Co-existent Phases in the One-Dimensional Static Theory of Elastic Bars

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Introduction

Nearly every material, when subjected to sufficiently large force, will change its phase. The phenomenon of change of phase is an intricate one, the general definition of phase itself being especially elusive. If the change is produced by deformation alone, which is the situation envisaged herein, large changes of shape or volume usually accompany the transformation. Consequently, although the phenomenon is well known through observation, careful experiments that isolate and quantify it are quite rare.

Most continuum theories also have failed to embody changes of phase; many specifically forbid it. Aside from the original work by GIBBS [1], a general approach to change of phase is absent. Special problems, however, have been successfully treated. The recent papers by ANTMAN [2], ANTMAN & CARBONE [3], cast within the framework of elastic rod theory, and the paper by ERICKSEN [4] on elastic bar theory, show that the phenomena of necking, change of phase, permanent deformation, hysteresis and yield are predicted by pure elasticity theories.

Motivated by the example of the van der Waals gas, and a calculation done some years earlier by DAFERMOS [5] which suggests that asymptotic limits of dynamic viscoelastic solutions may not be smooth, ERICKSEN relinquishes the belief that the constitutive relation must be convex. He works within the limits of elastic bar theory and he uses the criterion of least energy to assess the stability of equilibrium configurations. For a bar held in a hard loading device, he suggests that the notion of solution should also be generalized to allow for discontinuities in the strain. The approach is rather different from that of ANTMAN & CARBONE, who assume ellipticity and seek smooth solutions. Since rod theory allows the possibility that the constitutive function depend upon the configuration itself, as well as its derivative, instability of equilibrium solutions can result despite the assumption of ellipticity. After fixing end conditions appropriate for a straight rod extended by a dead load parallel to the axis of the bar, ANTMAN shows that, for sufficiently high loads, the obvious homogeneous deformation ceases to be even infinitesimally stable. Moreover, an absolutely stable, smooth solution is possible at these severe loads; that solution is periodic along the rod and has regions of reduced thickness or "necks".

KNOWLES & STERNBERG [6, 7, 8] have founded an entirely different approach on the observation that the differential equations of equilibrium change character from elliptic to hyperbolic as a consequence of a loss of ellipticity of the constitutive relation. Their motivation arises on the one hand from an attempt to describe the formation of Lüders bands in static experiments, and on the other from a critical study of a special constitutive relation proposed by BLATZ & KO. They observe that if the constitutive function fails to be elliptic, stationary shock surfaces may exist in a static solution. It is clear from their analysis [12; p. 2, 42, 53, 54] that they envisage solutions which have, at least on one side of the shock surface, deformation gradients at which ellipticity fails. We note that such solutions cannot be stable according to the energy criterion for stability. Though this observation casts some doubt on the applicability of KNOWLES & STERNBERG's results to the Lüders band phenomenon, especially for their Material 1, which has a convex domain of ellipticity, their general methods apply to stable solutions.

On the experimental side, the evidence shows that a variety of materials may undergo changes of phase involving permanent deformation. The polymers show less intricate behavior in this respect than metals. An elementary exposition of the observations may be found in HOLLIDAY & WARD [10]. In a uniaxial tension test the specimen appears to stretch more or less homogeneously until a certain value of the extension is reached; rather suddenly one or more (usually two) discontinuities form in the material. These boundaries move through the specimen as the load is increased, converting regions of moderate stretch into regions of high stretch. In a hard device a concomitant drop in load is observed as soon as the boundaries appear. In a soft device the load is maintained nearly constant or slightly increasing (KELLER & RIDER [11]) while the discontinuities are present. Relaxation of the load causes the boundaries eventually to stop, and permanent deformation results. Increased extension of polymer specimen causes the regions of high stretch to prevail; successive boundaries either coelesce, then disappear, or run to the edge of the specimen. Then increased extension of the material causes again a smooth deformation.

In metals, the observations are more difficult to assess, though the qualitative observations of the Savart-Masson effect are somewhat similar to those in the polymers. A description of the work of the poineers, as well as later experiments on the Savart-Masson effect, may be found in the article by BELL [9]. In a dead loading device, stress-extension curves have a staircase structure, the horizontal portion of the step being traversed rather quickly at constant load until the corner of the stair is reached, at which time the deformation halts until the load is increased. Creep appears to be absent when the load is held constant on the rising part of the stair, as long as the load is sufficiently below that which corresponds to the succeeding step.

Experiments in hard loading devices produce a sawtooth curve, the stress alternately rising and falling with abrupt changes of slope at the points of the sawteeth.

SHARP's [12] study of the effect in aluminum shows that stress-extension curves are reproducible for variable specimen diameters. He concluded that the Savart-Masson effect is caused by a material instability.

The original experiment by MCREYNOLDS [13] on dead loaded aluminum bars, and a later study by PHILLIPS, SWAIN & EBORALL [14] on aluminum-magnesium alloys, confirm that the deformation across the step is accompanied by the appearance of a slow, rather sharply defined wave, which passes through the specimen at the same time as the horizontal part of the stair is being transversed.

Here, I propose to study the problem of the stability of co-existent phases within the context of the theory of nonlinear, one dimensional elastic bars. The development rests upon a class of constitutive relations, each having a bounded domain and lacking invertibility. I allow for rather severe inhomogeneity resulting from variations in the cross-section or the material properties. The choice made for the constitutive class forces a specific choice for the underlying function space appropriate to the problem.

A definition of stability is given, based upon traditional ideas. A certain loading device, which can be hard or soft, is applied at one end of the bar, the other end being fixed. Body forces delivered by a potential are allowed.

The Weierstrass condition emerges as a necessary condition for metastability. The condition is not viewed as a restriction upon the constitutive function, but on the configurations which can be metastable. A weak form of the Euler equation is satisfied by metastable solutions in this function space. However, even if the constitutive function happens to be analytic, solutions of the Euler equation can be extremely rough. The bar can have stationary boundaries lying on any set of measure zero contained in the bar, a discontinuity in the strain occuring across each boundary.

A comparison of the latter result with common experimental observations prompts the question: Under what conditions do solutions have only a few discontinuities? I address this question by looking at the conditions under which most experiments on bars occur. Typically, the bar is inhomogeneous, usually because of a non-uniform cross section, and a gravitational body force is present. I find that such body forces and inhomogeneities tend to cause the absolutely stable solutions to display only a finite number of phase boundaries. For example, the absolutely stable solution for a bar shaped like the standard specimen for a uniaxial tensile test will contain exactly two phase boundaries. The region between the boundaries will be a region of high stretch, as compared with the region between each boundary and the end nearest to it. If the load is increased to a higher value, the boundaries will tend to reequilibrate further apart. For the most stable solutions, the positions of the phase boundaries corresponding to a given applied load are calculated.

