A Newtonian Development of the Mean-Axis Reference Frame for Flexible Aircraft

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I. Introduction

The objective of this work is to provide additional insight into the mean-axis equations of motion for flexible aircraft. Models for the flight dynamics of flexible aircraft can be formulated using body-referenced mean axes. The mean axes are defined by constraints on the internal translational and angular momentum of the aircraft relative to this frame. This implicitly defines the equations of motion for the body-referenced frame. The equations of motion for the flexible body are then frequently developed with Lagrangian mechanics. In this note, the translational and rotational equations of motion for the mean-axis reference frame are instead derived from first principles utilizing Newtonian mechanics. It is shown that the mean-axis constraints uniquely define the equations of motion for the body-referenced frame (up to an arbitrary, constant rotation of the frame). Moreover, the translational and rotational equations of motion formulated with the mean axes are similar to those for a rigid body.

This work is motivated by the increasing need for models of flexible aircraft that are appropriate for control design. As engineers seek to design and produce more fuel-efficient aircraft, the resulting trends are of reduced structural mass and increased wing aspect ratio. This leads to increasingly flexible aircraft, which present unique control challenges. Typically, aircraft are designed such that flutter and excessive vibrations will not occur within the flight envelope, and controllers are designed assuming a rigid aircraft. As aircraft become more flexible, however, they will require that controllers provide integrated rigid body and vibration control. In turn, these controllers require that models are developed which capture the essential dynamics of the system but provide the simplicity necessary for control design. The simplicity requirement translates into a model with a relatively low number of states, on the order of ten rather than thousands as may be found in high fidelity computational models. The requirement that the model capture the essential dynamics necessitates that the model accurately describes both the rigid body and the elastic degrees of freedom, as well as critical interactions between them such as body-freedom flutter.

When modeling the flight dynamics of flexible aircraft, the use of the mean-axis formulation of the equations of motion dates back to the early work of Milne in the mid 1960's [1]. The modeling approach has grown more

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popular with the advent of finite element methods for characterizing the free vibrations of the aircraft structure, and mean-axis models have been used in a wide variety of applications. Such applications include nonlinear real-time piloted simulations [2], flight dynamics and flutter analyses [3], structural loads analysis [4], and control law synthesis for active flutter suppression [5]. Some of the advantages of using mean-axis models are 1) the state vector is a direct extension of the state vector in rigid aircraft models, 2) the nonlinear dynamics of the rigid body degrees of freedom may be modeled, 3) the model may be parameterized using non-dimensional aerodynamic and aeroelastic coefficients, thus making the model valid over a region of the flight envelope rather than just one flight condition, 4) models of low dynamic order may be obtained that are especially attractive when using multivariable control techniques, and 5) the model structure and format is familiar to flight dynamicists. Recent work has applied the mean-axis to example aircraft of varying complexity [6, 7]. In addition, the validity of mean-axis models has been demonstrated by comparing flutter predictions to those obtained from computational models[3], and by comparing model-based transient responses with those obtained in flight tests [8].

Despite their benefits, mean-axis models present several challenges. For new users of the mean-axis modeling technique, it can be difficult to gain intuition for the physical meaning of the floating reference frame. To gain additional insight into the mean-axis technique, an alternate Newtonian derivation for a system of particles is presented for the rigid body degrees of freedom. The derivation is typically carried out using Lagrangian methods for a body with distributed mass [2, 3, 9, 10]. The Newtonian derivation may offer new insight as it approaches the dynamics from a momentum perspective, rather than an energy perspective. Furthermore, the simplicity of the system of particles, as opposed to a body with distributed mass, is meant to allow for additional insight into the derivation. The full equations of motion also include a set of vector equations that govern the elastic deformation of the body. Although not presented here, the full equations can be found in existing literature [2, 3, 9].

