IQC Analysis of Uncertain LTV Systems With Rational Dependence on Time

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Abstract— **The goal of this paper is to assess the robustness of an uncertain linear time-varying (LTV) system on a finite time horizon. The uncertain system is modeled as a connection of a known LTV system and a perturbation. The state matrices of the LTV system are assumed to be rational functions of time. This is used to model the uncertain LTV system as an connection of a time invariant system and an augmented perturbation that includes time. The input/output behavior of the perturbation is described by time-domain, integral quadratic constraints (IQCs). Static and dynamic IQCs are developed for the multiplication by time. A sufficient condition to bound the induced L_2 gain is formulated using dissipation inequalities and IQCs. The approach is demonstrated with two simple examples.**

I. Introduction

This paper focuses on robustness analysis for a class of uncertain linear-varying (LTV) systems. The analysis is performed on an uncertain LTV system modeled, as shown in Figure 1, by an interconnection of a known, nominal LTV system H_{LTV} and a perturbation Δ_H. This interconnection, denoted by $F_u(H_{LTV}, \Delta_H)$, is a standard tool for uncertainty modeling in robust control [1]. The perturbation can have block structure and is used to model difficult to analyze elements including nonlinearities and dynamic or parametric uncertainty. The analysis performed in this paper characterizes the input-output properties of Δ_H using integral quadratic constraints (IQCs) [2], [3].

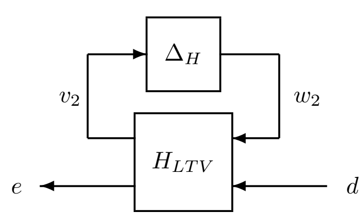

Fig. 1. Interconnection $F_u(H_{LTV}, \Delta_H)$ of a nominal LTV system H_{LTV} and perturbation Δ_H.

The state matrices of the LTV system H_{LTV} are assumed to be rational functions of time. This key assumption yields a useful reformulation for the uncertain system. Specifically, let Δ_t denote the operator that multiplies a signal by time, i.e. $w_1 = \Delta_t(v_1)$ is defined by $w_1(t) = t \cdot v_1(t)$ for all $t \geq 0$. Then, by assumption, the nominal LTV system H_{LTV} can be modeled by the interconnection of a linear time invariant (LTI) system G_{LTI} and Δ_t as shown on the left side of Figure 2. This further implies that the uncertain LTV system in Figure 1 is given by an interconnection of the LTI system G_{LTI} with an augmented perturbation $\Delta_G := \left[\begin{smallmatrix} \Delta_t & 0 \\ 0 & \Delta_H \end{smallmatrix}\right]$. In other words $F_u(H_{LTV}, \Delta_H) = F_u(G_{LTI}, \Delta_G)$ as shown on the right side of Figure 2.

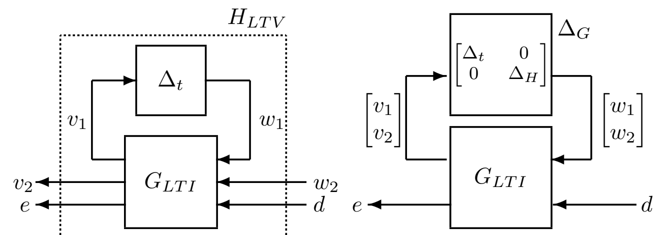

Fig. 2. Left: Nominal LTV system, $H_{LTV} = F_u(G_{LTI}, \Delta_t)$.
Right: Uncertain LTV system, $F_u(H_{LTV}, \Delta_H) = F_u(G_{LTI}, \Delta_G)$.

This paper presents a finite-dimensional linear matrix inequality (LMI) condition to upper bound the induced L_2 gain for this class of uncertain LTV systems. The analysis condition, stated as Theorem 1 in Section III-C, is a variation on the IQC theorem in [2]. There are two key technical issues. First, several IQCs are required to bound time operator Δ_t. These include static IQCs (Section III-A) as well as dynamic IQCs (Section III-B) based on a specific version of the swapping lemma [4]. Second, the main IQC stability theorem in [2] requires the perturbation to be a bounded operator. However, Δ_t is an unbounded operator on infinite time horizons. Theorem 1 in Section III-C uses a simple dissipation proof that avoids this boundedness assumption. Several simple examples are presented in Section IV to demonstrate the proposed analysis condition.