1. Theory of Static Elastic Bars

A one dimensional elastic bar is described by a single material co-ordinate

$$X \in [0, L], \tag{1.1}$$

L being the length of the undeformed bar, $[0, L]$ the bar interval. A **placement** of the bar is given by a function

$$y(X) \in \mathbb{R}^1, \qquad X \in [0, L], \tag{1.2}$$

which assigns the position $y(X)$ to the point X. For now, so as to introduce terminology, we are loose about domains of functions and assumptions of smoothness. A prime shall denote the derivative with respect to X. We call $u = y'(X)$ the **deformation** and $y'(X) - 1$ the **strain** at X. A **strain energy function** $W(u, X)$ will be prescribed. The **stress** may then be defined by

$$T = \frac{\partial W}{\partial u}(y'(X), X). \tag{1.3}$$

We shall also assume that a potential $P(y, X)$ delivers the **body force** b according to the rule

$$b = \frac{-\partial P}{\partial y}(y(X), X), \qquad b \in \mathbb{R}^1. \tag{1.4}$$

The bar may be inhomogeneous, as reflected by the explicit dependence of W upon X, for two reasons. Either cross-sections of identical size and shape at different points have different elasticities, or cross-sections with identical elasticities have different sizes or shapes, or some combination of these is present.

One end of the bar shall always be held fixed,

$$y(0) = 0, \tag{1.5}$$

whereas we allow the possibility that the other end be placed in a loading device. In a **hard device** the position $y(L)=l$ is assigned: all possible placements $y(X)$ are required to satisfy this restriction. A **soft device** produces a load at l given by

$$\sigma_0(l), \tag{1.6}$$

σ_0 being an assigned continuous function with a sufficiently large domain to include all possible end positions of the bar. When $\sigma_0=\text{const.}$ the device is termed a **dead loading device.**

Formally, the total energy for the soft device is defined by

$$E_S[y]\equiv\int_0^L \{W(y'(X),X)+P(y(X),X)\}\,dX-\int_L^{y(L)}\sigma_0(s)\,ds, \tag{1.7}$$

while for the hard device

$$E_H[y]\equiv\int_0^L \{W(y'(X),X)+P(y(X),X)\}\,dX. \tag{1.8}$$

In the latter the function $y(X)$ must satisfy $y(L)=l$. Customarily, the placements $y(X)$ admissible for E_S are not required to satisfy an additional end condition.

2. Specification of the Constitutive Function, Body Force Potential, and Loading Device

The constitutive function, body force potential, and loading device will now be described. Explicit smoothness assumptions will be deferred until the description is complete.

Let α_X, α_X^1, β_X^1, β_X be assigned functions on $[0,L]$ which satisfy

$$0<c_1<\alpha_X<\alpha_X^1<\beta_X^1<\beta_X<c_2<\infty, \tag{2.1}$$

for some constants c_1 and c_2. For fixed X the function $\frac{\partial W}{\partial u}(\cdot,X)$ will be defined on $[\alpha_X,\beta_X]$, and will be assumed

$$\text{strictly increasing on } [\alpha_X,\alpha_X^1],$$
$$\text{strictly decreasing on } [\alpha_X^1,\beta_X^1],$$

and

$$\text{strictly increasing on } [\beta_X^1,\beta_X]. \tag{2.2}$$

The body force function $\frac{\partial P}{\partial y}(y,X)$ will be an assigned function of (y,X) defined on $[0,\infty]\times[0,L]$.

If the bar is placed in a hard device, a constant l, the length of the deformed bar, will be assigned. If the bar is loaded in a soft device, a function $\sigma_0(s)$ defined on $[0,\infty]$ will be given. By combining the assumptions on the domains of W, P, and σ_0 it follows that the domain of the total energy functionals $E_S[\cdot]$ and $E_H[\cdot]$ is the set of all placements $y(X)$ such that the inequalities

$$\alpha_X \leqq y'(X) \leqq \beta_X \tag{2.3}$$

hold wherever $y'(X)$ exists.

A function of two variables is termed a **Carathéodory function** if it is continuous in the first argument for each fixed value of the second, measurable in the second argument for each fixed value of the first, and bounded. ROCKAFELLAR [17, pp. 157, 174–175] deduces some properties of Carathéodory functions which will be useful here. The following smoothness assumptions are made:

i) $\alpha_X, \alpha_X^1, \beta_X^1, \beta_X$ are measurable functions of X.

ii) $\dfrac{\partial W}{\partial u}(u, X)$ and $\dfrac{\partial P}{\partial y}(y, X)$ are Carathéodory functions.

iii) $\sigma_0(s)$ is continuous.

Figure 1 shows an example of such a constitutive function and its domain. For a homogeneous bar the domain of $\dfrac{\partial W}{\partial u}$ reduces to a rectangle and the functions α_X^1 and β_X^1 become constants. Otherwise the bar may be severely inhomogeneous, since $\dfrac{\partial W}{\partial u}(u, \cdot)$, $\alpha_\cdot$, $\alpha_\cdot^1$, $\beta_\cdot^1$, and $\beta_\cdot$ are only assumed to be measurable.

For sufficiently large deformation at $X(y'(X) > \beta_X)$, or sufficiently small deformation at $X(y'(X) < \alpha_X)$, the energy is undefined. Physically we have included the idea that for severe deformation the bar either breaks, or elasticity theory becomes an inadequate description. The constitutive function allows several possible values of the deformation $y'(X)$ for a given stress at X, as long as that stress falls in the range of $\dfrac{\partial W}{\partial u}(\cdot, X)$. Also, inhomogeneity arising from abrupt changes in the cross-section of the bar or from lamination is allowed.

No rational procedure seems to exist for choosing function spaces appropriate to problems in mathematical physics. Sometimes the choice is taken to be the function space that worked well for some associated linearized theory, where the domain of the operator was a linear vector space. Here, the definition of the constitutive equation suggests a more natural procedure.