II. Notation and Problem Formulation

A. Particle Dynamics

Consider a deformable body consisting of n particles with mass m_i $(i = 1, \ldots, n)$ as shown on the left of Figure 1. Each particle is free to translate in three directions, and the position of particle i in the inertial frame I is denoted by $\boldsymbol{r}_i$. Each particle is acted upon by internal and external forces. The external force on particle i is denoted by $\boldsymbol{F}_i$. The internal force on particle i due to particle j is denoted by $\boldsymbol{F}_{ij}$. By Newton's Third Law, the internal forces between particles i and j are assumed to be equal and opposite: $\boldsymbol{F}_{ij} = -\boldsymbol{F}_{ji}$. Moreover, the internal forces are assumed to act along the line between the two particles: $\boldsymbol{F}_{ij} = |\boldsymbol{F}_{ij}|(\boldsymbol{r}_j - \boldsymbol{r}_i)$. The dynamics for this deformable body are specified by

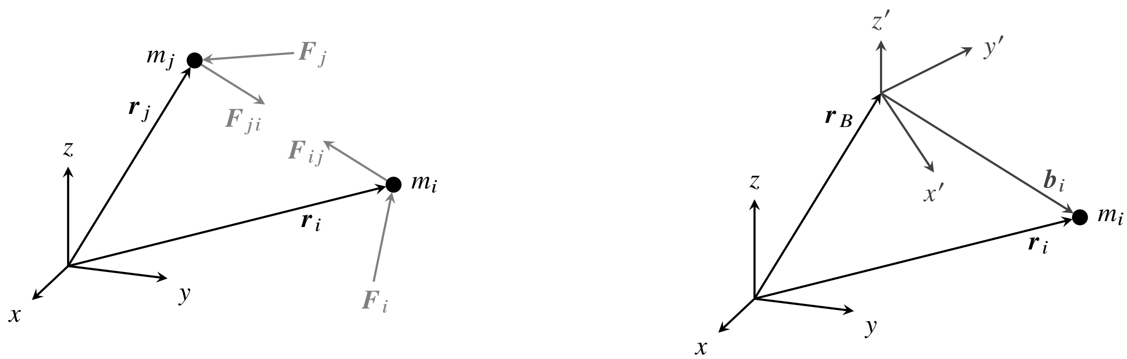

Fig. 1 Left: Notation for system of particles in an inertial (x, y, z) frame. Right: Notation for system using a body (x', y', z') frame.

Newton's Second Law:

$$
\begin{gathered}
m_i \ddot{\boldsymbol{r}}_i = \boldsymbol{F}_i + \sum_{j \neq i} \boldsymbol{F}_{ij} \qquad \text{for } i = 1, \ldots, n \\
\text{Initial Conditions: } \{\boldsymbol{r}_i(0)\}_{i=1}^n \text{ and } \{\dot{\boldsymbol{r}}_i(0)\}_{i=1}^n
\end{gathered} \tag{1}
$$

These equations of motion consist of n vector, second-order differential equations for the inertial positions $\boldsymbol{r}_i$. The initial conditions specify the position and velocity for each particle at $t = 0$. These dynamics can be rewritten as $3n$ scalar, second-order differential equations in terms of the (x, y, z) components of the various vectors. In this framework, internal forces are those that act only between particles. This implies that the body is unrestrained, meaning that it is not attached to a fixed point outside of the body.* The dynamic equations can be rewritten using a *body-reference* frame B. The right side of Figure 1 shows the inertial frame I, denoted (x, y, z), and the body frame B, denoted (x', y', z'). The body frame has origin at $\boldsymbol{r}_B$ and orientation given by an arbitrary Euler angle sequence. The body-reference frame moves with the body (in some manner that is not yet specified), but it is not necessarily attached to a particle or material point on the body. Hence, the particles may be located arbitrarily with respect to the origin of the reference frame $\boldsymbol{r}_B$. The vector $\boldsymbol{b}_i$ denotes the position of particle i relative to $\boldsymbol{r}_B$. The particle positions specified in the inertial and body frames are related as follows:

$$
\boldsymbol{r}_i = \boldsymbol{r}_B + \boldsymbol{b}_i \qquad \text{for } i = 1, \ldots, n \tag{2}
$$

Expressed using inertial derivatives, the acceleration of the i^{th} particle is simply:

$$
\ddot{\boldsymbol{r}}_i = \ddot{\boldsymbol{r}}_B + \ddot{\boldsymbol{b}}_i \qquad \text{for } i = 1, \ldots, n \tag{3}
$$

*Generally, only unrestrained bodies will be considered here. If connections to a fixed point do exist, however, they would be considered external forces in this framework.