The most closely related works on robustness of uncertain LTV systems are [5], [6], [7], [8], [9]. These works only assume that the state matrices of the nominal LTV system are piecewise continuous (not necessarily rational) functions of time. This is a more general class of uncertain LTV systems than consider here. The price for this generalization is that the analysis conditions are infinite dimensional (as opposed to the finite dimensional LMIs given here). The work in [8], [9] develops differential, time-dependent LMI analysis conditions. Numerical algorithms are developed by enforcing the LMIs on a finite time grid and using connections to Riccati differential equations. The work in [5] also uses differential LMIs developed for the special case of uncertain LTV systems with a single, full-block uncertainty. Both [6] and [7] propose optimizing over the IQC variables using a Riccati Differential Equation condition.

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II. BACKGROUND

A. Notation

Let $\mathbb{R}^{n\times m}$ and $\mathbb{S}^n$ denote the sets of n-by-m real matrices and n-by-n real, symmetric matrices. The finite-horizon L_2 norm of a signal $v : [0,\infty) \to \mathbb{R}^n$ is defined as $\|v\|_2 := \left(\int_0^\infty v(t)^T v(t) dt\right)^{1/2}$. If $\|v\|_2$ is finite then $v \in L_2$. The projection operator P_T maps any function v as follows: $(P_T v)(t) = v(t)$ for $t \le T$ and $(P_T v)(t) = 0$ otherwise. The extended space, denoted L_{2e}, is the set of functions v such that $P_T v \in L_2$ for all $T \ge 0$. $\mathbb{RL}_\infty$ denotes the set of rational functions with real coefficients that have no poles on the imaginary axis. $\mathbb{RH}_\infty$ is the subset of functions in $\mathbb{RL}_\infty$ that are analytic in the closed right-half of the complex plane.

B. Integral Quadratic Constraints (IQCs)

IQCs [2] are used to describe the input/output behavior of an operator Δ. They can be formulated in either the frequency or time domain. The time domain formulation is used in this paper and is based on the graphical interpretation in Figure 3. The inputs and outputs of Δ are filtered through an LTI system Ψ with zero initial condition $x_\psi(0) = 0$. The dynamics of Ψ are given as follows:

$$\begin{aligned} \dot{x}_\psi(t) &= A_\psi\, x_\psi(t) + B_{\psi 1}\, v(t) + B_{\psi 2}\, w(t) \\ z(t) &= C_\psi\, x_\psi(t) + D_{\psi 1}\, v(t) + D_{\psi 2}\, w(t) \end{aligned} \tag{1}$$

where $x_\psi \in \mathbb{R}^{n_\psi}$ is the state and $(A_\psi, B_\psi, C_\psi, D_\psi)$ denote the state matrices of Ψ. Moreover $B_\psi := [B_{\psi 1},\, B_{\psi 2}]$ and $D_\psi := [D_{\psi 1},\, D_{\psi 2}]$ are partitioned conformably with the dimensions of v and w. A time domain IQC is an inequality enforced on the output z over finite horizons. The formal definition is given next.

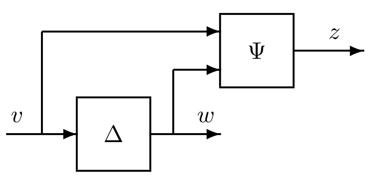

Fig. 3. Graphical interpretation for time domain IQCs

Definition 1. *Let $\Psi \in \mathbb{RH}_\infty^{n_z\times(n_v+n_w)}$ and $M \in \mathbb{S}^{n_z}$ be given. A bounded, causal operator $\Delta : \mathbf{L}_2^{n_v} \to \mathbf{L}_2^{n_w}$ satisfies the time domain IQC defined by (Ψ, M) if the following inequality holds for all $v \in L_2^{n_v}$, $w = \Delta(v)$, and $T \ge 0$:*

$$\int_0^T z(t)^T M z(t)\, dt \ \ge 0 \tag{2}$$

where z is the output of Ψ driven by inputs (v, w) with zero initial conditions $x_\psi(0) = 0$.

The notation $\Delta \in \mathcal{I}(\Psi, M)$ is used when Δ satisfies the corresponding IQC. Time domain IQCs, as defined above, require the constraint to hold over all finite time horizons. These are often referred to as hard IQCs [2]. An extensive library of IQCs is provided in [2] for various types of perturbations. Most IQCs are specified in the frequency domain using a multiplier Π. Under some mild assumptions, a valid time-domain IQC (Ψ, M) can be constructed from Π via a J-spectral factorization [10]. This allows the library of known (frequency domain) IQCs to be used. More general IQC parameterizations are not necessarily "hard" but can be handled with the method in [11].

