Estimating the region of attraction of uncertain systems with Integral Quadratic Constraints

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***Abstract*— A general framework for Region of Attraction (ROA) analysis is presented. The considered system consists of the feedback interconnection of a plant with polynomial dynamics and a bounded operator. The input/output behavior of the latter is characterized using an Integral Quadratic Constraint (IQC), for which it is assumed an hard factorization holds. This formulation allows to analyze problems involving hard-nonlinearities and uncertainties, adding to the state of practice typically limited to polynomial vector fields. An iterative algorithm based on Sum of Squares optimization is proposed to compute inner estimates of the ROA. The effectiveness of this approach is demonstrated on a numerical example featuring a nonlinear closed-loop system with saturated inputs.**

I. Introduction

The Region of Attraction (ROA) of an equilibrium point x^* is the set of all the initial conditions from which the trajectories of the system converge to x^* as time goes to infinity. Its knowledge is of practical interest to guarantee the safe operation of nonlinear systems, for which stability guarantees might hold only locally [1]. This paper proposes a new framework for the analysis of the region of attraction of systems with generic uncertainties. The considered problem consists of the feedback interconnection of a system G with polynomial vector field and a bounded casual operator Δ, which is described by Integral Quadratic Constraints (IQCs) [2]. The IQC paradigm characterizes a broad class of nonlinearities via so-called multipliers, and allows to refine the description of the uncertainties by specifying their nature. Therefore, the considered feedback interconnection G-Δ is quite general and covers a large class of nonlinear systems encountered in engineering applications.

The time domain interpretation of IQC, particularly its connection with dissipation inequalities [3], is instrumental to prove the main result of the paper. IQCs are typically expressed in the frequency domain, however a time domain counterpart exists via a (non-unique) factorization of the multipliers [2], [3], [4]. In here it will be assumed that the IQC is *hard*, in the sense that the time domain integral constraint holds over all finite times. This is not deemed overly restrictive, because many IQCs have factorizations holding this property. Indeed this applies to multipliers commonly used to characterize hard nonlinearities (e.g. sector and Zames-Falb multipliers) and various types of uncertainties (e.g. norm bounded, time varying and linear time invariant).

The state of practice to numerically calculate inner estimates of the ROA focuses on determining Lyapunov function level sets, which are contractive and invariant and thus are subsets of the ROA [1]. Even though non-Lyapunov methods have also been studied to reduce conservatism of the results [5], [6], a common feature of the approaches available in the literature is that they are applicable to polynomial vector fields only or that the algorithms used to compute ROA rely on Sum of Square (SOS) techniques, and this limits the types of nonlinearities that can be considered. Adding to this, a major drawback of the approaches usually employed to deal with uncertain systems is that they do not allow to specify the type of uncertainties [7]. This inherently leads to conservative results because they must hold for a larger set of uncertainties than the one actually affecting the system.

The technical contribution of this article is to propose a general and flexible framework for local stability analysis of nonlinear uncertain systems. The problem is formulated by defining an augmented plant which comprises the polynomial vector field G and the Linear Time Invariant (LTI) system provided by the factorization of the IQC. Building on this problem setup, Section III establishes the main result of the paper which gives certificates for regions of local stability. Specifically, the ROA is formulated as the level set of a polynomial function of generic degree (which is not necessarily a Lyapunov function for the system). Since the augmented plant and the sought function are polynomial, the problem is solved numerically via Sum of Squares (SOS) techniques [8]. In Section IV an aircraft polynomial short period model with actuator magnitude saturation is considered to showcase the capabilities of the framework. Strategies to exploit the bounds on the states provided by ROA are discussed with the aim of enhancing the IQC description and in doing so reducing the conservatism of the analyses.

II. Background

A. Notation

$\mathbb{RL}_\infty$ denotes the set of rational functions with real coefficients that are proper and have no poles on the imaginary axis. $\mathbb{RH}_\infty$ is the subset of functions in $\mathbb{RL}_\infty$ that are analytic in the closed right half of the complex plane. $\mathbb{RL}_\infty^{m\times n}$ and $\mathbb{RH}_\infty^{m\times n}$ denote the sets of $m\times n$ matrices whose elements are in $\mathbb{RL}_\infty$ and $\mathbb{RH}_\infty$ respectively. Vertical concatenation of two vectors $x\in\mathbb{R}^n$ and $y\in\mathbb{R}^m$ is denoted