Assuming that the translation of the reference frame $\boldsymbol{r}_B$ is known, substitute for $\ddot{\boldsymbol{r}}_i$ in Equation 1 to obtain an additional form of the dynamics:

$$\begin{gathered} m_i \left(\ddot{\boldsymbol{r}}_B + \ddot{\boldsymbol{b}}_i \right) = \boldsymbol{F}_i + \sum_{j \neq i} \boldsymbol{F}_{ij} \qquad \text{for } i = 1, \dots, n \\ \text{Initial Conditions: } \{\boldsymbol{b}_i(0)\}_{i=1}^n \text{ and } \{\dot{\boldsymbol{b}}_i(0)\}_{i=1}^n \end{gathered} \tag{4}$$

Expressing the dynamics with relative derivatives, rather than inertial derivatives, is often more natural when using a reference frame, although some additional work is required. Let ω and $\dot{\omega}$ denote the angular velocity and acceleration of frame B. The time derivative of a vector $\boldsymbol{b}_i$ in the body frame is given by the Transport Theorem [11]:

$$\dot{\boldsymbol{b}}_i = \mathring{\boldsymbol{b}}_i + \omega \times \boldsymbol{b}_i \tag{5}$$

Here $\dot{\boldsymbol{b}}_i = \frac{d}{dt}\Big|_I \boldsymbol{b}_i$ and $\mathring{\boldsymbol{b}}_i = \frac{d}{dt}\Big|_B \boldsymbol{b}_i$ denote time derivatives with respect to the inertial and body frames, respectively. Note that the Transport Theorem implies that $\dot{\omega} = \mathring{\omega}$, i.e. the derivative of ω is the same in the inertial and body frames since $\omega \times \omega = 0$. It follows from the Transport Theorem that the first and second derivatives of $\boldsymbol{r}_i$ are:

$$\dot{\boldsymbol{r}}_i = \dot{\boldsymbol{r}}_B + \mathring{\boldsymbol{b}}_i + \omega \times \boldsymbol{b}_i \tag{6}$$

$$\ddot{\boldsymbol{r}}_i = \ddot{\boldsymbol{r}}_B + \overset{\circ\circ}{\boldsymbol{b}}_i + \dot{\omega} \times \boldsymbol{b}_i + \omega \times (\omega \times \boldsymbol{b}_i) + 2\omega \times \mathring{\boldsymbol{b}}_i \tag{7}$$

Substitute for $\ddot{\boldsymbol{r}}_i$ in Equation 1 to obtain the dynamics expressed using the body frame B:

$$\begin{gathered} m_i \left(\ddot{\boldsymbol{r}}_B + \overset{\circ\circ}{\boldsymbol{b}}_i + \dot{\omega} \times \boldsymbol{b}_i + \omega \times (\omega \times \boldsymbol{b}_i) + 2\omega \times \mathring{\boldsymbol{b}}_i \right) = \boldsymbol{F}_i + \sum_{j \neq i} \boldsymbol{F}_{ij} \qquad \text{for } i = 1, \dots, n \\ \text{Initial Conditions: } \{\boldsymbol{b}_i(0)\}_{i=1}^n \text{ and } \{\mathring{\boldsymbol{b}}_i(0)\}_{i=1}^n \end{gathered} \tag{8}$$

The motion of the body frame appears in these dynamics due to its translational acceleration $\ddot{\boldsymbol{r}}_B$ as well as its angular velocity ω and acceleration $\dot{\omega}$. A specific choice for the body frame will be discussed in the subsequent sections. For now, assume the motion of the body frame is given. In this case, Equation 8 consists of n vector, second-order differential equations for the positions $\boldsymbol{b}_i$ in the body frame. Again, these dynamics can be rewritten as $3n$ scalar, second-order differential equations in terms of the (x', y', z') components of the various vectors. The initial conditions for Equation 8 are specified in the body frame. The following equations relate these body frame initial conditions to those given in the inertial frame:

$$\boldsymbol{b}_i(0) = \boldsymbol{r}_i(0) - \boldsymbol{r}_B(0) \tag{9}$$

$$\overset{\circ}{\boldsymbol{b}}_i(0) = \dot{\boldsymbol{r}}_i(0) - \dot{\boldsymbol{r}}_B(0) - \boldsymbol{\omega}(0) \times (\boldsymbol{r}_i(0) - \boldsymbol{r}_B(0)) \tag{10}$$