III. ROBUSTNESS ANALYSIS

The operator $\Delta_t : L_{2e}^{n_{v1}} \to L_{2e}^{n_{v1}}$ corresponds to multiplication by time. If $v_1 \in L_{2e}^{n_{v1}}$ and $w_1 = \Delta_t(v_1)$ then $w_1(t) = t \cdot v_1(t)$ for all $t \ge 0$. Note that this operator is passive pointwise in time, i.e. $w_1(t)^T v_1(t) \ge 0$ for all $t \ge 0$. This section exploits additional properties of time multiplication to derive static and dynamic IQCs for Δ_t. Then an LMI condition is presented to bound the induced L_2 gain of the uncertain LTV system.

A. Static IQCs for Δ_t

Let X and Y be any $n_{v1} \times n_{v1}$ matrices satisfying $X \succeq 0$ and $Y = -Y^T$. Then $w_1 = \Delta_t(v_1)$ implies:

$$\begin{aligned} &\begin{bmatrix} v_1(t) \\ w_1(t) \end{bmatrix}^T \begin{bmatrix} 0 & X+Y \\ X+Y^T & 0 \end{bmatrix} \begin{bmatrix} v_1(t) \\ w_1(t) \end{bmatrix} \\ &= t \cdot v_1(t)^T (2X + Y + Y^T) v_1(t) \\ &\ge 0 \quad \forall t \ge 0 \end{aligned} \tag{3}$$

Hence Δ_t satisfies the quadratic constraint in Equation 3 at each point in time. Thus Δ_t satisfies the time domain IQC defined by (Ψ, M) with

$$\Psi := I_{2n_{v1}} \text{ and } M := \begin{bmatrix} 0 & X+Y \\ X+Y^T & 0 \end{bmatrix}$$

This is a static IQC because the filter Ψ contains no dynamics. This IQC holds for all $T \in [0,\infty)$ as required by Definition 1 and hence it can be used for infinite-horizon analysis. It is also useful to define an IQC for finite-horizon analysis. Specifically, let X, Y, and Z be $n_{v1} \times n_{v1}$ matrices with $X, Z \succeq 0$ and $Y = -Y^T$. Then for any $T_0 \in [0,\infty)$,

$$\begin{aligned} &\begin{bmatrix} v_1(t) \\ w_1(t) \end{bmatrix}^T \begin{bmatrix} T_0^2 Z & X+Y \\ X+Y^T & -Z \end{bmatrix} \begin{bmatrix} v_1(t) \\ w_1(t) \end{bmatrix} \\ &= t \cdot v_1(t)^T (2X + Y + Y^T) v_1(t) \\ &+ (T_0^2 - t^2) \cdot v_1(t)^T Z v_1(t) \\ &\ge 0 \quad \forall t \in [0, T_0] \end{aligned} \tag{4}$$

This implies that Δ_t satisfies the time domain IQC defined by (Ψ, M) with

$$\Psi := I_{2n_{v1}} \text{ and } M := \begin{bmatrix} T_0^2 Z & X+Y \\ X+Y^T & -Z \end{bmatrix} \tag{5}$$

The IQC in Equation 2 only holds for $T \in [0, T_0]$. Hence it technically does not satisfy Definition 1 which requires the IQC to hold for all $T \ge 0$. However, the IQC in Equation 5 can be used to bound the induced L_2 gain over the finite horizon $[0, T_0]$.

B. Dynamic IQCs for Δ_t

Dynamic IQCs for Δ_t are derived in this section as a corollary of the Swapping Lemma [4]. First let $\delta : \mathbb{R} \to \mathbb{R}$ be a differentiable function of time and define the multiplication operator $\Delta_\delta : L_{2e}^{n_{v1}} \to L_{2e}^{n_{v1}}$ as follows: if $v_1 \in L_{2e}^{n_{v1}}$ and $w_1 = \Delta_\delta(v_1)$ then $w_1(t) = \delta(t) \cdot v_1(t)$ for all $t \geq 0$. The swapping lemma for Δ_δ is stated next.