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by $[x; y] \in \mathbb{R}^{n+m}$, whereas $x \cdot y$ indicates the scalar product between x and y. $\mathcal{L}_2^n$ is the space of all square integrable functions $v : [0, \infty] \to \mathbb{R}^n$, i.e. satisfying $\|v\|_2 < \infty$ where $||v||_2 := \left(\int_0^\infty v(t)^T v(t) dt \geq 0\right)^{1/2}$. The set of functions $g : \mathbb{R}^n \to \mathbb{R}$ which are m-times continuously differentiable is denoted by $\mathcal{C}^m$. $\mathbb{R}[x]$ indicates the set of all polynomials $r : \mathbb{R}^n \to \mathbb{R}$ in n variables, and $\partial(r)$ indicates the degree of r. Given a scalar $c > 0$, the level set of r is defined as $\Omega_{r,c} = \{x \in \mathbb{R}^n : r(x) \leq c\}$ and $\partial\Omega_{r,c}$ denotes its boundary. A polynomial g is said to be a Sum of Squares if there exists a finite set of polynomials $g_1, ..., g_k$ such that $g = \sum_{i=1}^k g_i^2$. The set of SOS polynomials in x will be denoted by $\Sigma[x]$.

B. Problem statement

Consider an autonomous nonlinear system of the form

$$\dot{x} = f(x), \quad x(0) = x_0 \tag{1}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is the vector field. The vector $x^* \in \mathbb{R}^n$ is called a *fixed* or *equilibrium* point of (1) if $f(x^*) = 0$. Let $\phi(t, x_0)$ denote the solution of (1) at time t with initial condition x_0. The ROA associated with x^* is defined as

$$\mathcal{R} := \left\{x_0 \in \mathbb{R}^n : \lim_{t\to\infty} \phi(t, x_0) = x^*\right\} \tag{2}$$

Thus $\mathcal{R}$ is the set of all initial states that converge to x^*.

A standard approach to compute inner estimates of the ROA consists in finding Lyapunov function level sets. Specifically, the next result (following directly from Lyapunov's direct method) is exploited:

Lemma 1: [1] Let $\mathcal{D} \subset \mathbb{R}^n$ and let $x^* \in \mathcal{D}$. If there exists a Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}$, with $V \in \mathcal{C}^1$ such that

$$\begin{aligned}
&V(x^*) = 0 \quad \text{and} \quad V(x) > 0 && \forall x \in \mathcal{D}\backslash x^* \\
&\nabla V(x) f(x) < 0 && \forall x \in \mathcal{D}\backslash x^* \\
&\Omega_{V,\gamma} = \{x \in \mathbb{R}^n : V(x) \leq \gamma\} && \text{is bounded and} \\
&\Omega_{V,\gamma} \subseteq \mathcal{D}
\end{aligned} \tag{3}$$

then $\Omega_{V,\gamma} \in \mathcal{R}$.

C. Sum of Squares optimization

If f is a polynomial vector field, a function V satisfying the conditions in Lemma 1 can be determined by means of Sums of Squares (SOS) techniques by exploiting their connection with convex optimization [8].

First, recall that $g \in \Sigma[x]$ if and only if there exists $Q{=}Q^T \succeq 0$ such that $g = z^T\ Q\ z$, where z gathers the monomials of g of degree less than or equal to $\partial(g)/2$. This problem can be recast as a semidefinite program and there are freely available toolboxes to solve this in an efficient manner [9]. An application of the Positivstellensatz (P-satz) Theorem, reported in Lemma 2, can then be applied to recast the set containment conditions (3) as SOS constraints.

Lemma 2: [8] Given $h, f_0, ..., f_r \in \mathbb{R}[x]$, the following set containment holds

$$\left\{x : h(x) = 0, f_1(x) \geq 0, ..., f_r(x) \geq 0\right\} \subseteq \left\{x : f_0(x) \geq 0\right\} \tag{4}$$

if there exist polynomials $p \in \mathbb{R}[x]$ and $s_1, ..., s_r \in \Sigma[x]$ such that

$$p(x)h(x) - \sum_{i=1}^r s_i(x) f_i(x) + f_0(x) \in \Sigma[x] \tag{5}$$

D. Integral Quadratic Constraints

IQCs provide a unified framework to assess the robustness of uncertain, nonlinear systems [2]. The basic idea is to describe the generic nonlinear uncertain operator Δ by means of IQCs on its input v and output w channels.

