domain multiplier associated with (Ψ, M). If $v \in \ell_2^{n_v}$ and $\psi(0) = 0$ then Parseval's theorem can be used to write $J_{\Psi,M}(v, 0, 0)$ in the frequency domain as:

$$J_{\Psi,M}(v, 0, 0) = \frac{1}{2\pi} \int_0^{2\pi} V(e^{j\omega})^* \, \Pi_{11}(e^{j\omega}) \, V(e^{j\omega}) d\omega \tag{28}$$

If $\Pi_{11}(e^{j\omega}) \geq \epsilon I$ for all $\omega \in [0, 2\pi]$ then $J_{\Psi,M}(v, 0, 0) \geq \epsilon \|v\|^2$. Similarly if $\Pi_{22}(e^{j\omega}) \leq -\epsilon I$ then $J_{\Psi,M}(0, w, 0) \leq -\epsilon \|w\|^2$ for all $w \in \ell_2^{n_w}$. Moreover, the Strict-PN condition actually implies that $J_{\Psi,M}$ is strictly convex in v and strictly concave in w. The following lemma constructs a Nash Equilibrium using the Strict-PN assumption.

Lemma 5. Let $\Pi = \Psi^\sim M \Psi \in \mathbb{RL}_\infty^{(n_v+n_w)\times(n_v+n_w)}$ be any factorization with $\Psi \in \mathbb{RH}_\infty^{n_r\times(n_v+n_w)}$ and $M = M^T \in \mathbb{R}^{n_r\times n_r}$. Define $Q := C_\psi^T M C_\psi$, $S := C_\psi^T M D_\psi$ and $R := D_\psi^T M D_\psi$ where $(A_\psi, B_\psi, C_\psi, D_\psi)$ are the state matrices of Ψ. If Π is a Strict-PN multiplier then

1. There exists a unique, real, stabilizing solution $X = X^T$ to $DARE(A_\psi, B_\psi, Q, R, S)$. In addition, $R + B_\psi^T X B_\psi$ is nonsingular.

2. For $\psi_0 \in \mathbb{R}^{n_\psi}$ define $v^\star \in \ell_2^{n_v}$ and $w^\star \in \ell_2^{n_w}$ by

$$\begin{bmatrix} v^\star(k) \\ w^\star(k) \end{bmatrix} := -K(A_\psi - B_\psi K)^k \psi_0 \tag{29}$$

   where $K := (R + B_\psi^T X B_\psi)^{-1}(A_\psi^T X B_\psi + S)^T$ is the stabilizing DARE gain. This input pair yields a value $J_{\Psi,M}(v^\star, w^\star, \psi_0) = \psi_0 X \psi_0$ for the two-player, LQ game in Equation (23). In addition, $(v^\star, w^\star)$ provides an open loop Nash equilibrium for this game, i.e.

$$J_{\Psi,M}(v^\star, w, \psi_0) \leq J_{\Psi,M}(v^\star, w^\star, \psi_0) \leq J_{\Psi,M}(v, w^\star, \psi_0), \; \forall v \in \ell_2^{n_v}, \; w \in \ell_2^{n_w} \tag{30}$$

3. $\bar{J}_{\Psi,M}(\psi_0) = \underline{J}_{\Psi,M}(\psi_0) = \psi_0^T X \psi_0$.

Proof
Statement 1 is a restatement of Lemma 7 in the appendix. If A_ψ is singular then Π has poles at $z = \infty$ and hence Π is non-proper. As a result, the proof of Lemma 7 requires the use of the descriptor system notation and results. This is the only technical lemma that requires descriptor systems and hence the proof is given in the appendix for readability.

To prove Statement 2, first note that $A_\psi - B_\psi K$ is a Schur stable matrix since X is the stabilizing solution of $DARE(A_\psi, B_\psi, Q, R, S)$. Hence $v^\star$ and $w^\star$ are ℓ_2 signals as claimed. The output of Ψ resulting from the inputs $(v^\star, w^\star)$ and initial condition ψ_0 is

$$r^\star(k) := C_\psi \psi^\star(k) + D_\psi \begin{bmatrix} v^\star(k) \\ w^\star(k) \end{bmatrix} \tag{31}$$

where $\psi^\star(k) := (A_\psi - B_\psi K)^k \psi_0$ is the state. This yields the following cost for the game:

$$J_{\Psi,M}(v^\star, w^\star, \psi_0) = \sum_{k=0}^{\infty} \begin{bmatrix} \psi^\star(k) \\ \begin{bmatrix} v^\star(k) \\ w^\star(k) \end{bmatrix} \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} \psi^\star(k) \\ \begin{bmatrix} v^\star(k) \\ w^\star(k) \end{bmatrix} \end{bmatrix} \tag{32}$$

Substitute for Q using the DARE and use $\begin{bmatrix} v^\star \\ w^\star \end{bmatrix} = -K\psi^\star$. After completing the square the cost is written as

$$J_{\Psi,M}(v^\star, w^\star, \psi_0) = \sum_{k=0}^{\infty} \left(\psi^\star(k)^T X \psi^\star(k) - \psi^\star(k+1)^T X \psi^\star(k+1) \right) \tag{33}$$

This is a telescoping sum which yields $J_{\Psi,M}(v^\star, w^\star, \psi_0) = \psi_0^T X \psi_0$.

Next let $\psi \in \ell_2^{n_\psi}$ denote the state of Ψ for initial condition ψ_0 and arbitrary inputs $v \in \ell_2^{n_v}$ and $w \in \ell_2^{n_w}$. Define deviation signals as:

$$\delta_\psi := \psi - \psi^\star, \quad \delta_v := v - v^\star, \quad \delta_w := w - w^\star \tag{34}$$

Note that δ_v belongs to ℓ_2 since it is a difference of ℓ_2 signals. Similarly, δ_w and δ_ψ are also in ℓ_2. By linearity, δ_ψ is the state of Ψ driven by inputs (δ_v, δ_w) from zero initial conditions ($\delta_\psi(0) = 0$). The cost for the game with inputs (v, w) and initial condition ψ_0 is:

$$J_{\Psi,M}(v, w, \psi_0) = \sum_{k=0}^{\infty} \begin{bmatrix} \psi^\star(k) + \delta_\psi(k) \\ \begin{bmatrix} v^\star(k)+\delta_v(k) \\ w^\star(k)+\delta_w(k) \end{bmatrix} \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} \psi^\star(k) + \delta_\psi(k) \\ \begin{bmatrix} v^\star(k)+\delta_v(k) \\ w^\star(k)+\delta_w(k) \end{bmatrix} \end{bmatrix} \tag{35}$$

This can be expanded into four quadratic terms involving $(\psi^\star, v^\star, w^\star)$ and $(\delta_\psi, \delta_v, \delta_w)$. Simplify using a similar completion of square and telescoping sum argument as above:

$$J_{\Psi,M}(v, w, \psi_0) = \psi_0^T X \psi_0 + \psi_0^T X \delta_\psi(0) + \delta_\psi(0)^T X \psi_0 + \sum_{k=0}^{\infty} \begin{bmatrix} \delta_\psi(k) \\ \begin{bmatrix} \delta_v(k) \\ \delta_w(k) \end{bmatrix} \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} \delta_\psi(k) \\ \begin{bmatrix} \delta_v(k) \\ \delta_w(k) \end{bmatrix} \end{bmatrix} \tag{36}$$

The second and third terms are zero because $\delta_\psi(0) = 0$. The fourth term is equal to $J_{\Psi,M}(\delta_v, \delta_w, 0)$. Therefore, Equation (36) can be rewritten as

$$J_{\Psi,M}(v, w, \psi_0) = J_{\Psi,M}(v^\star, w^\star, \psi_0) + J_{\Psi,M}(\delta_v, \delta_w, 0) \tag{37}$$