Lemma 1. *[4] Let $F \in \mathbb{RH}_\infty^{n_F \times n_{v1}}$ have a realization with state matrices (A_F, B_F, C_F, D_F). Then*

$$F\Delta_\delta = \Delta_\delta F - F_C \Delta_{\dot{\delta}} F_B$$

where the state realizations of F_B and F_C are given by $F_B := (A_F, B_F, I, 0)$ and $F_C := (A_F, I, C_F, 0)$.

The swapping lemma yields a class of dynamic IQCs for Δ_t as given in the next corollary.

Corollary 1. *Let $F \in \mathbb{RH}_\infty^{n_F \times n_{v1}}$ be given with realization (A_F, B_F, C_F, D_F) and define F_B and F_C as in Lemma 1. Δ_t satisfies the following time-domain IQC for any $X \succeq 0$ and $Y = -Y^T$ of appropriate dimension:*

$$\Psi := \begin{bmatrix} F & 0 \\ F_C F_B & F \end{bmatrix} \text{ and } M := \begin{bmatrix} 0 & X+Y \\ X+Y^T & 0 \end{bmatrix} \quad (6)$$

Proof. The derivative of time is simply one, i.e. $\Delta_{\dot{t}}(v_1) = v_1$. Hence the swapping lemma for Δ_t simplifies to:

$$F\Delta_t = \Delta_t F - F_C F_B \quad (7)$$

Let $w_1 = \Delta_t(v_1)$ for some $v_1 \in L_{2e}^{n_{v1}}$. Figure 4 provides a graphical interpretation of the swapping lemma result. The input to the dashed box is v_1 and the output is $F\Delta_t(v_1) = Fw_1$. This output is equal to $(\Delta_t F - F_C F_B)v_1$ by Equation 7 and as shown inside the dashed box of Figure 4. The internal signals $\tilde{v}_1$ and $\tilde{w}_1 = \Delta_t(\tilde{v}_1)$ satisfy the static quadratic constraint in Equation 3:

$$\begin{bmatrix} \tilde{v}_1(t) \\ \tilde{w}_1(t) \end{bmatrix}^T \begin{bmatrix} 0 & X+Y \\ X+Y^T & 0 \end{bmatrix} \begin{bmatrix} \tilde{v}_1(t) \\ \tilde{w}_1(t) \end{bmatrix} \geq 0 \quad \forall t \geq 0 \quad (8)$$

These internal signals satisfy $\tilde{v}_1 = Fv_1$ and $\tilde{w}_1 = F_C F_B v_1 + Fw_1$. Substitute these relations into Equation 8 to verify that Δ_t satisfies the dynamic IQC in Equation 6. $\square$

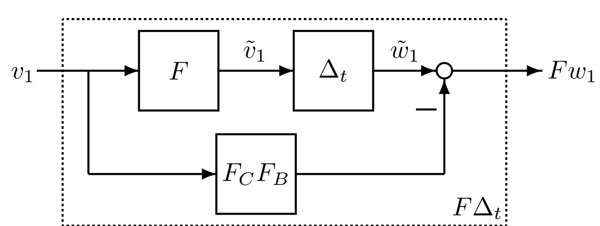

Fig. 4. Swapping Lemma for Δ_t

The dynamic IQC in Corollary 1 holds for all $T \geq 0$ and can be used for infinite-horizon analysis. A dynamic IQC for analysis on the finite horizon $[0, T_0]$ is given by:

$$\Psi := \begin{bmatrix} F & 0 \\ F_C F_B & F \end{bmatrix} \text{ and } M := \begin{bmatrix} T_0^2 Z & X+Y \\ X+Y^T & -Z \end{bmatrix} \quad (9)$$

where $X, Z \succeq 0$ and $Y = -Y^T$. This fact follows similar to Corollary 1 but using the static finite horizon constraint in Equation 4.

C. Robust Induced L_2 Gain

The uncertain LTV system is modeled by $F_u(G_{LTI}, \Delta_G)$ as shown in Figure 2. The nominal system G_{LTI} is described by the following state-space model:

$$\begin{aligned} \dot{x}_G(t) &= A_G\, x_G(t) + B_{G1}\, w(t) + B_{G2}\, d(t) \\ v(t) &= C_{G1}\, x_G(t) + D_{G11}\, w(t) + D_{G12}\, d(t) \\ e(t) &= C_{G2}\, x_G(t) + D_{G21}\, w(t) + D_{G22}\, d(t) \end{aligned} \quad (10)$$