In order that the descriptions of the total energies E_S and E_H make sense, $y'(X)$ must exist almost everywhere, and be measurable. Still, some possibilities are left open, depending upon the weakness of the derivative. On one hand, we must be cautious not arbitrarily to exclude any solutions, that is to say, arbitrarily exclude any possible physical behavior, by making $y(X)$ too smooth. Hence, the set of piecewise differentiable functions, though kinematically adequate, might be unnecessarily restrictive. The road builder's problem described by YOUNG [15, §61] illustrates the danger. On the other hand, if we try to weaken the derivative too much by, say, choosing the class of monotone, continuous functions for $y(X)$, then, though they are differentiable almost

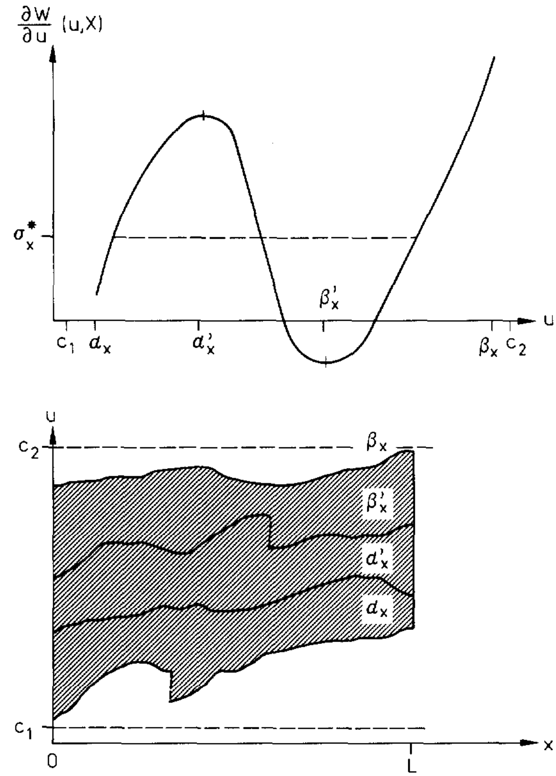

Fig. 1. The constitutive equation and its domain. The independent variable u is the deformation.

everywhere, we can find members of that class which produce an arbitrarily large extension of the bar with essentially zero strain (*i.e.*, $y'(X)=1$ a.e., but $y(L)$ is as large as desired). This is considered unreasonable. The reader is referred to HARDY, LITTLEWOOD & POLYA [16, p. 172, 173] and references therein for details.

The best alternative seems to be the class of absolutely continuous functions. For each absolutely continuous placement $y(X)$, there is an integrable function $f(X)$ such that

$$y(X)=\int_0^X f(Y)\,dY, \tag{2.4}$$

$y'(X)$ exists almost everywhere and is integrable, and $y'(X)=f(X)$, a.e. Integration by parts is possible with these functions.

In order that $y'(X)$ lie in the domain of the total energy functional given by (2.3), we must have,

$$\alpha_X \leqq y'(X) \leqq \beta_X \qquad \text{a.e.,} \tag{2.5}$$

which we may integrate according to (2.4), to obtain

$$\int_{X_1}^{X_2} \alpha_Y \, dY \leqq y(X_2) - y(X_1) \leqq \int_{X_1}^{X_2} \beta_Y \, dY, \qquad \forall X_1 < X_2 \quad \text{in } [0, L]. \tag{2.6}$$

Conversely, if $y(X)$ satisfies (2.6), then, because of the bounds given in (2.1), $y(X)$ is absolutely continuous. Moreover, $y'(X)$, wherever it exists, lies in the domain of the total energy functional, (1.7) or (1.8). Hence, two independent decisions led to the choice of function space, which we now write

$$\mathscr{F} \equiv \{y(X) | y(0) = 0$$

and

$$\int_{X_1}^{X_2} \alpha_Y \, dY \leqq y(X_2) - y(X_1) \leqq \int_{X_1}^{X_2} \beta_Y \, dY, \forall X_1 < X_2 \quad \text{in } [0, L]\}.$$

$\mathscr{F}$ is now the domain of the total energy functional.

The smoothness requirements set down for $W(u, X)$, $P(y, X)$ and $\sigma_0(s)$ assure that total energy functionals E_H and E_S are well defined for any of the placements in $\mathscr{F}$. Specifically, it follows from corollary 2B and proposition 2C in [17] that $W(y'(X), X)$ is a measurable, bounded function of X. $P(y(X), X)$ is clearly integrable, and $P(y, X)$ and $\sigma_0(s)$ have large enough domains to accomodate any member of $\mathscr{F}$.

3. Definitions of Metastability, Infinitesimal, Neutral, and Absolute Stability and Necessary Conditions

Suppose $E_S[y]$ represents the total energy for a soft device. A placement $y \in \mathscr{F}$ is termed **infinitesimally stable** in a soft device if there is a measurable, positive function ε_X such that

i) $y'(X) + \varepsilon_X \leqq \beta_X$, $y'(X) - \varepsilon_X \geqq \alpha_X$ a.e., and
ii) $E_S[y] \leqq E_S[z]$ whenever z is absolutely continuous and

$$|z'(X) - y'(X)| \leqq \varepsilon_X \text{ a.e.} \tag{3.1}$$

Note that because of (i), each z which satisfies (ii) belongs to $\mathscr{F}$; therefore $E_S[z]$ is well defined. A placement y is termed **metastable** in the soft device if it is infinitesimally stable in that device, and the function ε_X is essentially bounded away from zero. A placement y is **absolutely stable** in the soft device if it is metastable in that device, and if

$$E_S[y] < E_S[z], \qquad \text{whenever } z \in \mathscr{F} \quad \text{and} \quad z \not\equiv y. \tag{3.2}$$

A placement y is **neutrally stable** in the soft device if it is metastable and if

$$\begin{aligned} &\text{i) } E_S[y] \leqq E_S(z) && \text{for all } z \in \mathscr{F}, \text{ and} \\ &\text{ii) } E_S[y] = E_S[w] && \text{for some } w \in \mathscr{F},\ w \not\equiv y. \end{aligned} \tag{3.3}$$

In these definitions z is termed a **competitor.** Definitions of kinds of stability for the hard device are the same as the corresponding definitions for the soft device, except that the competitors z must satisfy $z(L)=y(L)$.

The definition of metastability chosen is of fundamental importance for the theory, since the results of the theory are extremely sensitive to changes in it. If y has less total energy than each member of a set of competitors, it might not have less energy than each member of a larger set of competitors. Since I do not wish to exclude metastable placements that occur in nature, even though one might have to perform extremely careful experiments in order to observe them, I have used a strong norm to fix the nearness of the competitors to y in the definition of metastability. It might be objected that the set of competitors is still too large, *e.g.* that z should satisfy not only $|z'(X)-y'(X)|\leqq\varepsilon$, but also $z'(X)-y'(X)\in C^{\infty}_{[0,L]}$. For the energy functional described in this paper, this alteration would make no difference; the definition of metastability given is equivalent to the same definition augmented by the condition that $z'(X)-y'(X)\in C^{\infty}_{[0,L]}$. The proof of this statement follows from (3.1); since the competitor z is absolutely continuous and $z'(X)\leqq\beta_x<c_2$, then given $\delta>0$ there is a $C^{\infty}_{[0,L]}$ function $p_\delta(X)$ for which

$$\begin{aligned}&|p_\delta(X)-z(X)|<\delta,\\ &|p'_\delta(X)-z'(X)|<\delta \quad \text{except on a set of}\\ &\text{measure}<\delta, \quad p_\delta(0)=0.\end{aligned} \tag{3.4}$$

RUDIN [18, thm. 2.23] and the Weierstrass-Stone approximation theorem provide the basis for a proof. Therefore, since $W(\cdot,X)$ and $P(\cdot,X)$ are continuously differentiable, $E[z]$ can be approximated by $E[p_\delta]$.