Let $\Pi : j\mathbb{R} \to \mathbb{C}^{(n_v+n_w)\times(n_v+n_w)}$ be a measurable Hermitian-valued function, commonly named multiplier. This multiplier is usually chosen among the rational functions bounded on the imaginary axis, i.e. $\Pi \in \mathbb{RL}_\infty^{(n_v+n_w)\times(n_v+n_w)}$. It is said that the two signals $v \in \mathcal{L}_2^{n_v}$ and $w \in \mathcal{L}_2^{n_w}$ satisfy the IQC defined by Π if

$$\int_{-\infty}^{+\infty} \begin{bmatrix} \hat{v}(j\omega) \\ \hat{w}(j\omega) \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} \hat{v}(j\omega) \\ \hat{w}(j\omega) \end{bmatrix} d\omega \geq 0 \tag{6}$$

where $\hat{v}$ and $\hat{w}$ indicate the Fourier transforms of the corresponding signals, and $*$ denotes their complex conjugate. A bounded and casual operator Δ: $\mathcal{L}_2^{n_v} \to \mathcal{L}_2^{n_w}$ is said to satisfy the frequency domain IQC defined by Π if the signals v and $w = \Delta(v)$ satisfy (6) for all v.

A library of IQCs exists for various types of uncertainties and nonlinearities as summarized in [2], [4], many of them conveniently derived in the frequency domain. However, for the purposes of the work, it is useful to connect frequency and time domain IQCs [3]. Let $\Pi \in \mathbb{RL}_\infty^{(n_v+n_w)\times(n_v+n_w)}$, and (Ψ, M) be a (non-unique) factorization of $\Pi = \Psi^\sim\ M\ \Psi$, where $M = M^T \in \mathbb{R}^{n_z\times n_z}$ and $\Psi \in \mathbb{RH}_\infty^{n_z\times(n_v+n_w)}$ is constructed from pre-selected basis transfer functions. Note that M is typically sign indefinite. By Parseval's theorem, substituting the proposed factorization of Π in (6), the following holds

$$\int_0^\infty z(t)^T M z(t) dt \geq 0 \tag{7}$$

where z is the output of the LTI system Ψ defined as

$$\begin{aligned}
\dot{x}_\Psi &= A_\Psi x_\Psi + B_{\Psi 1} v + B_{\Psi 2} w, \quad x_\Psi(0) = 0 \\
z &= C_\Psi x_\Psi + D_{\Psi 1} v + D_{\Psi 2} w
\end{aligned} \tag{8}$$

IQCs are a framework that generalizes the use of multipliers in the classical absolute stability problem [2], and Linear Matrix Inequalities involving state-space realizations of G and Ψ are typically employed to certify stability of the interconnection of G and Δ [3].

This work will consider only the class of bounded casual operators Δ satisfying the following time domain *hard* IQC

$$\int_0^T z(t)^T M z(t) dt \geq 0 \qquad \forall T \geq 0, \forall v \in \mathcal{L}_2^{n_v} \tag{9}$$

We will denote this by writing $\Delta \in \text{HardIQC}\ (\Psi, M)$. Note that the hard factorization property is stronger than the one in (7) (*soft*). A formulation of the ROA for the case when only the latter property holds is given in [10].

III. Region of Attraction estimation with IQC

A. Problem setup

The proposed framework aims to analyze the local stability of autonomous nonlinear systems of the form

$$\dot{x} = f(x, w) \tag{10a}$$
$$v = h(x, w) \tag{10b}$$
$$w = \Delta(v) \tag{10c}$$

where f and h are polynomial functions of x and w (defining the plant G), and Δ is a generic bounded operator (gathering nonlinearities and uncertainties with an IQC description). The prototype of systems considered by this work thus consists of the interconnection G-Δ (standard in robust control) where G is polynomial but not Δ in general, making therefore the combined system non-polynomial. In this work it will be assumed for simplicity, as commonly done in the literature [11], that the equilibrium point x^* is not a function of Δ. This could be relaxed, for example, with an extension of the algorithm proposed in [12], based on the concept of equilibrium-independent ROA. Without loss of generality, it will also be assumed $x^* = 0$.

Starting from the generic dynamics (10), the first step is to define the augmented plant sketched in Fig. 1. The feedback interconnection comprises the subsystems G (defined by (10a)-(10b)), Δ (10c), and Ψ (8). Introducing the vector $\tilde{x} = [x; x_\Psi]$, the plant can be reorganized as follows

$$\begin{aligned} \dot{\tilde{x}} &= F(\tilde{x}, w) \\ z &= H(\tilde{x}, w) \end{aligned} \tag{11}$$

where $F, H : \mathbb{R}^{n_{\tilde{x}}+n_w} \to \mathbb{R}^{n_{\tilde{x}}}$ are polynomial maps depending on G and Ψ. It is stressed that this manipulation of (10) does not make any assumption on Δ except the existence of a factorization Ψ for the associated multiplier Π.

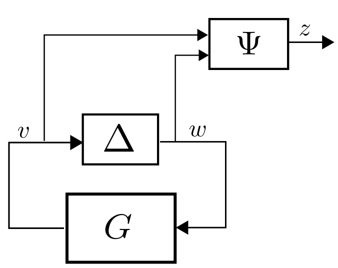

Fig. 1. Augmented plant for ROA analysis.