where $x_G \in \mathbb{R}^{n_G}$ is the state. The inputs are $w \in \mathbb{R}^{n_w}$ and $d \in \mathbb{R}^{n_d}$ while $v \in \mathbb{R}^{n_v}$ and $e \in \mathbb{R}^{n_e}$ are outputs. The partitioning $v := \left[\begin{smallmatrix} v_1 \\ v_2 \end{smallmatrix}\right]$ and $w := \left[\begin{smallmatrix} w_1 \\ w_2 \end{smallmatrix}\right]$ is consistent with the structure of the augmented perturbation $\Delta_G := \left[\begin{smallmatrix} \Delta_t & 0 \\ 0 & \Delta_H \end{smallmatrix}\right]$. Well-posedness of the interconnection $F_u(G_{LTI}, \Delta_G)$ is defined as follows.

Definition 2. *$F_u(G_{LTI}, \Delta_G)$ is well-posed if for all $x_G(0) \in \mathbb{R}^{n_G}$ and $d \in L_{2e}^{n_d}$ there exists unique solutions $x_G \in L_{2e}^{n_G}$, $v \in L_{2e}^{n_v}$, $e \in L_{2e}^{n_e}$, and $w \in L_{2e}^{n_w}$ satisfying Equation (10) and $w = \Delta(v)$ with a causal dependence on d.*

The objective is to assess the robustness of the uncertain LTV system $F_u(G_{LTI}, \Delta_G)$ as shown on the right of Figure 2. For a given Δ_G, the induced L_2 gain from d to e is defined as:

$$\|F_u(G_{LTI}, \Delta_G)\| := \sup_{\substack{0 \neq d \in L_2^{n_d}[0,\infty), \\ x_G(0)=0}} \frac{\|e\|_2}{\|d\|_2} \quad (11)$$

The robustness of $F_u(G_{LTI}, \Delta_G)$ is analyzed using the interconnection shown in Figure 5. The extended system of G_{LTI} (Equation 10) and the IQC filter Ψ (Equation 1) is governed by the following state space model:

$$\begin{aligned} \dot{x}(t) &= \mathcal{A}\, x(t) + \mathcal{B} \left[\begin{smallmatrix} w(t) \\ d(t) \end{smallmatrix}\right] \\ z(t) &= \mathcal{C}_1\, x(t) + \mathcal{D}_1 \left[\begin{smallmatrix} w(t) \\ d(t) \end{smallmatrix}\right] \\ e(t) &= \mathcal{C}_2\, x(t) + \mathcal{D}_2 \left[\begin{smallmatrix} w(t) \\ d(t) \end{smallmatrix}\right] \end{aligned} \quad (12)$$

The extended state vector is $x := \left[\begin{smallmatrix} x_G \\ x_\psi \end{smallmatrix}\right] \in \mathbb{R}^n$ where $n := n_G + n_\psi$. The state-space matrices are given by

$$\mathcal{A} := \begin{bmatrix} A_G & 0 \\ B_{\psi 1} C_{G1} & A_\psi \end{bmatrix}, \mathcal{B} := \begin{bmatrix} B_{G1} & B_{G2} \\ B_{\psi 1} D_{G11} + B_{\psi 2} & B_{\psi 1} D_{G12} \end{bmatrix}$$
$$\mathcal{C}_1 := \begin{bmatrix} D_{\psi 1} C_{G1} & C_\psi \end{bmatrix}, \mathcal{C}_2 := \begin{bmatrix} C_{G2} & 0 \end{bmatrix},$$
$$\mathcal{D}_1 := \begin{bmatrix} D_{\psi 1} D_{G11} + D_{\psi 2} & D_{\psi 1} D_{G12} \end{bmatrix}$$
$$\mathcal{D}_2 := \begin{bmatrix} D_{G21} & D_{G22} \end{bmatrix}$$

The actual system to be analyzed is $F_u(G_{LTI}, \Delta_G)$ with input d. The analysis is instead performed with the extended LTI system (Equation 12) and the constraint $\Delta_G \in \mathcal{I}(\Psi, M)$. The constrained extended system has inputs (d, w). The IQC implicitly constrains the input w such that the constrained extended system without Δ_G includes all behaviors of the original system $F_u(G_{LTI}, \Delta_G)$.