B. Mean-Axis Constraints

The dynamics expressed using a body-reference frame B (Equation 8) are valid for arbitrary translational and rotational motion of the frame. A particularly useful choice is a frame that satisfies the *mean-axis constraints* [1, 9, 12]. Specifically, the mean-axis constraints define a body frame for which there is no internal translational or angular momentum. Internal momentum is defined as momentum due to relative position and velocity with respect to the reference frame. To be precise, the internal translational momentum $\boldsymbol{P}_{int}$ and angular momentum $\boldsymbol{H}_{int}$ in frame B are given by:

$$\boldsymbol{P}_{int} := \sum_{i=1}^{n} m_i \overset{\circ}{\boldsymbol{b}}_i \tag{11}$$

$$\boldsymbol{H}_{int} := \sum_{i=1}^{n} m_i \boldsymbol{b}_i \times \overset{\circ}{\boldsymbol{b}}_i \tag{12}$$

The mean-axis constraints are $\boldsymbol{P}_{int}(t) = 0$ and $\boldsymbol{H}_{int}(t) = 0$ for all time $t \geq 0$. These constraints implicitly define the motion of the body-reference frame B. However, there is an ambiguity in the mean axes because these constraints are in terms of the internal translational and rotational momentum, implying constraints on velocity rather than position. In particular, the initial position and rotation of the axes are not specified. Hence, the constraints $\boldsymbol{P}_{int}(t) = 0$ and $\boldsymbol{H}_{int}(t) = 0$ only define the mean axes up to constant translational and rotational offsets. The ambiguity in the translational offset is removed by requiring the origin of the mean axes to initially be located at the center of mass. This specific translational offset plays a critical role in simplifying the equations of motion using the body-reference frame. To summarize, the mean axes are formally defined below.

Definition 1 (Mean Axes): The mean axes[†] are a body-reference frame B that satisfy the following two conditions:

A) *Translational Motion:* Frame B has no internal translational momentum, i.e. $\boldsymbol{P}_{int}(t) = 0$ for all $t \geq 0$. Moreover, the origin of B is located at the center of mass at the initial time, i.e. $\sum_{i=1}^{n} m_i \boldsymbol{b}_i(0) = 0$.

B) *Rotational Motion:* Frame B has no internal angular momentum, i.e. $\boldsymbol{H}_{int}(t) = 0$ for all $t \geq 0$. ■

As noted above, the mean-axis constraints implicitly define the motion of the mean axes. The next sections show that these constraints are equivalent to explicit equations of motion for the translation and rotation of the frame.

[†]Constant offsets in the angular orientation of the mean axes do not play a critical role in simplifying the equations of motion. Hence this definition allows for any angular orientation of the axes at the initial time. As a result there is a set of mean axes all related by constant rotational offsets. "The" mean axes refers to any one of these axes.

III. Translation of the Mean Axes

This section focuses on the translational mean-axis constraint in Definition 1.A. First, define the vector $\boldsymbol{p} := \sum_{i=1}^{n} m_i \boldsymbol{b}_i$ and the total mass of the body $m_{tot} := \sum_{i=1}^{n} m_i$. Note that $\frac{1}{m_{tot}}\boldsymbol{p}$ is the location of the center of mass in the body frame. Moreover, the internal translational momentum is $\boldsymbol{P}_{int} = \overset{\circ}{\boldsymbol{p}}$. Sum the n body-referenced differential equations with inertial derivatives only (Equation 4) to obtain the following differential equation for $\boldsymbol{p}$:

$$\begin{aligned} & m_{tot}\ddot{\boldsymbol{r}}_B + \ddot{\boldsymbol{p}} = \boldsymbol{F}_{ext} \qquad \text{for } i = 1, \dots, n \\ & \text{Initial Conditions: } \boldsymbol{p}(0) = \sum_{i=1}^{n} m_i \boldsymbol{b}_i(0) \text{ and } \dot{\boldsymbol{p}}(0) = \sum_{i=1}^{n} m_i \dot{\boldsymbol{b}}_i(0) \end{aligned} \tag{13}$$

where $\boldsymbol{F}_{ext} := \sum_{i=1}^{n} \boldsymbol{F}_i$ is the net external force. Note that the internal forces $\boldsymbol{F}_{ij}$ sum to zero since they are assumed to be equal and opposite. Additionally, the Transport Theorem yields a constraint on the solution of Equation 13. This constraint relates the absolute derivative, $\dot{\boldsymbol{p}}$, and the relative derivative, $\overset{\circ}{\boldsymbol{p}}$:

$$\begin{aligned} & \dot{\boldsymbol{p}} = \overset{\circ}{\boldsymbol{p}} + \boldsymbol{\omega} \times \boldsymbol{p} \qquad \text{for } i = 1, \dots, n \\ & \text{Initial Conditions: } \boldsymbol{p}(0) = \sum_{i=1}^{n} m_i \boldsymbol{b}_i(0) \text{ and } \overset{\circ}{\boldsymbol{p}}(0) = \sum_{i=1}^{n} m_i \overset{\circ}{\boldsymbol{b}}_i(0) = \boldsymbol{P}_{int}(0) \end{aligned} \tag{14}$$