B. Region of Attraction certificates with IQCs

1) Main result: The proposed estimation of invariant subsets of the ROA for the original system in (10) is based on the following theorem.

Theorem 1: Let F be the polynomial vector field defined in (11) and $\Delta : \mathcal{L}_2^{n_v} \to \mathcal{L}_2^{n_w}$ be a bounded, causal operator. Further assume:

1) $\Delta \in$ HardIQC (Ψ, M)
2) There exist a smooth, continuously differentiable function V: $\mathbb{R}^{n_{\tilde{x}}} \to \mathbb{R}$, and a matrix $M = M^T$ such that

$$V(0) = 0 \quad \text{and} \quad V(\tilde{x}) > 0 \qquad \forall \tilde{x} \backslash \{0\} \tag{12a}$$
$$\nabla V(\tilde{x}) F(\tilde{x}, w) + z^T M z < 0 \tag{12b}$$
$$\forall \tilde{x} \in \Omega_{V,\gamma} \backslash \{0\}, \forall w \in \mathbb{R}^{n_w}$$

Then the intersection of $\Omega_{V,\gamma}$ with the hyperplane $x_\Psi = 0$ is an inner estimate of the ROA of (10).

Proof: The theorem assumes that the inequality (12b) holds only over the set $\Omega_{V,\gamma}$. Hence, the proof must ensure first that all the trajectories originating in $\Omega_{V,\gamma}$ remain within for all finite time. Assume there exists a $T_1 > 0$ such that $\tilde{x}(T_1) \notin \Omega_{V,\gamma}$, and define $T_2 := \inf_{\tilde{x}(T) \notin \Omega_{V,\gamma}} T$. Since F and H are polynomial maps, solutions of the Ordinary Differential Equations in Eq. (11) are continuous. Then $\tilde{x}(T_2) \in \partial\Omega_{V,\gamma}$ and $\tilde{x}(t) \in \Omega_{V,\gamma} \forall t \in [0, T_2]$. Therefore, it is possible to integrate the inequality (12b) in this range

$$V(\tilde{x}(T_2)) - V(\tilde{x}(0)) + \int_0^{T_2} z^T M z dt < 0 \tag{13}$$

By using the fact that $\Delta \in$ HardIQC (Ψ, M) it thus holds

$$\gamma = V(\tilde{x}(T_2)) < V(\tilde{x}(0)) \leqslant \gamma \tag{14}$$

This is contradictory and hence the assumption that $\exists T_1 > 0$ such that $\tilde{x}(T_1) \notin \Omega_{V,\gamma}$ is not true. Thus $\tilde{x}(0) \in \Omega_{V,\gamma}$ implies $\tilde{x}(t) \in \Omega_{V,\gamma}$ for all finite time (invariance of the level set).

Next, it is required to prove that the equilibrium point is attractive. Note first that (12b) still holds if the term $\epsilon \tilde{x}^d \cdot \tilde{x}^d$ (with $\epsilon > 0$ sufficiently small and d sufficiently large) is added on the left-hand side. Thus, integrating in the interval $[0, T]$ gives now

$$V(\tilde{x}(T)) - V(\tilde{x}(0)) + \int_0^T z^T M z dt + \epsilon \int_0^T \tilde{x}^d \cdot \tilde{x}^d dt \leq 0 \tag{15}$$

It follows from $\Delta \in$ HardIQC (Ψ, M) and $V(\tilde{x}) > 0$ that

$$\epsilon \int_0^T \tilde{x}^d \cdot \tilde{x}^d dt \leq V(\tilde{x}(0)) \tag{16}$$

In other words, $\tilde{x} \in \mathcal{L}_2^{n_{\tilde{x}}}$ inside the level set. A similar perturbation argument can be used to show that $v \in \mathcal{L}_2^{n_v}$ and hence $w \in \mathcal{L}_2^{n_w}$, due to boundness of Δ.
Let us now define $y = [\tilde{x}; w]$ and $\mathcal{D}_y = \{y(\tilde{x}, w) : \tilde{x} \in \Omega_{V,\gamma}, w \in \mathbb{R}^{n_w}\}$. The vector field F is a polynomial function of $\tilde{x}$ and w. Therefore, F is locally Lipschitz [1]

$$\|F(y_2) - F(y_1)\| \leq L\|(y_2 - y_1)\| \quad \forall y_1, y_2 \in \mathcal{D}_y \tag{17}$$

with L a real constant. In particular, for $y_1 = [0; 0]$ and a generic y_2, it holds that