The following LMI is used to assess the robust perfor-

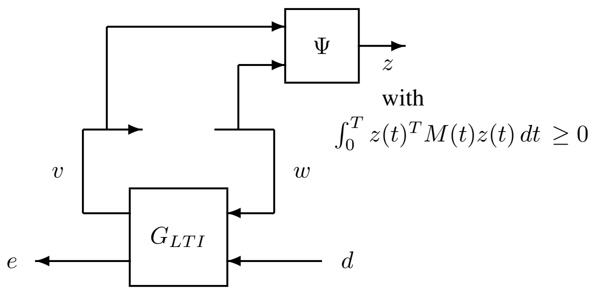

Fig. 5. Extended LTI system of G_{LTI} and filter Ψ.

mance of $F_u(G_{LTI}, \Delta_G)$ [1]:

$$\begin{bmatrix} \mathcal{A}^T P + P\mathcal{A} & P\mathcal{B} \\ \mathcal{B}^T P & \begin{bmatrix} 0_{n_w} & 0 \\ 0 & -\gamma^2 I_{n_d} \end{bmatrix} \end{bmatrix} + (\cdot)^T \begin{bmatrix} \mathcal{C}_2 & \mathcal{D}_2 \end{bmatrix} \\ + (\cdot)^T M \begin{bmatrix} \mathcal{C}_1 & \mathcal{D}_1 \end{bmatrix} \preceq -\epsilon I \tag{13}$$

This inequality is compactly denoted as $LMI_{Rob}(P, M, \gamma^2) \preceq -\epsilon I$. This notation emphasizes that the constraint is an LMI in (P, M, γ^2) for fixed (G, Ψ). The dependence on (G, Ψ) is not explicitly denoted but will be clear from context. The next theorem formulates a sufficient condition to bound the (robust) induced L_2 gain of $F_u(G_{LTI}, \Delta_G)$. The proof uses IQCs and a standard dissipation argument [12], [13], [14], [15].

Theorem 1. *Let G_{LTI} be an LTI system defined by* (10) *and $\Delta_G : L_{2e}^{n_v} \to L_{2e}^{n_w}$ be a causal operator. Assume $F_u(G_{LTI}, \Delta_G)$ is well-posed and $\Delta_G \in \mathcal{I}(\Psi, M)$. If there exists $\epsilon > 0$, $\gamma > 0$ and $P > 0$ such that*

$$LMI_{Rob}(P, M, \gamma^2) \preceq -\epsilon I \tag{14}$$

then $\|F_u(G, \Delta)\|_2 < \gamma$.

Proof. Let $d \in L_2$ and $x_G(0) = 0$ be given. By well-posedness, $F_u(G_{LTI}, \Delta_G)$ has a unique solution (x_G, v, w, e). Define a storage function by $V(x) := x^T P x$. Left and right multiply the LMI (13) by $[x^T, w^T, d^T]$ and its transpose to show that V satisfies the following dissipation inequality for all $t \in [0, T]$:

$$\dot{V} + e^T e + z^T M z \leq (\gamma^2 - \epsilon)\, d^T d \tag{15}$$

Integrate over $[0, T]$ and use $x(0) = 0$ to obtain:

$$x(T)^T P(T) x(T) + \int_0^T z^T(t) M(t) z(t) dt \\ - (\gamma^2 - \epsilon)\|d\|_{2,[0,T]}^2 + \|e\|_{2,[0,T]}^2 \leq 0.$$

Apply $P \succeq 0$ and $\Delta \in \mathcal{I}(\Psi, M)$ to conclude:

$$\|e\|_{2,[0,T]}^2 \leq (\gamma^2 - \epsilon)\|d\|_{2,[0,T]}^2 \tag{16}$$

Take the limit as $T \to \infty$ to conclude $\|F_u(G, \Delta)\|_2 < \gamma$. □

This is a standard dissipation inequality argument. This result does not require the perturbation Δ_G to be a bounded operator. This is important as Δ_t is not a bounded operator on an infinite horizon. Also note that the finite horizon IQCs given in the previous section are valid over the horizon $[0, T_0]$. These can be used to bound the gain of the uncertain LTV system over this finite horizon. The result is similar to Theorem 1 and is not formally stated.