The definition of metastability corresponds to the definition of a weak relative minimum [19, p. 48–56] in the calculus of variations. In this theory the weak relative minimum and the strong relative minimum (defined with the norm $|z(X)-y(X)|<\varepsilon$, $X\in[0,L]$) are *not* equivalent. It will be shown that a large class of placements which are metastable by the present definition are not strong relative minima for the total energy. In particular, the drop in load that occurs for a homogeneous bar held in a hard device during a stress-extension experiment could not be produced by strong relative minima of the total energy E_H.

The definition of metastability has been framed so that the metastable placement cannot lie on the boundary of the domain of the total energy functional. That is, if $y'(X)=\beta_X$ or $y'(X)=\alpha_X$ on any set of positive measure, then y cannot be metastable. Here, our intention has been to regard the domain of W as the largest domain where elastic bar theory applies; to enlarge the domain would demand the use of a more general theory, perhaps one in which placements need not be continuous. If $y'(X)=\beta_X$ on a set of positive measure, for example, there are placements which lie arbitrarily close to y in the norm $\sup\limits_{[0,L]}|z'(X)|$ which are not governed by the present theory.

The homogeneous bar held by a hard device most clearly illustrates the difficulty. At the homogeneous placement $y(X)=\beta X$ the inequality

$$E_H[y]\leqq E_H[z] \tag{3.5}$$

is trivially satisfied for all $z \in \mathscr{F}$ which satisfy the end condition $z(L) = y(L) = \beta L$. *Simply, there are no other functions in $\mathscr{F}$ which satisfy the end condition.* If there were one, say z, it would have to satisfy

$$z(L) - \beta L = \int_0^L (z'(X) - \beta)\, dX = 0. \tag{3.6}$$

But $z \in \mathscr{F}$ implies that $z'(X) \leqq \beta$ a.e., so from (3.6) $z'(X) = \beta$ a.e. That is, $z \equiv y$. Hence, (3.5) indicates stability, but for an empty class of competitors. For these reasons, I have framed the definition of metastability so that metastable placements lie in the center of the set of competitors.

We turn to the proof of the Euler and Weierstrass necessary conditions for metastability. Suppose $y(X)$ is a metastable placement, and let $E_S[y]$ represent the total energy for the soft loading device. Let $\gamma(X)$ be a Lipschitz function on $[0, L]$, with Lipschitz constant less than or equal to one. Suppose $\gamma(0) = 0$. Then, if $|\delta| < \varepsilon$,

$$z_\delta(X) \equiv y(X) + \delta\, \gamma(X)$$

is a competitor in the definition of metastability. Hence the expression

$$\begin{aligned} E_S(\delta) \equiv E_S[z_\delta] = \int_0^L \{ & W(y'(X) + \delta\, \gamma'(X), X) \\ & + P(y(X) + \delta\, \gamma(X), X)\}\, dX \\ & - \int_L^{y(L)+\delta\gamma(L)} \sigma_0(s)\, ds \end{aligned} \tag{3.7}$$

is well defined for $\delta\, \varepsilon(-\varepsilon, \varepsilon)$. $\dfrac{\partial W}{\partial u}(u, X)$ and $\dfrac{\partial P}{\partial y}(y, X)$ are Carathéodory functions by the assumptions laid down in Section 2. Hence, if the integrand of the first integral in (3.7) is viewed as a function of (δ, X), it is a continuously differentiable function of δ for fixed X, and a measurable bounded function of X. Therefore, the right hand side of (3.7) is a differentiable function of δ on $(-\varepsilon, \varepsilon)$, and the derivative may be carried under the integral sign (see, for example, HOBSON [20, p. 355] for a proof). Since $\sigma_0(s)$ has been assumed continuous, the second integral is also differentiable and

$$\begin{aligned} \left.\frac{dE_S(\delta)}{d\delta}\right|_{\delta=0} &= \int_0^L \left\{ \frac{\partial W}{\partial u}(y'(X), X)\, \gamma'(X) \right. \\ &\quad \left. + \frac{\partial P}{\partial y}(y(X), X)\, \gamma(X) \right\} dX - \sigma_0(y(L))\, \gamma(L) \\ &= \int_0^L \left\{ \left[\frac{\partial W}{\partial u}(y'(X), X) - \sigma_0(y(L)) \right] \gamma'(X) + \frac{\partial P}{\partial y}(y(X), X)\, \gamma(X) \right\} dX. \end{aligned} \tag{3.8}$$

Due to the absolute continuity of $\gamma(X)$, and the condition $\gamma(0)=0$, the body force term may be transformed by parts to yield

$$\left.\frac{dE_S(\delta)}{d\delta}\right|_{\delta=0}=\int_0^L\left\{\frac{\partial W}{\partial u}(y'(X),X)-\sigma_0(y(L))+\int_X^L\frac{\partial P}{\partial y}(y(Y),Y)\,dY\right\}\gamma'(X)\,dX. \tag{3.9}$$

Since $E_S(\delta)$ has a minimum for $\delta=0$ by the definition of metastability, (3.9) leads to the result

$$\int_0^L g(X)\,\gamma'(X)\,dX=0, \tag{3.10}$$

$$g(X)\equiv\frac{\partial W}{\partial u}(y'(X),X)-\sigma_0(y(L))+\int_X^L\frac{\partial P}{\partial y}(y(Y),Y)\,dY,$$

which must be satisfied by all Lipschitz functions $\gamma(X)$ having Lipschitz constant less than or equal to one, and vanishing at $X=0$. Following YOUNG [15, p. 19], we choose $\gamma(x)$ belonging to the set of "stump-shaped" functions. Let $A<B$ be two points in the bar interval $[0,L]$, and suppose $0<H<\frac{B-A}{2}$. Define

$$\gamma(X)\equiv\begin{cases}0 & \text{on } [0,A]\\ X-A & \text{on } [A,A+H]\\ H & \text{on } [A+H,B-H]\\ B-X & \text{on } [B-H,B]\\ 0 & \text{on } [B,L].\end{cases} \tag{3.11}$$

Upon substitution of this choice of γ into (3.10), dividing the resulting equation by H, taking the limit as $H\to 0$, and regarding A and B as any such points in $[0,L]$, we find that $g(X)=\text{const.}$ a.e. By replacing $g(X)$ into (3.10) and using any choice for γ' that has a non-vanishing integral, we find that, in fact,

$$g(X)=0 \quad \text{a.e.}$$

As a necessary condition for metastability in a soft device, a weak form of the Euler equation emerges:

$$\frac{\partial W}{\partial u}(y'(X),X)=-\int_X^L\frac{\partial P}{\partial y}(y(Y),Y)\,dY+\sigma_0(y(L)) \quad \text{a.e.}^\star \tag{3.12}$$