$$\|F(y_2)\| \leq L\|(y_2)\| \tag{18}$$

which proves that $\dot{\tilde{x}} \in \mathcal{L}_2^{n_{\tilde{x}}}$ inside the level set. Finally, $(\dot{\tilde{x}}, \tilde{x}) \in \mathcal{L}_2$ implies that $\tilde{x} \to 0$ as $T \to \infty$ [13]. Therefore, all the trajectories originated by initial conditions in $\Omega_{V,\gamma}$ stay in the set and eventually converge to the equilibrium point. That is, $\Omega_{V,\gamma}$ is a subset of the ROA of (11). Note finally that $x_\Psi(0)$=0 (8), thus the intersection of $\Omega_{V,\gamma}$ with the hyperplane $x_\Psi = 0$ is a subset of the ROA of (10). ■

Remark 1: Note that V is not a Lyapunov function of (11). In fact, it is possible for V to non-decrease at some points in time. This is a consequence of the term $z^T M z$ which in general only provides integral (and not pointwise-in-time) constraints and thus prevents from guaranteeing that $\dot{V} < 0$.

Remark 2: It is common practice to tackle the stability problem of systems subject to polynomial nonlinearities with Lyapunov techniques, and the study of systems subject to hard nonlinearities (and uncertainties) with multipliers-based techniques. The proposed result allows to consider the asymptotic stability problem [1] of systems generically described by (10) within a unified framework. To determine whether or not an equilibrium point x^* is asymptotically stable (without determining its ROA) it suffices indeed to satisfy Theorem 1 in any domain $\mathcal{D} \subset \mathbb{R}^n$ containing x^*.

2) An SOS-algorithm for estimates of ROA: Theorem 1 gives sufficient conditions to find inner estimates of the ROA in terms of set containment constraints. These constraints only involve polynomial functions and thus can be enforced by making use of SOS techniques. The following program exploits Lemma 2 to determine a function V which satisfies the set containments in Theorem 1.

Program 1:

$$\begin{aligned} &\max_{s_1 \in \Sigma[\tilde{x}, w]; V \in \mathbb{R}[\tilde{x}]} \gamma \\ &V - L_1 \in \Sigma[\tilde{x}] \\ &-(\nabla V f + z^T M z + L_2) - s_1(\gamma - V) \in \Sigma[\tilde{x}, w] \end{aligned} \tag{19}$$

where $L_1 = \epsilon_1 \tilde{x}^T \tilde{x}$ and $L_2 = \epsilon_2 [\tilde{x}; w]^T [\tilde{x}; w]$, with ϵ_1 and ϵ_2 small positive constants (e.g.$\simeq 10^{-6}$). Since Program 1 features bilinear terms in s_1, γ and V, the following 2-steps algorithm is proposed with the aim to enlarge the provable ROA by solving a sequence of convex programs.

Algorithm 1:

Outputs: the level set $\Omega_{V,\gamma}$.

Inputs: a polynomial V^0 satisfying (3) for some γ.

1) **γ-Step** : solve for s_1, γ, M

$$\begin{aligned} &\gamma^* = \max_{s_1 \in \Sigma[\tilde{x}, w]} \gamma \\ &-(\nabla V^0 F + z^T M z + L_2) - s_1(\gamma - V^0) \in \Sigma[\tilde{x}, w] \end{aligned}$$

2) **V-Step** : solve for s_2, V, M

$$\begin{aligned} &V - L_1 \in \Sigma[\tilde{x}, w]; \\ &-(\nabla V F + z^T M z + L_2) - \bar{s}_1(\gamma^* - V) \in \Sigma[\tilde{x}, w] \\ &(\gamma^* - V) - s_2(\gamma^* - V^0) \in \Sigma[\tilde{x}] \end{aligned}$$

$\bar{s}_1$ in the V-step is kept at the value optimised in the γ-Step, whereas V^0 in the γ-step holds the value calculated at the end of the previous iteration. If the origin is asymptotically stable, the linearization of the dynamics about the origin can be used to compute a Lyapunov function V_{LIN} that serves as input V^0 to Algorithm 1. This iterative scheme is inspired by the V-s iteration from [14], with the difference that here the function V^0 is used to enlarge the set $\Omega_{V,\gamma}$ (last constraint in the V-step), instead of a preset shape function, and that the scheme consists only of two steps.

IV. Numerical example

A closed-loop plant with actuator magnitude saturation is used to demonstrate the applicability of the proposed approach. First, Algorithm 1 is applied, and then two strategies to reduce the conservatism of the results are commented.