IV. Examples

A. Nominal LTV Analysis

Consider the following LTV system H_{LTV}

$$\dot{x}_H(t) = -\left(1 + \frac{0.9t}{1 + 0.9t}\right) x_H(t) + d(t) \tag{17}$$

$$e(t) = 5 x_H(t) \tag{18}$$

This is a nominal LTV system with no uncertainty. The IQCs developed in the previous section can be used to bound the induced L_2 gain for this LTV system. First, the nominal LTV system is expressed as $H_{LTV} = F_u(G_{LTI}, \Delta_t)$ where G_{LTI} is given by:

$$\dot{x}_G(t) = -x_G(t) - \sqrt{0.9} w_1(t) + d(t) \tag{19}$$

$$v_1(t) = \sqrt{0.9} x_G(t) - 0.9 w_1(t) \tag{20}$$

$$e(t) = 5 x_G(t) \tag{21}$$

The dynamic IQC given in Corollary 1 for Δ_t is used with the following filter

$$F(s) := \left[1,\ \frac{1}{s+2}, \ldots, \frac{1}{(s+2)^7}\right]^T \tag{22}$$

The matrices $X \succeq 0$ and $Y = -Y^T$ are decision variables solved by optimization. Specifically, gain γ is minimized subject to the LMI condition in Theorem 1. This is solved using CVX [16] yielding the bound on the (infinite-horizon) induced L_2 gain of $\gamma = 3.18$.

The dynamic IQC describes a set of uncertainties that include Δ_t. Hence the LMI condition is only sufficient, i.e. it yields an upper bound on the induced L_2 gain. There are necessary and sufficient Riccati Differential Equation conditions that can be used to compute exact bounds (within a bisection tolerance) on the induced L_2 gain of the nominal LTV system [17], [18], [19]. These exact bounds can be used to assess the conservativeness of the dynamic IQC developed for Δ_t. The Riccati Differential Equation condition yields an induced L_2 gain of 2.92 on a finite horizon of $[0, 6]$ sec. Longer time horizons yield the the same result to two decimal places. The IQC bound of 3.18 is 9% larger than this "exact" result. The conservatism introduced by using the dynamic IQC is relatively small for this example.

To continue this analysis, the induced gain was computed on different finite horizons $[0, T_0]$ with $T_0 \in \{0.01, 0.1, 0.5, 1, 2, \ldots, 6\}$. The exact results using the Riccati Differential Equation condition are shown as the blue solid curve in Figure 6. A bound on the gain was also computed using the dynamic IQC but with the additional finite horizon term shown in Equation 5. This yields the red dashed curve in Figure 6. It took approximately 22

[1]The notation $(\cdot)^T$ in (13) corresponds to an omitted factor required to make the corresponding term symmetric.

n_F	1	2	3	4	8
γ	5.00	3.71	3.37	3.25	3.18

TABLE I
INFINITE HORIZON INDUCED L_2 GAIN BOUNDS USING $F(s)$ IN EQ. 23

seconds on a standard laptop to compute the IQC bounds on all nine finite horizons. As noted above the curves for the Riccati and IQC conditions converge to 2.92 and 3.18, respectively as $T_0 \to \infty$. The IQC bound is slightly larger on all horizons indicating some conservatism in the IQC condition. However, the benefit of the IQC condition is that it can also be used to bound the gain for uncertain LTV systems. This is demonstrated in Section IV-B.

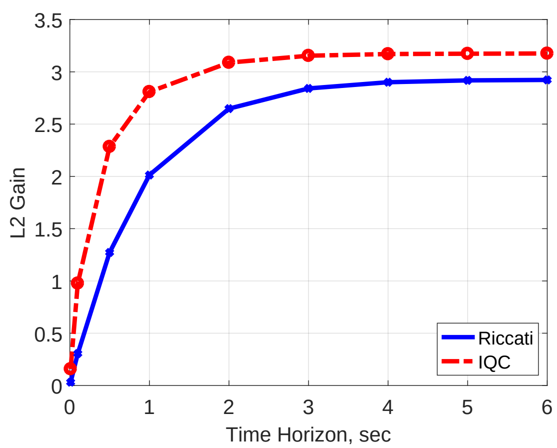

Fig. 6. Nominal Induced $\mathcal{L}_2$ Gain vs. Time Horizon using IQCs (red dashed) and Riccati Differential Equations (blue solid).

The gain bound computed using the LMI condition in Theorem 1 depends on the chosen filter $F(s)$. Consider filters of the following form where n_F is a fixed, positive integer:

$$F(s) := \left[1, \frac{1}{s+2}, \ldots, \frac{1}{(s+2)^{(n_F-1)}}\right]^T \tag{23}$$

Table I shows the infinite-horizon induced L_2 gain bounds obtained with the LMI condition in Theorem 1 for several values of n_F. The bound $\gamma = 3.18$ given for $n_F = 8$ was already reported above. The gain bound steadily improves with increasing values of n_F. The larger (worse) gain bound of $\gamma = 5.0$ was found using the static multiplier $F(s) = 1$ corresponding to $n_F = 1$. We also obtain $\gamma = 5.0$ on all finite horizons with the static multiplier $F(s) = 1$. It seems, based on this one example, that static multipliers are ineffective for finite-horizon analysis. No formal proof of this statement has been obtained as of yet. Additional details on IQC parameterizations can be found in [3].