Each metastable placement in $\mathscr{F}$ satisfies this equation. A similiar kind of argument shows that *if y is metastable in a hard device, there is a constant c such*

⋆ If the range of the mapping $\left\{X,\frac{\partial W}{\partial u}(u,X)\right\}$ is closed, then it is possible to adjust the value of $y'(X)$ on a set of measure zero, and thereby not alter the function y, so that (3.12) holds everywhere. The same remark applies to (3.13).

that

$$\frac{\partial W}{\partial u}(y'(X),X) = -\int_X^L \frac{\partial P}{\partial y}(y(Y),Y)\,dY + c \qquad \text{a.e.} \tag{3.13}$$

The Weierstrass condition, suitably qualified, also emerges as a necessary condition for metastability. Let ε be as in the definition of metastability and suppose that constants a_1 and a_2 are subject to the inequalities

$$\begin{aligned} 0 < a_1 \leqq \varepsilon, \\ 0 < a_2 \leqq \varepsilon, \end{aligned} \tag{3.14}$$

and let A be a point in $[0, L]$. Define, for H sufficiently small,

$$\gamma(X,H) = \begin{cases} 0, & X \in [0, A-H], \\ a_1(X+H-A), & X \in [A-H, A], \\ a_2(A-X)+a_1 H, & X \in \left[A, A+\dfrac{a_1}{a_2}H\right], \\ 0, & X \in \left[A+\dfrac{a_1}{a_2}H, L\right]. \end{cases} \tag{3.15}$$

Then the one parameter family of mappings $z(X,H) \equiv y(X) + \gamma(X,H)$ is a family of competitors for either the hard or soft device. Let $E[\cdot]$ represent the total energy of the hard or soft device. If $y(X)$ is metastable,

$$E[z] - E[y] \geqq 0 \tag{3.16}$$

for H sufficiently small. By expanding the inequality (3.16) and using (3.15) we obtain an integral inequality. This inequality is divided by H, and the limit as $H \to 0$ exists for almost every choice of A. By calculating this limit, we obtain the local inequality,

$$\frac{W(y'(X)+a_1,X) - W(y'(X),X)}{a_1} \geqq \frac{W(y'(X),X) - W(y'(X)-a_2,X)}{a_2}, \qquad \text{a.e.} \tag{3.17}$$

If we fix X and regard $y'(X)$ as a point in the domain of $W(\cdot, X)$, then the inequality (3.17) is equivalent to the geometrical statement that the forward secant which connects $Q = \{y'(X),\ W(y'(X),X)\}$ to $R = \{y'(X)+a_1,\ W(y'(X)+a_1,X)\}$ has greater slope than the backward secant which connects $P = \{y'(X)-a_2, W(y'(X)-a_2,X)\}$ to Q. Either from this interpretation, or analytically from (3.17) it is evident that with our assumptions concerning $W(\cdot, X)$ an equivalent statement is that the **Weierstrass condition** holds, *viz.*

$$W(y'(X)+a,X) - W(y'(X),X) - a\frac{\partial W}{\partial u}(y'(X),X) \geqq 0, \qquad \text{a.e.,} \quad \forall |a| < \varepsilon. \tag{3.18}$$

Geometrically, the curve $W(\cdot, X)$ lies above its tangent line at $y'(X)$ on an interval of length 2ε centered at $y'(X)$.

It may be true that in addition to being metastable, $y(X)$ may minimize $E[\cdot]$ with respect to competitors $y(X)+\gamma(X,H)$ with a_1 and a_2 unconstrained by the inequalities (3.14). Evidently, the Weierstrass condition would then hold for values of $|a|$ of magnitude larger than ε.

4. Analysis of the Weierstrass Condition

The Weierstrass condition (3.18) has been shown to be necessary for metastability. Henceforth, sufficient conditions that a placement be metastable will be sought. Toward this end we analyze the Weierstrass conditions in light of the assumed form for the constitutive class. The **Weierstrass excess function** is defined as usual by

$$\mathscr{E}(v,u;X)\equiv W(v,X)-W(u,X)-(v-u)\frac{\partial W}{\partial u}(u,X). \tag{4.1}$$

Plainly $\mathscr{E}(u,u;X)=0$, but $\mathscr{E}(v,u;X)$ is not antisymmetric with respect to exchange of u and v, unless $\frac{\partial W}{\partial u}(u,X)=\frac{\partial W}{\partial u}(v,X)$. We fix X throughout the discussion in this section. It follows from assumptions of smoothness on W that

$$\mathscr{E}(v,u;X)=\int_u^v\frac{\partial W}{\partial u}(\xi,X)\,d\xi-(v-u)\frac{\partial W}{\partial u}(u,X), \tag{4.2}$$

so in order for $\mathscr{E}(v,u;X)\geqq 0$ it is necessary and sufficient that the area under the graph of $\frac{\partial W}{\partial u}(\cdot,X)$ from u to v exceed the area of the rectangle of base $(v-u)$ and height $\frac{\partial W}{\partial u}(u,X)$. This geometrical interpretation of the excess function may ease the analytical treatment given below.

We now regard u as fixed somewhere in the interval $[\alpha_X,\beta_X]$. We seek the set of values of v which make the Weierstrass excess function non-negative; in particular we check to see if all v in a neighborhood of length ε centered at u make the excess function positive.

I shall refer to the constitutive relation restricted to the domain (α_X,α_X^1) as the **α-branch,** and the constitutive relation restricted to the domain (β_X^1,β_X) as the **β-branch** (see Figure 1). The constitutive function $\frac{\partial W}{\partial u}(\cdot,X)$, regarded as a function of u with X fixed, is invertible when restricted to either branch.