1) Estimates of the ROA: The closed-loop short period motion of the NASA's Generic Transport Model (GTM) is approximated as a 2 states polynomial system [15]

$$\begin{aligned} \dot{\alpha} &= f_\alpha(\alpha, q, \delta) \\ \dot{q} &= f_q(\alpha, q, \delta) \\ \delta_{CMD} &= Kq \end{aligned} \tag{20}$$

where α is the angle of attack, q is the pitch rate, δ is the elevator deflection (all angles expressed in radians), and K is a positive constant gain. The GTM steady-state solution consists of a locally stable equilibrium point at the origin, i.e. $x^* = 0$. Previous studies focused on Region of Attraction analysis of the Open (OL) and Closed-Loop (CL) system [15], while this work assumes that the elevator δ is subject to actuator magnitude saturation, that is

$$\delta = \begin{cases} \delta^{sat}; & |\delta_{CMD}| > \delta^{sat} \\ \delta_{CMD}; & |\delta_{CMD}| \leq \delta^{sat} \\ -\delta^{sat}; & |\delta_{CMD}| < -\delta^{sat} \end{cases} \tag{21}$$

where δ^{sat} is a constant defining the saturation level. The dynamics (20)-(21) thus falls into the category of non-polynomial systems G-Δ described in Sec. III-A.

Saturation can be characterized by means of IQCs holding as finite-horizon time domain constraints (i.e. hard IQCs). In this work it is exploited that saturation is a memoryless, bounded, nonlinearity within the sector [σ, η]. The sector multiplier Π_S enforcing these properties is given by [2]

$$\begin{aligned} &\Pi_S = \begin{bmatrix} -2\sigma\eta & \sigma + \eta \\ \sigma + \eta & -2 \end{bmatrix}; \\ &\psi_S = I_2; \quad M_S = \lambda_S \Pi_S \end{aligned} \tag{22}$$

with λ_S positive optimization variable. The factorization of Π is straightforward here since the multiplier is static, but in general it has an effect on the results as investigated in [10].

Algorithm 1 is applied to determine inner estimates of the ROA. Fig. 2 shows the results obtained using a quartic level set (i.e $\partial(V) = 4$) for the open-loop (*OL*) and the closed-loop with saturation described by Π_S (CL_S). The sector $[0, 1]$ was considered in the definition of Π_S. The curve *OL* was obtained with a revised version of the V-s iteration [14] (consistent with the 2-steps algorithm presented in Section III-B.2). Note that the curves CL_S and *OL* are similar. In fact, the definition of the sector given above includes the open-loop system (i.e. δ=0) as a particular case. Thus, it is expected that the resulting ROA estimate cannot be larger than the *OL* one. In view of this, the fact that the curves almost overlap (within the numerical tolerances of the problem) hints at the fact that the OL case is the worst (i.e. having the smallest ROA) among those covered by the Π_S.

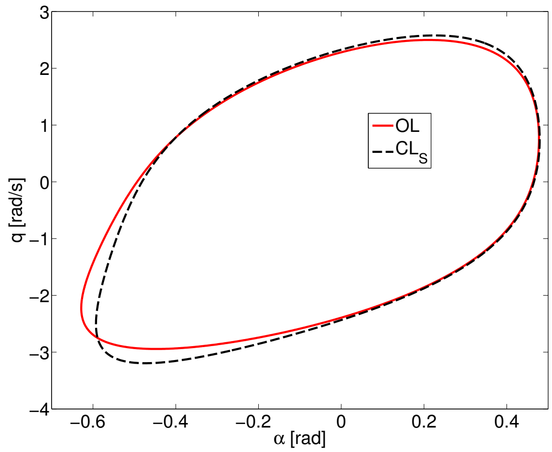

Fig. 2. Estimates of the region of attraction for the GTM model.

2) Relaxed sector constraints: As noted above, the sector IQC includes δ=0 as a special case. Moreover, the saturation level δ^{sat} does not influence the results. As a consequence the closed-loop ROA estimate is similar to the open-loop ROA. Two strategies are proposed to overcome these limitations. The premise for both, based on the notion of local IQCs [16], is sketched in Fig. 3, showing the relationship between commanded (δ_{CMD}) and saturated (δ) input. On the horizontal axis it is highlighted $\delta_{CMD}^{max} = Kq_{max}$, where q_{max} denotes the largest value for which q belongs to the region of attraction. It is then apparent that the lower bound σ of Π_S (22) has nonzero value, specifically $\sigma = \frac{\delta^{sat}}{K}\frac{1}{q_{max}}$.