This relation can be used to demonstrate that $(v^\star, w^\star)$ provides an open loop Nash equilibrium. Specifically, Equation (37) directly leads to

$$J_{\Psi,M}(v, w^\star, \psi_0) - J_{\Psi,M}(v^\star, w^\star, \psi_0) = J_{\Psi,M}(\delta_v, 0, 0) \tag{38}$$

As discussed before the lemma, the Strict-PN assumption implies that $J_{\Psi,M}(\delta_v, 0, 0) \geq 0$. Hence Equation (38) implies

$$J_{\Psi,M}(v^\star, w^\star, \psi_0) \leq J_{\Psi,M}(v, w^\star, \psi_0), \ \forall v \in \ell_2^{n_v} \tag{39}$$

The Strict-PN assumption and Equation (37) similarly implies that

$$J_{\Psi,M}(v^\star, w, \psi_0) \leq J_{\Psi,M}(v^\star, w^\star, \psi_0), \ \forall w \in \ell_2^{n_w} \tag{40}$$

This completes the proof of Statement 2.

Statement 3 follows from [18, Theorem 3.26]. The upper and lower values of the discrete-time linear quadratic game are both equal to the game value at the Nash equilibrium. □

Theorem 3.3 in [19] provides a related Nash equilibrium result for the continuous-time LQ game. The continuous-time result is more general in that it only requires (A_ψ, B_ψ) to be stabilizable. Lemma 5 requires the stronger assumption that A_ψ is stable. To the best of our knowledge, the discrete-time counterpart of Theorem 3.3 in [19] has not been established. However, the assumption that A_ψ is stable is sufficient for the IQC analysis considered in this paper. The proof for Statement 1 in Lemma 5 has some subtleties that do not appear in the continuous-time counterpart. In continuous-time, Π is assumed to be bounded on the closed imaginary axis and this implies that Π is proper. In discrete-time, Π is required to be bounded on the unit circle and hence Π can be improper. As a consequence, the discrete-time proof for Statement 1 in Lemma 5 cannot simply mimic its continuous-time counterpart. Instead a descriptor system representation of Π is required as is done for the proof of Lemma 7 in the appendix. Finally, notice that Statements 1 and 3 in Lemma 5 can also be proved by tailoring the operator-theoretic results in [11]. The operator-theoretic framework is more general while the linear algebra approach in this paper is more closely aligned with possible numerical implementations.

3.3. *J-Spectral Factorization for Strict-PN Multipliers*

Lemma 6 provides a simple frequency domain condition on Π that is sufficient for the existence of a J-spectral factor. In addition, this lemma provides several useful properties of the J-spectral factorization. The Strict-PN assumption again plays a key role in the result.

Lemma 6. Let $\Pi = \Psi^\sim M \Psi \in \mathbb{RL}_\infty^{(n_v+n_w)\times(n_v+n_w)}$ be any factorization with $\Psi \in \mathbb{RH}_\infty^{n_r \times (n_v+n_w)}$ and $M = M^T \in \mathbb{R}^{n_r \times n_r}$. Define $Q := C_\psi^T M C_\psi$, $S := C_\psi^T M D_\psi$ and $R := D_\psi^T M D_\psi$ where $(A_\psi, B_\psi, C_\psi, D_\psi)$ are the state matrices of Ψ. If Π is a Strict-PN multiplier then

1. Π has a J-spectral factorization $(\hat{\Psi}, \hat{J})$ with $\hat{J} := diag(I_{n_v}, -I_{n_w})$. Moreover, this J-spectral factorization can be constructed from the unique stabilizing solution X of $DARE(A_\psi, B_\psi, Q, R, S)$. Let $\hat{D}_\psi$ satisfy $\hat{D}_\psi^T \hat{J} \hat{D}_\psi = R + B_\psi^T X B_\psi$ and define $\hat{C}_\psi := \hat{J} \hat{D}_\psi^{-T}(B_\psi^T X A_\psi + S^T)$. Then $(\hat{\Psi}, \hat{J})$ is a J-spectral factorization of Π with
$$\hat{\Psi} := \left[\begin{array}{c|c} A_\psi & B_\psi \\ \hline \hat{C}_\psi & \hat{D}_\psi \end{array}\right] \tag{41}$$
2. $\hat{X} = 0$ is the unique stabilizing solution of $DARE(A_\psi, B_\psi, \hat{Q}, \hat{R}, \hat{S})$ where $\hat{Q} := \hat{C}_\psi^T \hat{J} \hat{C}_\psi$, $\hat{S} := \hat{C}_\psi^T \hat{J} \hat{D}_\psi$, and $\hat{R} := \hat{D}_\psi^T \hat{J} \hat{D}_\psi$.
3. $\bar{J}_{\hat{\Psi},\hat{J}}(\psi_0) = \underline{J}_{\hat{\Psi},\hat{J}}(\psi_0) = 0, \forall \psi_0 \in \mathbb{R}^{n_\psi}$.
4. $\Delta \in$ HardIQC$(\hat{\Psi}, \hat{J})$ for any bounded, casual operator $\Delta \in$ IQC(Π).
5. For any $G \in \mathbb{RH}_\infty^{(n_v+n_e)\times(n_w+n_d)}$, $P = P^T$, and γ,
$$LMI_{(G,\Psi)}(P, M, \gamma) = LMI_{(G,\hat{\Psi})}(\hat{P}, \hat{J}, \gamma) \tag{42}$$
where $\hat{P} := P - \left[\begin{smallmatrix} 0 & 0 \\ 0 & X \end{smallmatrix}\right]$. Moreover, if $LMI_{(G,\hat{\Psi})}(\hat{P}, \hat{J}, \gamma) < 0$ then $\hat{P} \geq 0$.

Proof
The existence of the stabilizing solution X follows from Lemma 5. Recall the stabilizing gain is given by $K := (R + B_\psi^T X B_\psi)^{-1}(A_\psi^T X B_\psi + S)^T$. A J-spectral factorization of Π can be constructed from X using a standard expansion technique [44]. First express Π as:

$$\Pi(z) = \begin{bmatrix} (zI - A_\psi)^{-1} B_\psi \\ I \end{bmatrix}^\sim \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} (zI - A_\psi)^{-1} B_\psi \\ I \end{bmatrix} \tag{43}$$

Use the DARE and the definition of K to show:

$$\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} = \begin{bmatrix} K^T \\ I \end{bmatrix} (R + B_\psi^T X B_\psi) \begin{bmatrix} K & I \end{bmatrix} - \begin{bmatrix} A_\psi^T X A_\psi - X & A_\psi^T X B_\psi \\ B_\psi^T X A_\psi & B_\psi^T X B_\psi \end{bmatrix} \tag{44}$$

Substitute Equation (44) into the expression for Π to obtain

$$\Pi(z) = \begin{bmatrix} (zI - A_\psi)^{-1} B_\psi \\ I \end{bmatrix}^\sim \left(\begin{bmatrix} K^T \\ I \end{bmatrix} (R + B_\psi^T X B_\psi) \begin{bmatrix} K & I \end{bmatrix} \right) \begin{bmatrix} (zI - A_\psi)^{-1} B_\psi \\ I \end{bmatrix} \tag{45}$$