Equations 13 and 14, which govern $\boldsymbol{p}$, are used in the next theorem to provide an explicit equation of motion corresponding to the translational mean-axis constraint.

Theorem 1: The frame B satisfies the translational mean-axis constraint (Definition 1.A) if and only if $\boldsymbol{r}_B$ satisfies the following equation of motion:

$$\begin{aligned} & m_{tot}\ddot{\boldsymbol{r}}_B = \boldsymbol{F}_{ext} \\ & \text{Initial Conditions: } \boldsymbol{r}_B(0) = \frac{1}{m_{tot}} \sum_{i=1}^{n} m_i \boldsymbol{r}_i(0) \text{ and } \dot{\boldsymbol{r}}_B(0) = \frac{1}{m_{tot}} \sum_{i=1}^{n} m_i \dot{\boldsymbol{r}}_i(0) \end{aligned} \tag{15}$$

Proof: ($\Rightarrow$) Assume the frame B satisfies the translational mean-axis constraint (Definition 1.A). This implies that $\overset{\circ}{\boldsymbol{p}}(t) = \boldsymbol{P}_{int}(t) = 0$ for all $t \geq 0$ and $\boldsymbol{p}(0) = \sum_{i=1}^{n} m_i \boldsymbol{b}_i(0) = 0$. Equation 14 reduces to the following constraint:

$$\dot{\boldsymbol{p}}(t) = \boldsymbol{\omega}(t) \times \boldsymbol{p}(t) \tag{16}$$

The initial condition is $\boldsymbol{p}(0) = 0$. The unique solution to this differential equation is $\boldsymbol{p}(t) = 0$ for all $t \geq 0$ and for all $\boldsymbol{\omega}$. This further implies $\ddot{\boldsymbol{p}}(t) = 0$ for all $t \geq 0$. In this case Equation 13 simplifies to $m_{tot}\ddot{\boldsymbol{r}}_B = \boldsymbol{F}_{ext}$. Moreover, it follows from the relation $\boldsymbol{r}_i = \boldsymbol{r}_B + \boldsymbol{b}_i$ that:

$$\sum_{i=1}^{n} m_i \boldsymbol{r}_i(0) = m_{tot} \boldsymbol{r}_B(0) + \boldsymbol{p}(0) \tag{17}$$

Therefore $\boldsymbol{p}(0) = 0$ (assumed by the translational mean-axis constraint) implies $\boldsymbol{r}_B(0) = \frac{1}{m_{tot}} \sum_{i=1}^{n} m_i \boldsymbol{r}_i(0)$. It can similarly be shown that $\dot{\boldsymbol{r}}_B(0) = \frac{1}{m_{tot}} \sum_{i=1}^{n} m_i \dot{\boldsymbol{r}}_i(0)$.

($\Leftarrow$) Assume the frame B satisfies the ODE and initial conditions in Equation 15. In this case Equation 13 simplifies to the following unforced ODE:

$$\ddot{\boldsymbol{p}}(t) = 0 \tag{18}$$

The initial conditions in Equation 15 can be rewritten as $\boldsymbol{p}(0) = \dot{\boldsymbol{p}}(0) = 0$. Based on these initial conditions, the unique solution to the unforced ODE in Equation 18 is $\boldsymbol{p}(t) = 0$ for all $t \geq 0$. This implies that $\boldsymbol{p}(t) = \boldsymbol{P}_{int}(t) = 0$ for all $t \geq 0$ and $\boldsymbol{p}(0) = \sum_{i=1}^{n} m_i \boldsymbol{b}_i(0) = 0$. Thus the translational mean-axis constraints are satisfied. ■

This theorem provides a single vector, second-order differential equation (Equation 15) for the translational motion of the mean axes. This corresponds to three scalar, second-order differential equations when expressed in component form. Because $\boldsymbol{p}(t) = 0$ for all $t \geq 0$, $\boldsymbol{r}_B(t) = \frac{1}{m_{tot}} \sum_{i=1}^{n} m_i \boldsymbol{r}_i(t)$ for all $t \geq 0$. Therefore, the differential equation and initial conditions specify that the origin of the mean axes is located at the instantaneous center of mass of the flexible body. This conclusion is reinforced by the fact that the mean-axis translational equation of motion (Equation 15) is identical to the equation of motion for the center of mass of a system of particles given in standard dynamics references. Furthermore, it is also identical to the equation of motion for the center of mass of a rigid body [11].