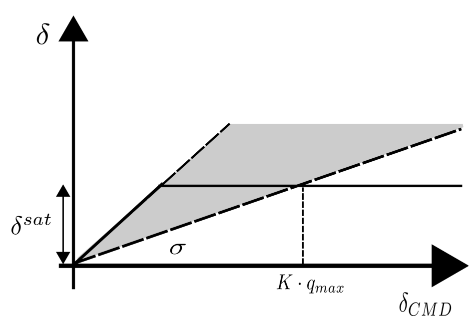

Fig. 3. Relaxed sector constraint exploiting bounds on the states.

The first strategy (*Strategy 1*) consists in writing the sector multiplier as the sum of two terms. The first is the standard one with fixed bounds $[0, 1]$, while the second has the refined sector description, that is

$$\Pi_S|_{[\sigma=0;\eta=1]} + \Pi_S|_{\left[\sigma=\frac{\delta^{sat}}{K}\frac{1}{q_{max}};\eta=1\right]} \tag{23}$$

where q_{max} is related to the coefficients of V. Because of the link between q_{max} and V, the second term in (23) cannot be multiplied by λ_S as done in the case of fixed bounds, since this would lead to bilinear terms. It is for this reason that also $\Pi_S|_{[\sigma=0;\eta=1]}$ is employed for the multiplier defined in (23).

The relationship between q_{max} and the polynomial V is discussed next. For quadratic V there is a known result which allows to express the maximum value held by a linear combination of the states on the level set $\Omega_{V,\gamma}$ via an LMI [17]. In case of generic ∂V, this can be enforced via SOS constraints. To this purpose, let us rewrite the second multiplier in (23) as

$$\begin{aligned}\Pi_S|_{\left[\sigma=\frac{\delta^{sat}}{K}\frac{1}{q_{max}};\eta=1\right]} &= \begin{bmatrix} -2\frac{\delta^{sat}}{K}\frac{1}{q_{max}} & \frac{\delta^{sat}}{K}\frac{1}{q_{max}}+1 \\ \frac{\delta^{sat}}{K}\frac{1}{q_{max}}+1 & -2 \end{bmatrix} \\ &= \begin{bmatrix} -2 & R_\sigma+1 \\ R_\sigma+1 & -2R_\sigma \end{bmatrix} \\ \text{with} \qquad R_\sigma &= \frac{K}{\delta^{sat}}q_{max} = \frac{1}{\sigma}\end{aligned} \tag{24}$$

where it has been used the property that if Δ satisfies the IQC given by Π, this still holds for $k\Pi$, with k positive scalar. The following algorithm, including also R_σ as decision variable, exploits the relaxed sector definition outlined above.

Algorithm 2:
Outputs: the level set $\Omega_{V,\gamma}$.
Inputs: a polynomial V^0 satisfying (3) for some γ.

1) γ-Step : solve for $s_1, s_{f_+}, s_{f_-}, R_\sigma, \gamma, M_i$

$$\gamma^* = \max_{s_1\in\Sigma[\tilde{x},w], s_{f_+}, s_{f_-}\in\Sigma[\tilde{x}], R_\sigma} \gamma$$
$$-(\nabla V^0 F + z^T M z + L_2) - s_1(\gamma - V^0) \in \Sigma[\tilde{x}, w]$$
$$f_+ - s_{f_+}(\gamma - V^0) \in \Sigma[\tilde{x}]$$
$$f_- - s_{f_-}(\gamma - V^0) \in \Sigma[\tilde{x}]$$

2) V-Step : solve for s_2, R_σ, V, M_i

$$V - L_1 \in \Sigma[\tilde{x}, w]; \quad V(0) = 0;$$
$$-(\nabla V F + z^T M z + L_2) - \bar{s}_1(\gamma^* - V) \in \Sigma[\tilde{x}, w]$$
$$(\gamma^* - V) - s_2(\gamma^* - V^0) \in \Sigma[\tilde{x}]$$
$$f_+ - \bar{s}_{f_+}(\gamma - V) \in \Sigma[\tilde{x}]$$
$$f_- - \bar{s}_{f_-}(\gamma - V) \in \Sigma[\tilde{x}]$$

where

$$\begin{aligned} f_+ &= q + \frac{\delta^{sat}}{K}R_\sigma \\ f_- &= -q + \frac{\delta^{sat}}{K}R_\sigma \end{aligned} \tag{25}$$

With this definition of f_+ and f_-, Algorithm 2 guarantees that at each iteration $-q_{max} \le q \le q_{max}$, and in doing so it enables to employ a less conservative sector.