The Strict-PN conditions imply that $\Pi(e^{j\omega})$ has n_v positive eigenvalues and n_w negative eigenvalues for all $\omega \in [0, 2\pi]$. This follows from the Courant-Fischer minimax theorem [45]. Moreover, $(R + B_\psi^T X B_\psi)$ must have the same signature as Π by Equation (45). Thus there exists a nonsingular matrix $\hat{D}_\psi$ such that $\hat{D}_\psi^T \hat{J} \hat{D}_\psi = R + B_\psi^T X B_\psi$ with $\hat{J} := diag(I_{n_v}, -I_{n_w})$. Finally, it can be verified from Equation (45) that $\hat{\Psi}$ as defined in the lemma satisfies $\Pi = \hat{\Psi}^\sim \hat{J} \hat{\Psi}$. It remains to show that $\hat{\Psi}^{-1}$ is stable. A realization for the inverse is

$$\hat{\Psi}^{-1} := \left[\begin{array}{c|c} A_\psi - B_\psi \hat{D}_\psi^{-1} \hat{C}_\psi & B_\psi \hat{D}_\psi^{-1} \\ \hline -\hat{D}_\psi^{-1} \hat{C}_\psi & \hat{D}_\psi^{-1} \end{array} \right] \tag{46}$$

The state matrix is $A_\psi - B_\psi \hat{D}_\psi^{-1} \hat{C}_\psi = A_\psi - B_\psi K$. This is a Schur stable matrix because K is the stabilizing gain. Hence $\hat{\Psi}^{-1}$ is a stable system and this completes the proof of Statement 1.

To prove Statement 2, first note that $(\hat{Q}, \hat{S}, \hat{R})$ as defined can be written as:

$$\begin{bmatrix} \hat{Q} & \hat{S} \\ \hat{S}^T & \hat{R} \end{bmatrix} = \begin{bmatrix} K^T \\ I \end{bmatrix} (R + B_\psi^T X B_\psi) \begin{bmatrix} K & I \end{bmatrix} \tag{47}$$

$\hat{R} = R + B_\psi^T X B_\psi$ is nonsingular as shown above and $\hat{Q} - \hat{S}\hat{R}^{-1}\hat{S}^T = 0$. Hence $\hat{X} = 0$ is a solution of $DARE(A_\psi, B_\psi, \hat{Q}, \hat{R}, \hat{S})$. The corresponding gain $\hat{K} = \hat{R}^{-1}\hat{S}^T = K$ is stabilizing since $A_\psi - B_\psi \hat{K} = A_\psi - B_\psi K$ is a Schur stable matrix. Thus $\hat{X} = 0$ is the unique stabilizing solution of $DARE(A_\psi, B_\psi, \hat{Q}, \hat{R}, \hat{S})$.

Next, note that $\bar{J}_{\hat{\Psi},\hat{J}}(\psi_0) = \underline{J}_{\hat{\Psi},\hat{J}}(\psi_0) = \psi_0^T \hat{X} \psi_0$ by Lemma 5. Hence Statement 3 follows from the fact $\hat{X} = 0$.

To prove Statement 4, note $\bar{J}_{\hat{\Psi},\hat{J}}(\psi_0) = 0$ for all $\psi_0 \in \mathbb{R}^{n_\psi}$. The factorization is hard if $\bar{J}_{\hat{\Psi},\hat{J}}(\psi_0) \leq 0$ for all $\psi_0 \in \mathbb{R}^{n_\psi}$ by Lemma 3. Hence $(\hat{\Psi}, \hat{J})$ is a hard factorization of Π.

To show Statement 5, first express the "M" term of $LMI_{(G,\Psi)}(P, M, \gamma)$ as follows:

$$\begin{bmatrix} \mathcal{C}_1^T \\ \mathcal{D}_{11}^T \\ \mathcal{D}_{12}^T \end{bmatrix} M \begin{bmatrix} \mathcal{C}_1^T \\ \mathcal{D}_{11}^T \\ \mathcal{D}_{12}^T \end{bmatrix}^T = L^T \begin{bmatrix} C_\psi^T \\ D_\psi^T \end{bmatrix} M \begin{bmatrix} C_\psi & D_\psi \end{bmatrix} L \tag{48}$$

where L is given by

$$L = \left[\begin{array}{cccc} 0 & I & 0 & 0 \\ \hline C_{G_1} & 0 & D_{G11} & D_{G12} \\ 0 & 0 & I & 0 \end{array} \right] \tag{49}$$

Next use (44), (47) and the definitions of Q, S, R, $\hat{Q}$, $\hat{S}$, $\hat{R}$ to show

$$\begin{bmatrix} C_\psi^T \\ D_\psi^T \end{bmatrix} M \begin{bmatrix} C_\psi & D_\psi \end{bmatrix} = \begin{bmatrix} \hat{C}_\psi^T \\ \hat{D}_\psi^T \end{bmatrix} \hat{J} \begin{bmatrix} \hat{C}_\psi & \hat{D}_\psi \end{bmatrix} - \begin{bmatrix} A_\psi^T X A_\psi - X & A_\psi^T X B_\psi \\ B_\psi^T X A_\psi & B_\psi^T X B_\psi \end{bmatrix} \tag{50}$$

Substitute this expression into the "M" term of $LMI_{(G,\Psi)}(P, M, \gamma)$ (Equation (48)). Some lengthy but straightforward algebraic manipulations yield $LMI_{(G,\Psi)}(P, M, \gamma) = LMI_{(G,\hat{\Psi})}(\hat{P}, \hat{J}, \gamma)$. Finally, it remains to show that the additional assumption $LMI_{(G,\hat{\Psi})}(\hat{P}, \hat{J}, \gamma) < 0$ implies $\hat{P} \geq 0$. By Lemma 5, $\underline{J}_{\hat{\Psi},\hat{J}}(\psi_0) = \psi_0^T \hat{X} \psi_0$ and by Lemma 4 $\hat{P} \geq 0$ if $\underline{J}_{\hat{\Psi},\hat{J}}(\psi_0) \geq 0$ for all $\psi_0 \in \mathbb{R}^{n_\psi}$. Thus $\hat{P} \geq 0$ since $\hat{X} = 0$ as already shown. □

The above result complements the minimax theorems in [20]. In particular, [20, Theorem 2.1] states a sufficient condition to ensure $\bar{J}_{\Psi,M}(\psi_0) = \underline{J}_{\Psi,M}(\psi_0)$. Statement 3 in Lemma 6 states the J-spectral factorization ensures the upper and lower game values are, in fact, both equal to zero. Moreover, [20, Theorem 2.2] states a sufficient condition which can be sued to check whether a given factorization is hard. This paper shows that a J-spectral factorization is hard and satisfies the extra "storage function" property mentioned in Statement 5 of Lemma 6.

4. MODIFIED DISSIPATION INEQUALITY APPROACH

The modified dissipation inequality result is now stated as Theorem 2 below. The proof of this theorem relies on a J-spectral factorization results contained in Lemma 6.

Theorem 2. Let $G \in \mathbb{RH}_\infty^{(n_v+n_e)\times(n_w+n_d)}$ be a stable LTI system defined by (3) and $\Delta : \ell_{2e}^{n_v} \to \ell_{2e}^{n_w}$ be a bounded, causal operator. Assume $F_u(G,\Delta)$ is well-posed and $\Delta \in \text{SoftIQC}(\Psi, M(\lambda))$ for all λ in some set Λ. If $\exists$ a matrix $P = P^T$, vector $\lambda \in \Lambda$, and non-negative scalar γ such that $\Psi^\sim M(\lambda)\Psi$ is a PN multiplier and $LMI_{(G,\Psi)}(P, M(\lambda), \gamma) < 0$, then

1. *(Worst-case Gain):* $\|F_u(G,\Delta)\|_{2\to 2} < \gamma$
2. *(Robust Asymptotic Stability):* $x_G \in \ell_2^{n_G}$ and $\lim_{k\to\infty} x_G(k) = 0$ for any $x_G(0) \in \mathbb{R}^{n_G}$ and $d \in \ell_2^{n_d}$.
3. *(Robust Uniform Stability):* There exists c such that $\|x_G(k)\| \leq c\|x_G(0)\|$ holds for all $x_G(0) \in \mathbb{R}^{n_G}$ and $d = 0$.