The branch with domain $[\alpha_X^1,\beta_X^1]$ will be called the **unstable branch,** because placements with values of their deformation in $[\alpha_X^1,\beta_X^1]$ can never satisfy the Weierstrass necessary condition (3.18). That is, if $u\in[\alpha_X^1,\beta_X^1]$, there is some $\varepsilon>0$ such that either $[u,u+\varepsilon]\subset[\alpha_X^1,\beta_X^1]$ or $[u-\varepsilon,u]\subset[\alpha_X^1,\beta_X^1]$. Suppose the former be true. Since $\frac{\partial W}{\partial u}(\cdot,X)$ is strictly decreasing and continuous on $[u,u+\varepsilon]$, the mean value theorem implies that

$$\int_u^{u+\delta} \frac{\partial W}{\partial u}(\xi, X)\, d\xi < \delta \frac{\partial W}{\partial u}(u, X), \qquad 0 < \delta \leqq \varepsilon. \tag{4.3}$$

Hence $\mathscr{E}(v, u; X) < 0$ for $0 < (v-u) \leqq \varepsilon$. In the latter case the inequality (4.3) holds with $-\varepsilon \leqq \delta \leqq 0$. Therefore, *the Weierstrass condition is violated on the unstable branch.*

The same reasoning shows that if u lies on the domain of the α-branch, or on the domain of the β-branch, the inequality (4.3) is reversed; there is some $\varepsilon > 0$ such that if u belongs to either the α- or β-branch,

$$\mathscr{E}(v, u; X) > 0, \qquad v \in [u-\varepsilon, u+\varepsilon], \qquad v \neq u. \tag{4.4}$$

Some of the constitutive equations in the constitutive class will have the property that

$$\begin{gathered} \frac{\partial W}{\partial u}(\beta_X, X) \geqq \frac{\partial W}{\partial u}(\alpha_X^1, X) \\ \text{and} \\ \frac{\partial W}{\partial u}(\alpha_X, X) \leqq \frac{\partial W}{\partial u}(\beta_X^1, X). \end{gathered} \tag{4.5}$$

For these constitutive relations the ranges of both the α- and β-branches contain the range of the unstable branch, or simply, the graph of the constitutive function forms a complete S. In practice, constitutive relations disobeying (4.5) are found [21], and a special analysis is required for those. The inequalities (4.5) provide sufficient conditions that the Maxwell line, which I shall describe shortly, can be drawn, and they ease other explanations which will follow. I term this subclass of constitutive relations the **S-class.**

Consider a constitutive relation belonging to the S-class. Then, the intersection of the ranges of the α- and β-branches is an interval (σ_1, σ_2). If $\sigma \in (\sigma_1, \sigma_2)$ the equation

$$\frac{\partial W}{\partial u}(u, X) = \sigma \tag{4.6}$$

will have unique, continuous solutions

$$u = \pi_\alpha(\sigma) \quad \text{and} \quad u = \pi_\beta(\sigma) \quad \sigma \in (\sigma_1, \sigma_2), \tag{4.7}$$

respectively, on the α-branch and β-branch. The inverses (4.7) can be extended to the closed interval $[\sigma_1, \sigma_2]$ by continuity. The excess function $\mathscr{E}(\pi_\beta(\sigma), \pi_\alpha(\sigma); X)$ is a strictly decreasing, continuous function of σ, as may be seen easily by differentiating it with respect to σ. Also,

$$\begin{gathered} \mathscr{E}(\pi_\beta(\sigma_1), \pi_\alpha(\sigma_1); X) > 0 \quad \text{and} \\ \mathscr{E}(\pi_\beta(\sigma_2), \pi_\alpha(\sigma_2); X) < 0. \end{gathered} \tag{4.8}$$

Hence there is a unique value σ_X^* such that

$$\mathscr{E}(\pi_\beta(\sigma_X^*), \pi_\alpha(\sigma_X^*); X) = 0, \tag{4.9}$$

which, with (4.6) and the remark following (4.1), implies that

$$\mathscr{E}(\pi_\alpha(\sigma_X^*), \pi_\beta(\sigma_X^*); X) = 0. \tag{4.10}$$

I shall define $\alpha_X^* \equiv \pi_\alpha(\sigma_X^*)$ and $\beta_X^* \equiv \pi_\beta(\sigma_X^*)$. The line connecting the points $P = (\alpha_X^*, \sigma_X^*)$ to $Q = (\beta_X^*, \sigma_X^*)$ in the plot of $\frac{\partial W}{\partial u}(u, X)$ vs. u (see Figure 1) is known as the *Maxwell line*★. By virtue of the interpretation given after equation (4.2), it follows that the Maxwell line cuts off equal areas of the curve above and below.

Having constructed the Maxwell line, one easily determines the sign of $\mathscr{E}(v, u; X)$ for any values of u and v in $[\alpha_X, \beta_X]$. The results are summarized below without proof.★★

a. As we have already shown, if u lies on the α- or β-branch, there is a neighborhood of u such that if v belongs to this neighborhood, and v is unequal to u, $\mathscr{E}(v, u; X) > 0$.
b. If u belongs to $[\alpha_X, \alpha_X^*)$ or $(\beta_X^*, \beta_X]$, $\mathscr{E}(v, u; X)$ is strictly positive for any★★★ $v \neq u$.
c. If $u = \alpha_X^*$, $\mathscr{E}(v, u; X) > 0$ unless $v = u$ or $v = \beta_X^*$ (recall that $\mathscr{E}(\beta_X^*, \alpha_X^*; X) = \mathscr{E}(\alpha_X^*, \beta_X^*; X) = 0$).
d. If $u = \beta_X^*$, $\mathscr{E}(v, u; X) > 0$ unless $v = u$ or $v = \alpha_X^*$.
e. If u belongs to (α_X^*, α_X^1), let v_u and v_β be the points on the unstable branch and the β-branch, respectively, which satisfy

$$\frac{\partial W}{\partial u}(v_u, X) = \frac{\partial W}{\partial u}(v_\beta, X) = \frac{\partial W}{\partial u}(u, X).$$

Then there are unique points $\bar{v} \in (v_u, v_\beta)$ and $\bar{\bar{v}} \in (v_\beta, \beta_X)$ so that[§]

$$\mathscr{E}(v, u; X) \begin{cases} > 0 & \text{for } v \in [\alpha_X, \bar{v}) \cup (\bar{\bar{v}}, \beta_X], \\ < 0 & \text{for } v \in (\bar{v}, \bar{\bar{v}}). \end{cases}$$

f. If u belongs to (β_X^1, β_X^*) an analogous result holds. Let points v_u and v_α on the unstable and α-branches, respectively, satisfy

$$\frac{\partial W}{\partial u}(v_u X) = \frac{\partial W}{\partial u}(v_\alpha, X) = \frac{\partial W}{\partial u}(u, X).$$

★ MAXWELL [22] analyzed a non-invertible constitutive relation for the pressure as a function of density similar to one that had been introduced earlier by JAMES THOMSON [23]. From a thermodynamic argument he deduced that the pressure corresponding to this line would be the equilibrium pressure for the two phases.

★★ The proofs of all of these assertions follow from (4.2) by use of the definition of the Maxwell line and the piecewise strict monotonicity of the constitutive function.

★★★ v is assumed throughout this summary to belong to the basic domain $[\alpha_X, \beta_X]$.

§ The most useful of these inequalities is the second one, which shows that $\mathscr{E}(v_\beta, u; X) < 0$.