This approach has the appealing feature that the lower bound of the sector is part of the optimization (via the decision variable R_σ). However, it has the drawback that the degree of freedom given by the decision variable λ_S cannot be fully exploited, as noted above. Prompted by this, an alternative strategy (*Strategy 2*) is devised. The idea is to determine q_{max} at the end of each iteration of Algorithm 1, and based on that update the lower bound σ. The expression of Π_S used at iteration $n+1$ is thus given by

$$\Pi_S|^{n+1} = \Pi_S|_{\left[\sigma=\frac{\delta^{sat}}{K}\frac{1}{q_{max}|^n};\eta=1\right]} \tag{26}$$

where the value $q_{max}|^n$ can be computed at the end of iteration n of Algorithm 1 with the following SOS program

$$\begin{aligned} q_{max}|^n = & \max_{s_{f_+}, s_{f_-} \in \Sigma[\tilde{x}]} q_{max} \\ & q + q_{max} - s_{f_+}(\gamma|^n - V|^n) \in \Sigma[\tilde{x}] \\ & -q + q_{max} - s_{f_-}(\gamma|^n - V|^n) \in \Sigma[\tilde{x}] \end{aligned} \tag{27}$$

This strategy has the desired property that the sector employed at iteration $n+1$ (function of $q_{max}|^n$) is always consistent with the ROA computed at the same iteration. This results from the fact that the computed ROA is non-decreasing throughout the iterations, therefore $q_{max}|^{n+1} \geq q_{max}|^n$. It is also noted that an iterative approach similar to the one in *Strategy 2* was employed in [18] to obtain less conservative predictions in the context of worst-case performance analysis of saturated systems.

Fig. 4 displays the estimates of the region of attraction obtained with Strategy 1 (*Str. 1*) and Strategy 2 (*Str. 2*) commented in this section. Two levels of saturation (δ^{sat}=[0.05, 0.1] rad) are considered. The curves for the open-loop *OL* (same as in Fig. 2) and closed-loop *CL* (i.e. the dynamic in (20) without saturation on the elevator) cases are also reported for comparison.

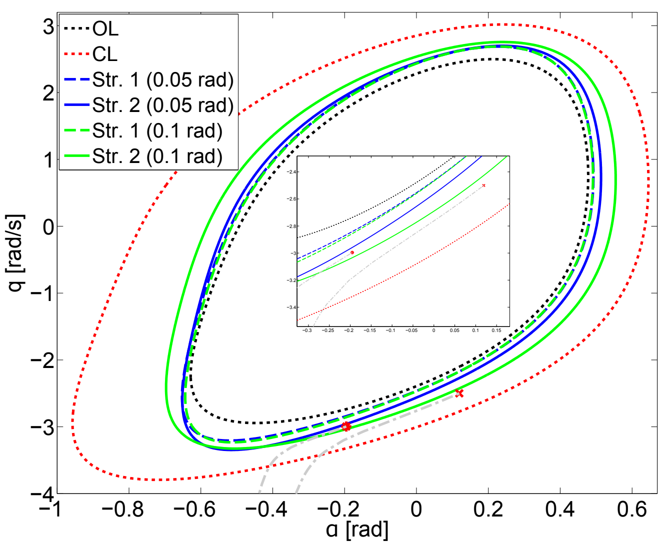

Fig. 4. Estimates of the ROA with relaxations on the sector IQC.

The first remark is that all the curves obtained with the proposed approaches are larger than the *OL* curve, i.e. they provide less conservative results than the estimate CL_S in Fig. 2. It is also worth noting that the provided estimates of ROA do now depend on δ^{sat}, which was not possible previously. Specifically, as the value of δ^{sat} is increased, the corresponding curves get closer to the closed-loop one, as expected. A measure of the conservatism of the results is also provided by means of extensive simulation campaigns. Fig. 4 shows two markers (cross for δ^{sat}=0.1 rad and asterisk for δ^{sat}=0.05 rad) corresponding to initial conditions for which the dynamics was found unstable, and the relative escaping trajectories in dashed-dotted lines. The zoom in the plot allows to appreciate the small gap between markers and corresponding ROA, suggesting that the effect of saturation is quantitatively predicted by the analyses.

V. Conclusions

This paper presents a new framework for region of attraction analysis of systems affected by generic uncertainties. Non-polynomial nonlinearities and uncertainties are described by means of Integral Quadratic Constraints for which an hard factorization exists. The main result, achieved by exploiting the connection between IQC and dissipation inequalities, gives sufficient conditions to determine inner estimates of the ROA which are not necessarily level sets of a Lyapunov function. An iterative algorithm based on SOS techniques is proposed to enlarge the provable ROA. This is applied to a case study with polynomial nonlinearities and saturation. Two strategies which combine the bounds on the states provided by the ROA with the definition of the sector multiplier are discussed. It is shown that they both enable to provide larger stability regions than those obtained by using the standard sector definition.

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