Proof
First assume that $\Psi^\sim M(\lambda)\Psi$ is a Strict-PN multiplier. By Statement 1 of Lemma 6, this multiplier has a J-spectral factorization $(\hat{\Psi}, \hat{J})$ constructed from the stabilizing solution X of a related DARE. By Statement 4 of Lemma 6, if $\Delta \in \text{SoftIQC}(\Psi, M(\lambda))$ then $\Delta \in \text{HardIQC}(\hat{\Psi}, \hat{J})$. In other words, the J-spectral factorization provides a time domain hard IQC for Δ. By Statement 5 of Lemma 6, $LMI_{(G,\hat{\Psi})}(\hat{P}, \hat{J}, \gamma) = LMI_{(G,\Psi)}(P, M(\lambda), \gamma) < 0$ where $\hat{P} := P - \left[\begin{smallmatrix} 0 & 0 \\ 0 & X \end{smallmatrix}\right]$. Thus the LMI condition can be rewritten using the J-spectral factorization. Finally, Statement 5 of Lemma 6 also implies that $\hat{P} \geq 0$. Hence the dissipation conditions hold using the hard IQC $(\hat{\Psi}, \hat{J})$ and storage matrix $\hat{P} \geq 0$. The analysis conclusions follow from the standard dissipation result in Theorem 1.

A perturbation argument is needed if $\Psi^\sim M(\lambda)\Psi$ is a PN multiplier. Δ is a bounded operator, by assumption, and hence it satisfies the constant multiplier $\Pi_0 := diag(\|\Delta\|_{2\to 2} I_{n_v}, -I_{n_w})$. For all $\epsilon > 0$, the perturbed multiplier $\Psi^\sim M(\lambda)\Psi + \epsilon\Pi_0$ is a Strict-PN multiplier that defines a valid frequency domain IQC for Δ. In addition, it can be factorized as:

$$\Psi_{pert}^\sim M_{pert}(\lambda,\epsilon)\Psi_{pert} := \begin{bmatrix} \Psi \\ I \end{bmatrix}^\sim \begin{bmatrix} M(\lambda) & 0 \\ 0 & \epsilon\Pi_0 \end{bmatrix} \begin{bmatrix} \Psi \\ I \end{bmatrix} \tag{51}$$

By Lemma 2, $\Delta \in \text{SoftIQC}(\Psi_{pert}, M_{pert}(\lambda, \epsilon))$. Moreover, $LMI_{(G,\Psi)}(P, M(\lambda), \gamma) < 0$ implies that $LMI_{(G,\Psi_{pert})}(P, M_{pert}(\lambda, \epsilon), \gamma) < 0$ holds for sufficiently small $\epsilon > 0$. The result now follows using the arguments above with the Strict-PN multiplier given in Equation (51). $\square$

Theorem 2 removes the constraint $P \geq 0$ and allows for time domain soft IQCs. Moreover, Theorem 2 provides a time domain condition which can be generalized to cases where the nominal system G is time-varying and/or nonlinear. This enables robustness analysis for uncertain (finite-horizon) time-varying systems [46, 47], linear parameter varying systems [48, 49], and nonlinear systems. The time domain soft IQCs required in Theorem 2 are equivalent to frequency domain IQCs by Lemma 2. In exchange for these generalizations, Theorem 2 requires the multiplier satisfy the additional PN conditions in Definition 4. The PN conditions are satisfied by most multipliers, as noted above, and hence the modified dissipation inequality typically reduces conservatism in the analysis relative to the standard dissipation inequality result. A numerical example will be presented in Section 5 to demonstrate this fact. It is worth mentioning that the modified dissipation inequality reduces analysis conservatism only from a practical viewpoint. From a theoretical perspective, there always exists a standard dissipation inequality with a hard IQC yielding the same analysis result as the modified dissipation inequality. However, how to parameterize the IQC multiplier to include this specific hard IQC in the search space beforehand is a non-trivial practical issue. The modified dissipation inequality approach enables searches over a larger parameter space and hence can lead to less conservative results in practice.

For the worst-case gain computation, the standard homotopy approach in [1] can be used to obtain similar LMI conditions with some minor changes in the technical assumptions. In particular, the homotopy method leads to the following frequency domain analysis condition:

$$\begin{aligned} 0 > & \begin{bmatrix} G_{21}(e^{j\omega}) & G_{22}(e^{j\omega}) \\ 0 & I \end{bmatrix}^* \begin{bmatrix} I & 0 \\ 0 & -\gamma^2 I \end{bmatrix} \begin{bmatrix} G_{21}(e^{j\omega}) & G_{22}(e^{j\omega}) \\ 0 & I \end{bmatrix} \\ & + \begin{bmatrix} G_{11}(e^{j\omega}) & G_{12}(e^{j\omega}) \\ I & 0 \end{bmatrix}^* \Psi(e^{j\omega})^\sim M(\lambda) \Psi(e^{j\omega}) \begin{bmatrix} G_{11}(e^{j\omega}) & G_{12}(e^{j\omega}) \\ I & 0 \end{bmatrix}, \ \forall \omega \in [0, 2\pi] \end{aligned} \tag{52}$$

where $G := \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}$ is partitioned according to the inputs (w, d) and outputs (v, e). The KYP Lemma [50] implies that for any factorization $(\Psi, M(\lambda))$ this frequency domain inequality is equivalent to the existence of $P = P^T$ satisfying $LMI_{(G,\Psi)}(P, M(\lambda), \gamma) < 0$. The homotopy method does not require $P \geq 0$ nor does it require the factorization to be hard. Theorem 2 demonstrates that a valid dissipation inequality argument is still possible. It requires the LMI to be rewritten using a special J-spectral factorization. The term X appearing in the proof can be interpreted as hidden energy stored in the original (non-hard) IQC factorization. The dissipativity approach handles robust asymptotic stability and robust uniform stability more transparently. The homotopy method can also be used to certificate robust asymptotic stability and robust uniform stability under different mild conditions, e.g. G being controllable and observable. Both the homotopy method and the modified dissipation inequality approach require G to be stable. The homotopy approach may also be generalized to uncertain LTV or nonlinear systems, although such generalizations are not pursued in this paper. Finally, the robust uniform stability condition in Corollary 1 has a similar modified form as stated next.

Corollary 2. Let $G \in \mathbb{RH}_\infty^{(n_v+n_e)\times(n_w+n_d)}$ be a stable LTI system defined by (3) and $\Delta : \ell_{2e}^{n_v} \to \ell_{2e}^{n_w}$ be a bounded, causal operator. Assume $F_u(G, \Delta)$ is well-posed and $\Delta \in \text{SoftIQC}(\Psi, M(\lambda))$ for all λ in some set Λ. If $\exists$ a matrix $P = P^T$ and vector $\lambda \in \Lambda$ such that $\Psi^\sim M(\lambda)\Psi$ is a PN multiplier and

$$\begin{bmatrix} \mathcal{A}^T P \mathcal{A} - P & \mathcal{A}^T P \mathcal{B}_1 \\ \mathcal{B}_1^T P \mathcal{A} & \mathcal{B}_1^T P \mathcal{B}_1 \end{bmatrix} + \begin{bmatrix} \mathcal{C}_1^T \\ \mathcal{D}_{11}^T \end{bmatrix} M(\lambda) \begin{bmatrix} \mathcal{C}_1^T \\ \mathcal{D}_{11}^T \end{bmatrix}^T < 0 \tag{53}$$

then there exists c such that $\|x_G(k)\| \leq c\|x_G(0)\|$ holds for all $x_G(0) \in \mathbb{R}^{n_G}$ and $d = 0$, i.e. the system has *Robust Uniform Stability*.