Then there are unique points $\hat{\hat{v}}\in[\alpha_X, v_\alpha)$ and $\hat{v}\in(v_\alpha, v_u)$ such that

$$\mathscr{E}(v,u;X)\begin{cases} >0 & \text{for } v\in[\alpha_X,\hat{\hat{v}})\cup(\hat{v},\beta_X], \\ <0 & \text{for } v\in(\hat{\hat{v}},\hat{v}). \end{cases}$$

5. Analysis of the Equilibrium Equation

A weak form of the Euler equation emerged as a necessary condition for metastability of a placement in $\mathscr{F}$. As in the discussion of the preceding section, we aim for sufficient conditions for metastability. The Euler equation and the Weierstrass condition will turn out to form the basis of those conditions.

a. Soft Device

For the soft device the equation of equilibrium is

$$\frac{\partial W}{\partial u}(y'(X),X) = -\int_X^L \frac{\partial P}{\partial y}(y(Y),Y)\,dY + \sigma_0(y(L)) \quad \text{a.e.} \tag{5.1}$$

The bar is conceived as pinned at $X=0$, loaded at $X=L$ with a stress $\sigma_0(y(L))$ which depends on the length of the bar, and pulled in the direction $[0,L]$ by a body force per unit length $-\frac{\partial P}{\partial y}(y(X),X)$.

Solutions of (5.1) are sought from among placements in $\mathscr{F}$. For each of the placements in $\mathscr{F}$, the terms in (5.1) are well defined almost everywhere. It is evident from the assumptions on $P(y,X)$ that a necessary condition for a placement to satisfy (5.1) is that the stress derived from it be an absolutely continuous function of X, even though $y'(\cdot)$ and $\frac{\partial W}{\partial u}(u,\cdot)$ need only be measurable.

Since placements with values of $y'(X)$ in the domain of the unstable branch can never be metastable, we shall only look for solutions with deformations on the α- or β-branches. If on a subinterval of the bar an equilibrium solution has values of its deformation on the α-branch (β-branch) only, then we shall say that the subinterval is in the *α-phase* (*β-phase*). Suppose $\frac{\partial W}{\partial u}(u,\cdot)$ is a smooth function of X. The argument put forth in the preceding paragraph combined with the invertibility of $\frac{\partial W}{\partial u}(\cdot,X)$ on either branch by itself shows that in a particular phase, $y(X)$ possesses an absolutely continuous first derivative.

For simplicity, the constitutive function will be taken from the S-class (see equation (4.5)).

The general proof of existence for (5.1) proceeds by inversion of the left hand side and then integration. The inversion problem can be solved in simple cases, and then can be generalized to handle (5.1). Since the simple cases hold some interest by themselves, and the proof is not essentially lengthened by considering them, we adopt a method of proof which proceeds by solving successively more difficult problems.

The first problem has been solved by ERICKSEN [4]. The bar is assumed homogeneous, the loading device dead, and the body force null. The problem reduces to finding solutions of

$$\frac{dW}{du}(y'(X))=\sigma_0. \tag{5.2}$$

$\frac{dW}{du}(\cdot)$ is continuous on $[\alpha,\beta]$ and the constant σ_0 is assigned. If σ_0 does not belong to the range of $\frac{dW}{du}(\cdot)$, there is no solution. Assume σ_0 belongs to the range of $\frac{dW}{du}(\cdot)$. At σ_0, $\frac{dW}{du}(\cdot)$ may have a single or double valued inverse. In the former case, there is one and only one value $\gamma\in[\alpha,\beta]$ such that

$$\frac{dW}{du}(\gamma)=\sigma_0 \tag{5.3}$$

The unique solution of (5.3) which lies in $\mathscr{F}$ is the homogeneous placement $y(X)=\gamma X$. In the latter case, there are exactly two values $\mu<\nu$ that satisfy (5.2). Therefore, every solution has the property that

$$y'(X)=\begin{cases}\mu, & X\in S_\alpha,\\ \nu, & X\in S_\beta,\end{cases}\qquad S_\alpha\cup S_\beta=[0,L], \tag{5.4}$$

S_α and S_β being disjoint subsets of the bar. In order that $y\in\mathscr{F}$, it is necessary and sufficient that S_α and S_β be measurable sets. Every solution may then be written

$$y(X)=\int_{[0,X]\cap S_\alpha}\mu\,dX+\int_{[0,X]\cap S_\beta}\nu\,dX. \tag{5.5}$$

A given stress, therefore, corresponds to a continuum of solutions.

The bar will now be assumed inhomogeneous and the body force gravitational, but the loading device shall remain dead. The equation of equilibrium becomes

$$\frac{\partial W}{\partial u}(y'(X),X)=\int_X^L\rho(Y)\,g\,dY+\sigma_0\equiv B(X,\sigma_0). \tag{5.6}$$

The right hand side of (5.6) is denoted by $B(X,\sigma_0)$, g is a non-negative constant, and the bounded measurable function $\rho(Y)$ is essentially bounded away from zero. Again looms the possibility of non-existence. Solutions will exist if and only if $B(X,\sigma_0)$ belongs to the range $\frac{\partial W}{\partial u}(\cdot,X)$ for almost all $X\in[0,L]$:

$$\frac{\partial W}{\partial u}(\alpha_X,X)\leqq\int_X^L\rho(Y)\,g\,dY+\sigma_0\leqq\frac{\partial W}{\partial u}(\beta_X,X)\qquad\text{a.e.} \tag{5.7}$$

Assuming (5.7) to hold, we define sets $\mathcal{S}_\alpha$ and $\mathcal{S}_\beta$ by

$$\begin{aligned}\mathcal{S}_\alpha &\equiv \left\{ X \in [0,L] \,|\, B(X,\sigma_0) < \frac{\partial W}{\partial u}(\alpha_X^1, X) \right\},\\ \mathcal{S}_\beta &\equiv \left\{ X \in [0,L] \,|\, B(X,\sigma_0) > \frac{\partial W}{\partial u}(\beta_X^1, X) \right\}.\end{aligned} \tag{5.8}$$

Since the set on which a measurable function is positive is measurable, then these sets are measurable, and from (5.7) their union is the interval $[0,L]$. On $\mathcal{S}_\alpha$ the inverse of $\frac{\partial W}{\partial u}(\cdot, X)$ corresponding to the α-branch may be used, and on $\mathcal{S}_\beta$ the inverse corresponding to the β-branch may be used. Typically $\mathcal{S}_\alpha \cap \mathcal{S}_\beta \neq \emptyset$, and so we define $S = \mathcal{S}_\alpha \cap \mathcal{S}_\beta$. On S we have an ambiguous choice for an inverse, either the α- or β-branch will do. Hence we arbitrarily divide S into two disjoint, measurable sets:

$$S = S_\alpha \cup S_\beta, \qquad S_\alpha \cap S_\beta = \emptyset. \tag{5.9}$$

On $(\mathcal{S}_\alpha - S) \cup S_\alpha$ the inverse for the α-branch will be used, and on $(\mathcal{S}_\beta - S) \cup S_\beta$ the inverse for the β-branch will be used.