IV. Rotation of the Mean Axes

This section focuses on the rotational mean-axis constraint in Definition 1.B. The total angular momentum about the origin of frame B is:

$$\boldsymbol{H}_{tot} = \sum_{i=1}^{n} \boldsymbol{b}_i \times m_i \dot{\boldsymbol{r}}_i \tag{19}$$

The inertial derivative of $\boldsymbol{H}_{tot}$ is given by:

$$\dot{\boldsymbol{H}}_{tot} = \sum_{i=1}^{n} \left(\dot{\boldsymbol{b}}_i \times m_i \dot{\boldsymbol{r}}_i + \boldsymbol{b}_i \times m_i \ddot{\boldsymbol{r}}_i \right) \tag{20}$$

To simplify this expression, use the equations of motion for the particles in the inertial frame (Equation 1) to substitute for $m_i \ddot{\boldsymbol{r}}_i$. In addition, substitute $\dot{\boldsymbol{r}}_i = \dot{\boldsymbol{r}}_B + \dot{\boldsymbol{b}}_i$. This yields the following form for $\dot{\boldsymbol{H}}_{tot}$ after some re-arrangement:

$$\dot{\boldsymbol{H}}_{tot} = \left(\sum_{i=1}^{n} m_i \dot{\boldsymbol{b}}_i\right) \times \dot{\boldsymbol{r}}_B + \sum_{i=1}^{n} m_i \left(\dot{\boldsymbol{b}}_i \times \dot{\boldsymbol{b}}_i\right) + \sum_{i=1}^{n} \boldsymbol{b}_i \times \left(\boldsymbol{F}_i + \sum_{j \neq i} \boldsymbol{F}_{ij}\right) \tag{21}$$

The second term is equal to zero because $\dot{\boldsymbol{b}}_i \times \dot{\boldsymbol{b}}_i = 0$. Moreover, the third term simplifies to $\boldsymbol{M}_{ext} := \sum_{i=1}^{n} \boldsymbol{b}_i \times \boldsymbol{F}_i$ because the internal forces are assumed to be equal, opposite, and acting along the line between the particles. The vector $\boldsymbol{M}_{ext}$ is the net moment about the origin of frame B due to the external forces. Finally, the translational mean-axis condition implies the first term is zero. Specifically, $\sum_{i=1}^{n} m_i \dot{\boldsymbol{b}}_i$ is equal to $(\sum_{i=1}^{n} m_i \mathring{\boldsymbol{b}}_i) + \boldsymbol{\omega} \times (\sum_{i=1}^{n} m_i \boldsymbol{b}_i)$ by the Transport Theorem. The translational mean-axis condition implies $\sum_{i=1}^{n} m_i \boldsymbol{b}_i = 0$ and $\sum_{i=1}^{n} m_i \mathring{\boldsymbol{b}}_i = 0$ as shown in Theorem 1. Hence the first term is zero. As a result of these simplifications, the inertial derivative of $\boldsymbol{H}_{tot}$ simplifies to

$$\dot{\boldsymbol{H}}_{tot} = \boldsymbol{M}_{ext} \tag{22}$$

In other words, the rate of change of the total angular momentum about the center of mass is equal to the total moment. This result is consistent with equations from standard dynamics references [11].