It is emphasized that the modified dissipation inequality approach was previously developed for uncertain continuous-time, linear parameter varying systems [16]. In addition, similar results have been obtained for uncertain (polynomial) nonlinear systems in [21]. For simplicity and conciseness, this paper considers only the case where G is a discrete-time LTI system. The results presented here can be extended to discrete-time, linear parameter varying systems with mainly notational changes as in [16, 21]. In addition, the stability concepts of discrete-time, linear parameter varying systems are quite similar to their continuous-time counterparts [48, 49].

5. NUMERICAL EXAMPLE

This section provides a simple analysis example to demonstrate the main results. The calculations were performed in Matlab using CVX [51,52] with the solver SDPT3 [53,54]. The uncertain system is given by a nominal LTI system G under a perturbation Δ described by two IQC multipliers Π_1 and Π_2. The objective is to compute a bound on the worst case gain of $F_u(G, \Delta)$. Both the standard and modified dissipation inequalities will be used to compute this bound.

The nominal system G is given by

$$\begin{aligned} x_G(k+1) &= -0.5\, x_G(k) + \begin{bmatrix} 0.5 & 0.4 \end{bmatrix} \begin{bmatrix} w(k) \\ d(k) \end{bmatrix} \\ \begin{bmatrix} v(k) \\ e(k) \end{bmatrix} &= \begin{bmatrix} 2.5 \\ 2 \end{bmatrix} x_G(k) + \begin{bmatrix} 0 & 0.6 \\ 0 & 0.9 \end{bmatrix} \begin{bmatrix} w(k) \\ d(k) \end{bmatrix}. \end{aligned} \tag{54}$$

The first IQC multiplier is $\Pi_1 = \Psi_1^{\sim} M \Psi_1$ with $M = \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$ and Ψ_1 given by:

$$\begin{aligned} \psi(k+1) &= -0.3\, \psi(k) + \begin{bmatrix} 1.3 & 0 \end{bmatrix} \begin{bmatrix} v(k) \\ w(k) \end{bmatrix} \\ r(k) &= \begin{bmatrix} 0 \\ -0.1 \end{bmatrix} \psi(k) + \begin{bmatrix} 0.2 & 0 \\ 0 & -0.1 \end{bmatrix} \begin{bmatrix} v(k) \\ w(k) \end{bmatrix}. \end{aligned} \tag{55}$$

The second IQC multiplier is $\Pi_2 = \Psi_2^{\sim} M \Psi_2$ with $\Psi_2 = \left[\begin{smallmatrix} -0.5 & 0.3 \\ 0 & 1.7 \end{smallmatrix}\right]$. This is a static multiplier with no dynamics in Ψ_2. Both Ψ_1 and Ψ_2 are stable with stable inverses. Thus $\{(\Psi_i, M)\}_{i=1}^2$ are both J-spectral factorizations. Using the game-theoretic perspectives presented in Section 3, one can easily verify that $\{(\Psi_i, M)\}_{i=1}^2$ are both hard IQC factorizations.

The standard dissipation inequality (Theorem 1) with the IQC parameterization $\Pi(\lambda) = \lambda_1 \Pi_1 + \lambda_2 \Pi_2$ yields a worst case gain bound of $\gamma_1 = 6.16$. This is solved by minimizing γ subject to the LMI conditions $P \geq 0$, $\lambda \geq 0$, and $LMI_{(G,\Psi)}(P, M(\lambda), \gamma) < 0$. This corresponds to the use of $\Psi := [\Psi_1^T, \Psi_2^T]^T$ and $M(\lambda) := diag(\lambda_1 M, \lambda_2 M)$. Note that the vertically stacked multiplier Ψ is not square and $(\Psi, M(\lambda))$ is not a J-spectral factorization.

Using instead the modified dissipation inequality (Theorem 2), i.e. dropping the constraint $P \geq 0$, results in $\gamma_2 = 5.01$. The optimal decision variables in this case are $P^* = \left[\begin{smallmatrix} 39.83 & 2.40 \\ 2.40 & -0.46 \end{smallmatrix}\right]$, $\lambda_1^* = 64.76$, and $\lambda_2^* = 6.90$. The resulting P^* has eigenvalues at 39.98 and -0.61 and is therefore indefinite. The combined multiplier $\Pi(\lambda^*)$ is a Strict-PN multiplier. By Lemma 6, a J-spectral factorization $(\hat{\Psi}, \hat{J})$ of the combined multiplier $(\Psi, M(\lambda^*))$ can be constructed. The stabilizing solution of the corresponding DARE is $X = -0.72$. yields a storage function $\hat{P} = P^* - \left[\begin{smallmatrix} 0 & 0 \\ 0 & X \end{smallmatrix}\right] = \left[\begin{smallmatrix} 39.83 & 2.40 \\ 2.40 & 0.26 \end{smallmatrix}\right]$, as described in the proof of Theorem 2. As expected, $\hat{P} \geq 0$ with eigenvalues at 39.98 and 0.11. It can be verified that $LMI_{(G,\hat{\Psi})}(\hat{P}, \hat{J}, \gamma_2) < 0$.

This simple example demonstrates that enforcing $P \geq 0$ with multiple IQCs can yield conservative analysis results. Theorem 2 provides a valid dissipation inequality proof (under additional technical assumptions on $\Pi(\lambda)$) even if the constraint $P \geq 0$ and the hard IQC assumption are dropped.

6. CONCLUSIONS

The paper presented a discrete-time formulation for robustness analysis that combines integral quadratic constraints (IQCs) and dissipativity theory. A modified dissipation inequality was given that requires neither non-negative storage functions nor hard IQCs. The proof of this result used game-theoretic concepts and J-spectral factorizations. A simple numerical example was used to demonstrate that the modified dissipation inequality approach can lead to less conservative results than the standard dissipation inequalities.

ACKNOWLEDGMENTS

The authors acknowledge helpful comments and discussions with Joaquin Carrasco, Andrew Packard, Sei Zhen Khong, Mazen Farhood, and Anders Rantzer. The work was supported by the National Science Foundation Grant No. NSF-CMMI-1254129 entitled "CAREER: Probabilistic Tools for High Reliability Monitoring and Control of Wind Farms" and the NASA Langley NRA Cooperative Agreement NNX12AM55A entitled "Analytical Validation Tools for Safety Critical Systems Under Loss-of Control Conditions", Dr. Christine Belcastro technical monitor. It was also supported by the São Paulo Research Foundation (FAPESP) grant 2015/00269-5.