B. Robust LTV Analysis

Consider the feedback system shown in Figure 7. The plant P is given by the following second-order, LTV system:

$$\dot{x}_P(t) = \begin{bmatrix} 0 & 1 \\ -4 & -2\zeta(t) \end{bmatrix} x_P(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \tag{24}$$

$$y(t) = \begin{bmatrix} 4 & 0 \end{bmatrix} x_P(t) \tag{25}$$

where $\zeta(t) := 0.2 + \frac{0.63t}{1+0.9t}$. The controller is a simple proportional gain $K = 2$. The uncertainty Δ_H is assumed to be a causal operator with norm-bound $\|\Delta_H\| \le b$. The goal is to compute a bound on the gain from disturbance d to error e. The closed-loop uncertain LTV system has a rational dependence on time and hence it can be modeled by $F_u(G_{LTI}, \Delta_G)$ as shown on the right of Figure 2.

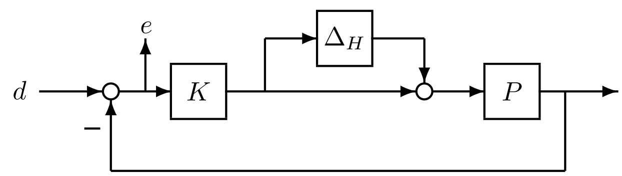

Fig. 7. Uncertain LTV Feedback System

The dynamic IQC for Δ_t (Corollary 1) is used with the filter $F(s)$ given in Equation 22. The following static IQC is used for Δ_H:

$$\Psi := I_2 \text{ and } M := \begin{bmatrix} b^2 & 0 \\ 0 & -1 \end{bmatrix} \tag{26}$$

An (infinite-horizon) upper bound on the induced L_2 gain was computed using the LMI condition in Theorem 1. Figure 8 shows the gain bound for twelve different values of the uncertainty bound b. It took approximately 24 seconds on a standard laptop to compute these IQC bounds. The IQC condition yields a bound on the nominal gain ($b = 0$) of 2.09. For comparison, the Riccati Differential Equation condition yields an induced L_2 gain of 1.90 on a finite horizon of $[0, 10]$ sec. Longer time horizons yield the the same value to two decimal places. This value is shown as the red circle in Figure 8. The IQC condition yields a nominal gain that is about 10% higher than the exact value from the Riccati Differential Equation. However, the IQC condition can also be used to assess the robust performance when the uncertainty Δ_H is introduced into the feedback loop.

V. CONCLUSIONS

This paper derived LMI conditions to bound the induced L_2 gain of an uncertain LTV system. The uncertain system was assumed to have state matrices with rational dependence on time. This enables the uncertain system to be formulated as an interconnection of a time invariant system and an augmented uncertainty that includes time. Static and dynamic integral quadratic constraints were introduced to bound the time operator. These can be used as part of a dissipation inequality condition to bound the induced L_2 gain on either finite or infinite horizons. Two simple examples were provided to demonstrate the approach. Future work will include additional studies on the conservatism of

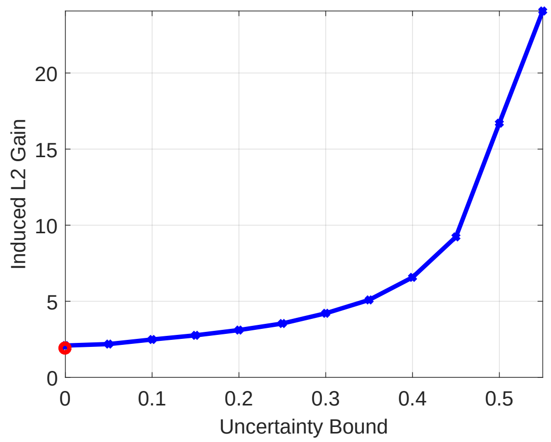

Fig. 8. Robust Induced $\mathcal{L}_2$ Gain vs. Uncertainty Bound.

the proposed approach and possible extensions to synthesis conditions.

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