Before stating the rotational mean-axis result, it is useful to rewrite the total angular momentum in an alternative form that involves the internal angular momentum $\boldsymbol{H}_{int}$. First substitute for $\dot{\boldsymbol{r}}_i$ in the definition of $\boldsymbol{H}_{tot}$ (Equation 19) using the expression derived from the Transport Theorem (Equation 6):

$$\boldsymbol{H}_{tot} = \sum_{i=1}^{n} \boldsymbol{b}_i \times m_i \left(\dot{\boldsymbol{r}}_B + \mathring{\boldsymbol{b}}_i + (\omega \times \boldsymbol{b}_i)\right) \tag{23}$$

$$= \left(\sum_{i=1}^{n} m_i \boldsymbol{b}_i\right) \times \dot{\boldsymbol{r}}_B + \sum_{i=1}^{n} m_i \boldsymbol{b}_i \times \mathring{\boldsymbol{b}}_i + \sum_{i=1}^{n} m_i \left(\boldsymbol{b}_i \times (\omega \times \boldsymbol{b}_i)\right) \tag{24}$$

The translational mean-axis condition implies $\sum_{i=1}^{n} m_i \boldsymbol{b}_i = 0$ as shown in Theorem 1. Hence the first term involving $\dot{\boldsymbol{r}}_B$ drops out of the expression. The second term is simply the internal angular momentum $\boldsymbol{H}_{int}$ as defined in Equation 12. The third term can be rewritten using the vector triple product identity:

$$\sum_{i=1}^{n} m_i \left(\boldsymbol{b}_i \times (\omega \times \boldsymbol{b}_i)\right) = \sum_{i=1}^{n} m_i \left(|\boldsymbol{b}_i|^2 \omega - (\boldsymbol{b}_i \cdot \omega)\boldsymbol{b}_i\right) \tag{25}$$

This term is simply $\boldsymbol{J}\boldsymbol{\omega}$ where $\boldsymbol{J}$ is the instantaneous moment of inertia tensor.[‡] $\boldsymbol{J}\boldsymbol{\omega}$ represents the angular momentum associated with the rotation of the frame itself. The moment of inertia tensor $\boldsymbol{J}$ depends on the particle locations $\boldsymbol{b}_i$ expressed in the body frame B. The vectors $\boldsymbol{b}_i$ can vary in time due to deformations and hence $\boldsymbol{J}$ can also vary in time.

[‡]Let $[\omega_x, \omega_y, \omega_z]^T$ and $[x_i, y_i, z_i]^T$ be the components of the vectors $\boldsymbol{\omega}$ and $\boldsymbol{b}_i$ expressed in the body frame. Then the components of the

To summarize, if the translational mean-axis condition holds then the total angular momentum is:

$$\boldsymbol{H}_{tot} = \boldsymbol{J}\boldsymbol{\omega} + \boldsymbol{H}_{int} \tag{27}$$

Combining Equations 22 and 27 yields the following dynamic equation:

$$\frac{d}{dt}\Big|_I (\boldsymbol{J}\boldsymbol{\omega} + \boldsymbol{H}_{int}) = \boldsymbol{M}_{ext} \tag{28}$$

This differential equation is used in the next theorem to provide an explicit equation of motion corresponding to the rotational mean-axis constraint.

Theorem 2: Assume the frame B satisfies the translational mean-axis constraint (Definition 1.A). Then B also satisfies the rotational mean-axis constraint (Definition 1.B) if and only if $\boldsymbol{\omega}$ satisfies the following equation of motion:

$$\begin{aligned} &\frac{d}{dt}\Big|_I (\boldsymbol{J}\boldsymbol{\omega}) = \boldsymbol{M}_{ext} \\ &\text{Initial Condition: } \boldsymbol{\omega}(0) = \boldsymbol{J}^{-1}(0)\boldsymbol{H}_{tot}(0) \end{aligned} \tag{29}$$

Proof: ($\Rightarrow$) Assume the frame B satisfies the rotational mean-axis constraint (Definition 1.B), i.e. $\boldsymbol{H}_{int}(t) = 0$ for all $t \geq 0$. Hence $\boldsymbol{H}_{tot} = \boldsymbol{J}\boldsymbol{\omega}$ by Equation 27. Thus $\boldsymbol{H}_{tot}(0) = \boldsymbol{J}(0)\boldsymbol{\omega}(0)$ and the dynamics in Equation 29 follow by simplifying Equation 28.