REFERENCES

1. Megretski A, Rantzer A. System analysis via integral quadratic constraints. *IEEE Transactions on Automatic Control* 1997; **42**:819–830.
2. Zhou K, Doyle J, Glover K. *Robust and Optimal Control*. Prentice-Hall, 1996.
3. Skogestad S, Postlethwaite I. *Multivariable Feedback Control*. John Wiley and Sons: Chichester, 2005.
4. Lessard L, Recht B, Packard A. Analysis and design of optimization algorithms via integral quadratic constraints. *SIAM Journal on Optimization* 2016; **26**(1):57–95.
5. Zames G, Falb P. Stability conditions for systems with monotone and slope-restricted nonlinearities. *SIAM Journal of Control* 1968; **6**(1):89–108.
6. Carrasco J, Turner M, Heath W. Zames-falb multipliers for absolute stability: from osheas contribution to convex searches. *European Control Conference*, 2015; 1261–178.
7. Jönsson U. Lectures on input-output stability and integral quadratic constraints. *Technical Report*, Royal Institute of Technology (KTH) 2001.
8. Willems J. Dissipative dynamical systems part I: General theory. *Archive for Rational Mech. and Analysis* 1972; **45**(5):321–351.
9. Willems J. Dissipative dynamical systems part II: Linear systems with quadratic supply rates. *Archive for Rational Mech. and Analysis* 1972; **45**(5):352–393.
10. Bart H, Gohberg I, Kaashoek M. *Minimal Factorization of Matrix and Operator Functions*. Birkhäuser, 1979.
11. Ionescu V, Oară C, Weiss M. *Generalized Riccati Theory and Robust Control: A Popov Function Approach*. Wiley, 1999.
12. Carrasco J, Seiler P. Integral quadratic constraint theorem: A topological separation approach. *Accepted to IEEE Conf. on Decision and Control*, 2015.
13. Pfifer H, Seiler P. Robustness analysis of linear parameter varying systems using integral quadratic constraints. *International Journal of Robust and Nonlinear Control* 2015; **25**(15):2843–2864.
14. Veenman J, Scherer C. Stability analysis with integral quadratic constraints: A dissipativity based proof. *IEEE Conf. on Decision and Control*, 2013; 3770–3775.
15. Seiler P. Stability analysis with dissipation inequalities and integral quadratic constraints. *IEEE Transactions on Automatic Control* 2015; **60**(6):1704–1709.
16. Pfifer H, Seiler P. Less conservative robustness analysis of linear parameter varying systems using integral quadratic constraints. To appear in International Journal of Robust and Nonlinear Control, 2016.
17. Meinsma G. J-spectral factorization and equalizing vectors. *Systems and Control Letters* 1995; **25**:243–249.
18. Engwerda J. *LQ Dynamic Optimization and Differential Games*. 1st edn., Wiley, 2005.
19. Engwerda J. Uniqueness conditions for the affine open-loop linear quadratic differential game. *Automatica* 2008; **44**:504–511.
20. Megretski A. KYP lemma for non-strict inequalities and the associated minimax theorem. Arxiv 2010.
21. Pfifer H, Seiler P. Integral quadratic constraints for delayed nonlinear and parameter-varying systems. *Automatica* 2015; **56**:36–43.
22. Turner M, Kerr M. L_2 gain bounds for systems with sector bounded and slope-restricted nonlinearities. *International Journal of Robust and Nonlinear Control* 2012; **22**(13):1505–1521.
23. Kao C. On stability of discrete-time LTI systems with varying time delays. *IEEE Transactions on Automatic Control* 2012; **57**:1243–1248.

24. Kao C, Chen M. Robust estimation with dynamic integral quadratic constraints: the discrete-time case. *IET Control Theory and Applications* 2013; **7**:15991608.
25. Wang S, Heath W, Carrasco J. A complete and convex search for discrete-time noncausal FIR Zames-Falb multipliers. *IEEE Conf. on Decision and Control*, 2014; 3918 – 3923.
26. Cantoni M, Jönsson U, Khong S. Robust stability analysis for feedback interconnections of time-varying linear systems. *SIAM J. of Control Optim.* 2013; **51**(1):353–379.
27. Khong SZ, Lovisari E, Rantzer A. A unifying framework for robust synchronisation of heterogeneous networks via integral quadratic constraints. To appear in IEEE Transactions on Automatic Control, 2016.
28. Dai L. *Singular Control Systems*, *Lecture Notes in Control and Information Sciences*, vol. 118. Springer, 1989.
29. Khalil H. *Nonlinear Systems*. Third edn., Prentice Hall, 2001.
30. Hu B, Seiler P. Exponential decay rate conditions for uncertain linear systems using integral quadratic constraints. To appear in IEEE Transactions on Automatic Control, 2017.
31. Vinnicombe G. *Uncertainty and Feedback - $\mathcal{H}_\infty$ loop-shaping and the ν-gap metric*. Imperial College Press, 2001.
32. Heath W, Wills A. Zames-Falb multipliers for quadratic programming. *IEEE Conf. on Decision and Control*, 2005; 963–968.
33. Kao C, Lincoln B. Simple stability criteria for systems with time-varying delays. *Automatica* 2004; **40**:1429–1434.
34. Francis B. *A Course in H_∞ Control Theory*. Springer-Verlag, 1987.
35. Jones B, Kerrigan E, Morrison J. A modeling and filtering framework for the semi-discretised Navier-Stokes equations. *European Control Conference*, 2009; 1215–1220.
36. Varga A. Task I.A.1 - Selection of basic software tools for standard and generalized state-space systems and transfer matrix factorizations. *Technical Report*, Subroutine Library in Systems and Control Theory (SLICOT) 1998.
37. Kågström B, Poromaa P. Computing eigenspaces with specified eigenvalues of a regular matrix pair (A, B) and condition estimation: theory, algorithms and software. *Numerical Algorithms* 1996; **12**(2):369–407.
38. Schaft A. *L_2-gain and passivity in nonlinear control*. Springer-Verlag New York, Inc., 1999.
39. Jönsson U. Stability analysis with Popov multipliers and integral quadratic constraints. *Systems* & *Control Letters* 1997; **31**(2):85 – 92.
40. Helmersson A. An IQC-based stability criterion for systems with slowly varying parameters. *Technical Report LiTH-ISY-R-1979*, Linköping University 1997.
41. Fu M, Dasgupta S, Soh Y. Integral quadratic constraint approach vs. multiplier approach. *Automatica* 2005; **41**:281–287.
42. Başar T, Olsder G. *Dynamic Noncooperative Game Theory*. 2nd edn., SIAM, 1999.
43. Başar T, Bernhard P. *H^∞-Optimal Control and Related Minimax Design Problems*. 2nd edn., Birkhäuser, 1995.
44. Molinari B. The stabilizing solution of the discrete algebraic Riccati equation. *IEEE Transactions on Automatic Control* 1975; **20**(3):396–399.
45. Horn R, Johnson C. *Matrix Analysis*. Cambridge University Press, 1990.
46. Farhood M, Dullerud G. Control of nonstationary LPV systems. *Automatica* 2008; **44**(8):2108 – 2119.
47. Farhood M. LPV control of nonstationary systems: A parameter-dependent Lyapunov approach. *IEEE Transactions on Automatic Control* 2012; **57**(1):209–215.
48. Wu F. Control of linear parameter varying systems. PhD Thesis, University of California, Berkeley 1995.
49. Wu F, Yang X, Packard A, Becker G. Induced $\mathcal{L}_2$ norm control for LPV systems with bounded parameter variation rates. *International Journal of Robust and Nonlinear Control* 1996; **6**:983–998.
50. Rantzer A. On the Kalman-Yakubovich-Popov lemma. *Systems and Control Letters* 1996; **28**(1):7–10.
51. CVX Research I. CVX: Matlab software for disciplined convex programming, version 2.0. `http://cvxr.com/cvx` Aug 2012.
52. Grant M, Boyd S. Graph implementations for nonsmooth convex programs. *Recent Advances in Learning and Control*, Blondel V, Boyd S, Kimura H (eds.). Lecture Notes in Control and Information Sciences, Springer-Verlag Limited, 2008; 95–110.
53. Tutuncu R, Toh K, Todd M. Solving semidefinite-quadratic-linear programs using SDPT3. *Mathematical Programming Ser. B* 2003; **95**:189–217.
54. Toh K, Todd M, Tutuncu R. SDPT3 - a matlab software package for semidefinite programming. *Optimization Methods and Software* 1999; **11**:545–581.
55. Katayama T. (J, J')-Spectral factorization and conjugation for discrete-time descriptor system. *Circuits, Systems and Signal Processing* 1996; **15**(5):649–669.
56. Ionescu V, Oara C, Weiss M. General matrix pencil techniques for the solution of algebraic Riccati equations: a unified approach. *IEEE Transactions on Automatic Control* 1997; **42**(8):1085–1097.
57. Ionescu V, Weiss M. On computing the stabilizing solution of the discrete-time Riccati equation. *Linear Algebra and its Applications* 1992; **174**:229 – 238.