($\Leftarrow$) Assume the frame B satisfies the ODE and initial conditions in Equation 29. Then Equation 22 simplifies to the following unforced ODE:

$$\frac{d}{dt}\Big|_I (\boldsymbol{H}_{int}) = 0 \tag{30}$$

Moreover, $\boldsymbol{H}_{int}(0) = \boldsymbol{H}_{tot}(0) - \boldsymbol{J}(0)\boldsymbol{\omega}(0) = 0$ by the assumed initial condition for $\boldsymbol{\omega}$. Based on these initial conditions, the solution to the unforced ODE in Equation 30 is $\boldsymbol{H}_{int}(t) = 0$ for all $t \geq 0$. ■

This theorem provides a single vector, second-order differential equation (Equation 29) for the rotational motion of the mean axes. This corresponds to three scalar, second-order differential equations when expressed in components. The

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vector $\sum_{i=1}^n m_i \left(|\boldsymbol{b}_i|^2\boldsymbol{\omega} - (\boldsymbol{b}_i \cdot \boldsymbol{\omega})\boldsymbol{b}_i\right)$ are given by:

$$\begin{bmatrix} J_{xx} & J_{xy} & J_{xz} \\ J_{yx} & J_{yy} & J_{yz} \\ J_{zx} & J_{zy} & J_{zz} \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \tag{26}$$

where $J_{xx} = \sum_{i=1}^n m_i(y_i^2 + z_i^2)$, $J_{xy} = -\sum_{i=1}^n m_i(x_i y_i)$, etc.

differential equation for $\boldsymbol{\omega}$ can be expanded using the Transport Theorem:

$$\boldsymbol{J}\dot{\boldsymbol{\omega}} + \overset{\circ}{\boldsymbol{J}}\,\boldsymbol{\omega} + \boldsymbol{\omega} \times (\boldsymbol{J}\boldsymbol{\omega}) = \boldsymbol{M}_{ext} \tag{31}$$

This is similar to the standard rotational equations of motion for a rigid body, except that $\boldsymbol{J}$ can vary in time due to deformations of the body. These time variations introduce the term $\overset{\circ}{\boldsymbol{J}}\boldsymbol{\omega}$ where $\overset{\circ}{\boldsymbol{J}}$ denotes the rate of change of the moment of inertia tensor measured in the body frame. For small deformations, the changes to the inertia tensor $\boldsymbol{J}$ may become negligibly small. In this case, it may be assumed that $\overset{\circ}{\boldsymbol{J}}\,\boldsymbol{\omega}$ is zero and Equation 31 reduces to $\boldsymbol{J}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times (\boldsymbol{J}\boldsymbol{\omega}) = \boldsymbol{M}_{ext}$. This is identical in form to the standard Newton-Euler equations for the rotational motion of a rigid body [11]. The ODE in Theorem 2 specifies an initial condition on the rate $\boldsymbol{\omega}$ but not on the initial orientation of the frame. As a result, the mean axes are only unique up to constant offsets in the orientation based on the specified initial conditions.

It should also be noted that the rotational equations are completely derived without specifying a particular Euler angle rotation sequence. Although this is possible with the Lagrangian method, it requires an advanced technique using quasi-coordinates that is described in [13]. Thus, the Newtonian approach allows for a high level of abstraction throughout the derivation while only relying on basic first-principles. This level of abstraction is preferred so that the equations and assumptions can be stated in the most general form.

Several observations may be made about the similarities and differences of the mean-axis frame and rigid body equations of motion. The distinction between a rigid body reference frame and the mean-axis frame is important for a deformable body. Not only is the body deformable, the reference frame B is a body-reference frame but not a *body-fixed* frame. The reference frame for a rigid body is typically body-fixed, meaning that it is attached to a material point on the body. As previously mentioned, the reference frame B is more abstractly related to the body using momentum constraints, which do not necessarily constrain the frame to a fixed material point. The mean-axis constraints result in equations of motion for the mean axes that are similar in form to a body-fixed frame for a rigid body, but a body-fixed frame for a deformable body may have significantly different equations of motion. However, the mean-axis frame becomes indistinguishable from a body-fixed frame if the stiffness increases without bound (i.e. the stiffness of the deformable body increases until it becomes a rigid body). Equivalently stated, the mean-axis reference frame for a rigid body is in fact a body-fixed frame.

V. Conclusion

The flexible aircraft translational and rotational equations of motion presented in this note are fully derived from first principles using Newton's Laws. The equations of motions are formulated for a system of particles with a body-referenced frame. The mean-axis constraints are then considered in this framework, which define properties of the internal momentum relative to the mean-axis reference frame. It is shown that the motion of the body-referenced frame is uniquely defined (up to a constant rotational offset) if the mean-axis constraints are satisfied. In addition, similarities are highlighted between the resulting mean-axis equations of motion and rigid body equations of motion.

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