APPENDIX

A. IQC MULTIPLIERS AND DARE STABILIZING SOLUTIONS

This appendix presents one key lemma stating that if Π satisfies the Strict-PN condition then there exists a stabilizing solution to a related DARE. The multiplier Π is assumed to be bounded on the unit circle but can, in general, be non-proper. Moreover, the feedthrough matrix can be singular. Hence the proof requires descriptor system notation and matrix pencil techniques to resolve both these issues. Some background on descriptor form and matrix pencil techniques can be found in [28, 55–57]. The proof in this appendix also relies on the connection between the invariant subspace of a Hamiltonian matrix and its related Riccati equation. A summary of the developments on this connection can be found in [56, Section III].

A few basic facts regarding descriptor form are provided before stating and proving the lemma. Consider a discrete-time system H in descriptor form:

$$\begin{aligned} Ex(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) + Du(k) \end{aligned} \tag{56}$$

This system has the transfer function $H(z) := C(zE - A)^{-1}B + D$. The matrix inversion lemma can be used to show $H^{\sim}(z) = -zB^T(zA^T - E^T)^{-1}C^T + D^T$. Thus $H^{\sim}$ has the following descriptor representation:

$$\begin{aligned} A^T x(k+1) &= E^T x(k) - C^T u(k) \\ y(k) &= B^T x(k+1) + D^T u(k) \end{aligned} \tag{57}$$

Next, a descriptor realization for $H^{-1}(z)$ is

$$\begin{aligned} \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} x_{in}(k+1) &= \begin{bmatrix} A & B \\ C & D \end{bmatrix} x_{in}(k) + \begin{bmatrix} 0 \\ -I \end{bmatrix} y(k) \\ u(k) &= \begin{bmatrix} 0 & I \end{bmatrix} x_{in}(k) \end{aligned} \tag{58}$$

where $x_{in}^T := [x^T \;\; u^T]$ is the state of the inverse system H^{-1}.

These facts are used to construct the required descriptor representations for the multiplier Π and its inverse Π^{-1}. Let $\Pi = \Psi^{\sim} M \Psi$ be any factorization with Ψ stable. Define $Q := C_\psi^T M C_\psi$, $S := C_\psi^T M D_\psi$ and $R := D_\psi^T M D_\psi$ where $(A_\psi, B_\psi, C_\psi, D_\psi)$ are the state matrices of Ψ. A descriptor representation for Π is given by:

$$\begin{aligned} E_\pi x_\pi(k+1) &= A_\pi x_\pi(k) + B_\pi u(k) \\ y(k) &= C_\pi x_\pi(k) + D_\pi u(k) \end{aligned} \tag{59}$$

where $x_\pi \in \mathbb{R}^{2n_\psi}$ is the state of Π and the descriptor matrices are defined as:

$$E_\pi := \begin{bmatrix} I & 0 \\ 0 & A_\psi^T \end{bmatrix}, \quad \left[\begin{array}{c|c} A_\pi & B_\pi \\ \hline C_\pi & D_\pi \end{array}\right] := \left[\begin{array}{cc|c} A_\psi & 0 & B_\psi \\ -Q & I & -S \\ \hline S^T & zB_\psi^T & R \end{array}\right] \tag{60}$$

Notice $C_\pi = \left[\, S^T \;\; zB_\pi^T \,\right]$ and hence $y(k)$ in (59) partially depends on $x_\pi(k+1)$. This is similar to Equation (57).

A descriptor representation for Π^{-1} is given by:

$$\begin{aligned} E_{in} x_{in}(k+1) &= A_{in} x_{in}(k) + B_{in} y(k) \\ u(k) &= C_{in} x_{in}(k) + D_{in} y(k) \end{aligned} \tag{61}$$

where $x_{in} := [x_\pi^T, \;\; u^T]^T \in \mathbb{R}^{2n_\psi + (n_v + n_w)}$ is the state of Π^{-1} and the matrices are defined as:

$$E_{in} := \begin{bmatrix} I & 0 & 0 \\ 0 & A_\psi^T & 0 \\ 0 & -B_\psi^T & 0 \end{bmatrix}, \quad \left[\begin{array}{c|c} A_{in} & B_{in} \\ \hline C_{in} & D_{in} \end{array}\right] := \left[\begin{array}{ccc|c} A_\psi & 0 & B_\psi & 0 \\ -Q & I & -S & 0 \\ S^T & 0 & R & -I \\ \hline 0 & 0 & I & 0 \end{array}\right] \tag{62}$$

It is emphasized that the filter Ψ is proper but the descriptor notation is required because A_ψ and/or R may be singular. In particular, if A_ψ is singular then $\Psi^{\sim}$ (and hence Π) is non-proper. In addition, if R is singular then Π^{-1} is non-proper. The lemma is now stated.

Lemma 7. Let $\Pi = \Psi^{\sim} M \Psi \in \mathbb{RL}_{\infty}^{(n_v+n_w)\times(n_v+n_w)}$ be any factorization with $\Psi \in \mathbb{RH}_{\infty}^{n_r\times(n_v+n_w)}$ and $M = M^T \in \mathbb{R}^{n_r\times n_r}$. Define $Q := C_{\psi}^T M C_{\psi}$, $S := C_{\psi}^T M D_{\psi}$ and $R := D_{\psi}^T M D_{\psi}$ where $(A_{\psi}, B_{\psi}, C_{\psi}, D_{\psi})$ are the state matrices of Ψ. If Π is a Strict-PN multiplier then there exists a unique, real, stabilizing solution $X = X^T$ to $DARE(A_{\psi}, B_{\psi}, Q, R, S)$. In addition, $R + B_{\psi}^T X B_{\psi}$ is nonsingular.

Proof
The multiplier Π has $2n_{\psi}$ zeros (possibly at $z = \infty$) where n_{ψ} is the state dimension of Ψ. These zeros are symmetric about the unit disk because $\Pi = \Pi^{\sim}$. The block-determinant formula yields

$$\det\left(\Pi\left(e^{j\omega}\right)\right) = \det\left(\Pi_{22}\left(e^{j\omega}\right)\right)\det\left(\Pi_{11}\left(e^{j\omega}\right) - \Pi_{12}\left(e^{j\omega}\right)\Pi_{22}^{-1}\left(e^{j\omega}\right)\Pi_{12}^{*}\left(e^{j\omega}\right)\right)$$

Then the Strict-PN conditions imply that Π is nonsingular, i.e. contains no zeros, on the unit circle. Therefore Π has n_{ψ} zeros strictly inside the unit circle. The poles of Π^{-1} are the zeros of Π and thus the matrix pencil $\lambda E_{in} - A_{in}$ has n_{ψ} generalized eigenvalues inside the unit disk. The generalized stable eigenspace of (E_{in}, A_{in}) is spanned by the columns of some matrix $X_s \in \mathbb{R}^{(2n_{\psi}+n_v+n_w)\times n_{\psi}}$. Hence there exists a Schur stable matrix $\Lambda \in \mathbb{R}^{n_{\psi}\times n_{\psi}}$ such that

$$A_{in} X_s = E_{in} X_s \Lambda \tag{63}$$

Partition $X_s = \left[X_1^T,\, X_2^T,\, X_3^T\right]^T$ compatibly with the blocks of A_{in} so that $X_1, X_2 \in \mathbb{R}^{n_{\psi}\times n_{\psi}}$ and $X_3 \in \mathbb{R}^{(n_v+n_w)\times n_{\psi}}$.

Next it is shown by contradiction that X_1 is nonsingular. Assume that X_1 is singular and let $\psi_0 \in \mathbb{R}^{n_{\psi}}$ denote a non-trivial vector in the null space of X_1. This vector cannot lie in the null space of $\left[\begin{smallmatrix} X_2 \\ X_3 \end{smallmatrix}\right]$ otherwise X_s would not span an n_{ψ}-dimensional space. Define the signals u, y, x_{π} as follows:

$$u(k) = \begin{cases} 0 & \text{for } k < 0 \\ X_3 \Lambda^k \psi_0 & \text{for } k \geq 0 \end{cases} \tag{64}$$

$$y(k) = \begin{cases} B_{\psi}^T (A_{\psi}^T)^{-k-1} X_2 \psi_0 & \text{for } k < 0 \\ 0 & \text{for } k \geq 0 \end{cases} \tag{65}$$

$$x_{\pi}(k) = \begin{cases} \left[\begin{smallmatrix} 0 \\ (A_{\psi}^T)^{-k} X_2 \psi_0 \end{smallmatrix}\right] & \text{for } k < 0 \\ \left[\begin{smallmatrix} X_1 \\ X_2 \end{smallmatrix}\right] \Lambda^k \psi_0 & \text{for } k \geq 0 \end{cases} \tag{66}$$

The signals u, y, and x_{π} are all in ℓ_2 ¶ since A_{ψ} and Λ are Schur stable matrices. In addition, u, x_{π}, and y are input, state, and output solutions for Π (Equation (59)) with boundary condition $x_{\pi}(0) = \left[\begin{smallmatrix} 0 \\ X_2 \psi_0 \end{smallmatrix}\right]$. This can be directly verified for $k < 0$. For $k \geq 0$, define $x_{in}(k) := X_s \Lambda^k \psi_0$. Use Equation (63) to show that $x_{in}(k)$ is a forward solution of Π^{-1} (Equation (61)) with initial condition $x_{in}(0) = X_s \psi_0$ and input $y(k) = 0$. This verifies that the signals u, y, and x_{π} defined above are also a solution to Π for $k \geq 0$. Therefore, the Fourier transforms of u and y, denoted as U and Y, satisfy

$$Y(e^{j\omega}) = \Pi(e^{j\omega}) U(e^{j\omega}) \;\; \forall \omega \in [0, 2\pi]$$

Partition the signals as $u = \left[\begin{smallmatrix} u_1^T & u_2^T \end{smallmatrix}\right]^T$ and $y = \left[\begin{smallmatrix} y_1^T & y_2^T \end{smallmatrix}\right]^T$ such that $u_1, y_1 \in \ell_2^{n_v}$ and $u_2, y_2 \in \ell_2^{n_w}$. By construction, the inner products satisfy $\langle u_1, y_1\rangle = \langle u_2, y_2\rangle = 0$. Use Parseval's theorem and the Strict-PN sign-definiteness conditions to show‖:

$$\begin{aligned} 0 = \langle u_1, y_1\rangle = \langle u_1, \Pi_{11} u_1 + \Pi_{12} u_2\rangle \geq \langle u_1, \Pi_{12} u_2\rangle \\ 0 = \langle u_2, y_2\rangle = \langle u_2, \Pi_{21} u_1 + \Pi_{22} u_2\rangle \leq \langle u_2, \Pi_{21} u_1\rangle \end{aligned}$$

¶A slight abuse of notation is used here as these are two-sided signals.

‖The inner product $\langle u_1, \Pi_{11} u_1\rangle$ can be interpreted, via Parseval's theorem, in the frequency domain. For example, $\langle u_1, \Pi_{11} u_1\rangle = \frac{1}{2\pi}\int_0^{2\pi} U_1(e^{j\omega})^* \Pi_{11}(e^{j\omega}) U_1(e^{j\omega}) d\omega$.

This immediately implies $\langle u_1, \Pi_{11} u_1 \rangle = \langle u_2, \Pi_{22} u_2 \rangle = 0$ because $\langle u_1, \Pi_{12} u_2 \rangle = \langle u_2, \Pi_{21} u_1 \rangle$. The Strict-PN conditions then yield $u_1 = u_2 = 0$ and hence $u = y = 0$. As a consequence $0 = u(0) := X_3 \psi_0$ and it must be that $X_2 \psi_0$ is non-trivial. In addition, $u = 0$ implies that $x_\pi(k)$ for $k \geq 0$ satisfies

$$E_\pi x_\pi(k+1) = A_\pi x_\pi(k) + B_\pi u(k) = A_\pi x_\pi(k)$$

This is impossible since the nontrivial initial condition $x_\pi(0) = \left[\begin{smallmatrix} 0 \\ X_2 \psi_0 \end{smallmatrix}\right]$ is in the antistable eigenspace of the pair (E_π, A_π) and this initial condition cannot yield a forward ℓ_2 solution $x_\pi(k) = \left[\begin{smallmatrix} X_1 \\ X_2 \end{smallmatrix}\right] \Lambda^k \psi_0$ for $k \geq 0$.

By contradiction, X_1 is nonsingular. Define $X := X_2 X_1^{-1}$. It follows from [57, Section 4] that $R + B_\psi^T X B_\psi$ is nonsingular and X is the unique stabilizing solution to $DARE(A_\psi, B_\psi, Q, R, S)$. This is a standard result and the remainder of the proof is only sketched. Define $K := -X_3 X_1^{-1}$ and $\tilde{X}_s := \left[\begin{smallmatrix} I & X^T & -K^T \end{smallmatrix}\right]^T$. Equation (63) is equivalent to

$$A_{in} \tilde{X}_s = E_{in} \tilde{X}_s \tilde{\Lambda} \tag{67}$$

where $\tilde{\Lambda} := X_1 \Lambda X_1^{-1}$ is a Schur stable matrix. This leads to the following three equations:

$$A_\psi - B_\psi K = \tilde{\Lambda} \tag{68}$$

$$-Q + X + SK = A_\psi^T X \tilde{\Lambda} \tag{69}$$

$$S^T - RK = -B_\psi^T X \tilde{\Lambda} \tag{70}$$

Substituting the expression for $\tilde{\Lambda}$ (Equation (68)) into Equation (70) yields $K = (R + B_\psi^T X B_\psi)^{-1} \left(A_\psi^T X B_\psi + S \right)^T$. This expression along with Equations (68) and (69) can be used to show, via standard manipulations, that X satisfies the $DARE(A_\psi, B_\psi, Q, R, S)$. Based on (68), $A_\psi - B_\psi K$ is a Schur stable matrix. Therefore, X is a stabilizing solution to the $DARE$. The above steps require a few additional facts to be demonstrated, e.g. X is symmetric and $R + B_\psi^T X B_\psi$ is nonsingular. These details can be found in [57]. $\square$

As mentioned before, the above lemma can also be proved by connecting the PN multipliers defined here with the coercive operator theory in [11, Theorem 4.12.8].