We now construct those inverses. Over the domain of the α-branch, $\frac{\partial W}{\partial u}(\cdot, X)$ is invertible; its inverse will be called $\pi_\alpha(\sigma, X)$. It is not hard to prove that $\pi_\alpha(\cdot,\cdot)$ is a Carathéodory function. π_α is clearly continuous in σ for fixed X, and bounded. It remains to prove that $\pi_\alpha(u,\cdot)$ is measurable. For this proof it is convenient to extend the domain of $\frac{\partial W}{\partial u}(\cdot, X)$ momentarily to all of $\mathbb{R}^1$ and then to invert. That is, we attach linear increasing functions at $\frac{\partial W}{\partial u}(\alpha_X, X)$ and at $\frac{\partial W}{\partial u}(\alpha_X^1, X)$ to construct a Carathéodory function $\frac{\partial W^e}{\partial u}(u, X)$ defined on $\mathbb{R}^1 \times [0,L]$. The restriction of $\frac{\partial W^e}{\partial u}(\cdot, X)$ to the domain (α_X, α_X^1) is the α-branch. Then we invert $\frac{\partial W^e}{\partial u}(\cdot, X)$ and call its inverse $\pi_\alpha^e(\cdot, X)$. If $\pi_\alpha^e(u,\cdot)$ is measurable, then so is $\pi_\alpha(u,\cdot)$. But

$$\{X \,|\, \pi_\alpha^e(\sigma, X) > c\} = \left\{ X \,\middle|\, \frac{\partial W^e}{\partial u}(c, X) < \sigma \right\},$$

and the second set is measurable by definition. Hence the first is also measurable. This completes the proof that π_α is a Carathéodory function. The Carathéodory function $\pi_\beta(\sigma, X)$ is defined to be the inverse of the β-branch.

Solutions of (5.6) may now be constructed. Define

$$y'(X) = \begin{cases} \pi_\alpha(B(X,\sigma_0), X), & X \in (\mathcal{S}_\alpha - S) \cup S_\alpha, \\ \pi_\beta(B(X,\sigma_0), X), & X \in (\mathcal{S}_\beta - S) \cup S_\beta, \end{cases} \tag{5.10}$$

and integrate to obtain a solution $y \in \mathscr{F}$. As in the case of the homogeneous bar with null body force, there are, typically, many solutions corresponding to a given load. Unless the bar has some peculiar kind of inhomogeneity, S will not be null. Any two disjoint, measurable subsets of S will determine a solution. It is evident by reversing the proof that every solution of (5.6) must be delivered by (5.10) for some choices of S_α and S_β.

With these results in hand extracted from special cases, it is possible to tackle the general problem (5.1).

Theorem 1. *Suppose that for every* $y \in \mathscr{F}$ *satisfying*

$$\hat{c}_1 X \leqq y(X) \leqq \hat{c}_2 X \quad \forall X \in [0, L],$$

$$\frac{\partial W}{\partial u}(\alpha_X, X) \leqq -\int_X^L \frac{\partial P}{\partial y}(y(Y), Y)\, dY + \sigma_0(y(L)) \leqq \frac{\partial W}{\partial u}(\beta_X, X) \tag{5.11}$$

for each $X \in [0, L]$. *Then* (5.1) *has a solution* $y(X)$ *which belongs to* $\mathscr{F}$.

A sketch of the proof is provided in the appendix.

b. Hard Device

For a bar loaded in a hard device, a constant l is assigned. A solution of the equation of equilibrium is a placement $y \in \mathscr{F}$ which satisfies both

$$y(L) = l \tag{5.29}$$

and

$$\frac{\partial W}{\partial u}(y'(X), X) = -\int_X^L \frac{\partial P}{\partial y}(y(Y), Y)\, dY + c \qquad \text{a.e.} \tag{5.30}$$

for some constant c. We regard c as disposable; if a constant c can be found so that (5.29) and (5.30) hold for $y \in \mathscr{F}$, then y is a solution. Again, we only seek solutions whose values of y' lie on the domains of α- and β-branches.

The simple case of a homogeneous bar acted upon by a null body force will be treated first. The appropriate equation is

$$\frac{dW}{du}(y'(X)) = c. \tag{5.31}$$

I shall drop X from the notation whenever dependence upon it is trivial; for a homogeneous bar α_X, α_X^1, β_X, *etc.* become α, α^1, β, *etc.* If the assigned length l is too large or too small, no solutions will exist. Explicitly, if

$$l > \beta L, \tag{5.32}$$

or, equivalently, if

$$\int_0^L (y'(X) - \beta)\, dX > 0 \tag{5.33}$$

then $y'(X) > \beta$ on a set of positive measure, which contradicts the assumption that $y \in \mathscr{F}$. Hence, no solutions are possible if $l > \beta L$. The same conclusion follows if $l < \alpha L$. Henceforth we assume that

$$\alpha L < l < \beta L. \tag{5.34}$$

It will be useful for the present discussion to name two additional points in the domain of the constitutive function. Let $\hat{\alpha}$ and $\hat{\beta}$ be the unique points which satisfy

$$\begin{aligned} \frac{dW}{du}(\hat{\alpha}) &= \frac{dW}{du}(\beta^1), \quad \hat{\alpha} \neq \beta^1, \\ \frac{dW}{du}(\hat{\beta}) &= \frac{dW}{du}(\alpha^1), \quad \hat{\beta} \neq \alpha^1. \end{aligned} \tag{5.35}$$

It has been tacitly assumed that the constitutive function belongs to the S-class. We distinguish several possible assignments of l:

1. $\hat{\beta} L < l < \beta L$. In this case

$$\int_0^L y'(X) - \hat{\beta}\, dX > 0, \tag{5.36}$$

so that $y'(X) > \hat{\beta}$ on a set of positive measure. This fact and the equilibrium equation (5.31) imply that $c > \frac{dW}{du}(\hat{\beta})$. But then $\frac{\partial W}{\partial u}(\cdot)$ has a unique inverse at c,

$$\frac{dW}{du}(v) = c, \quad \hat{\beta} < v \leq \beta. \tag{5.37}$$

Therefore the unique solution of (5.31) is the homogeneous placement $y(X) = vX$, in which $v = \frac{l}{L}$.

2. $\alpha L \leq l \leq \hat{\alpha} L$. This case is completely analogous to the preceding one. There is only one solution,

$$y(X) = \frac{l}{L} X. \tag{5.38}$$

3. $\hat{\alpha} L \leq l \leq \hat{\beta} L$. Here $y'(X) \leq \hat{\beta}$ on a set of positive measure, and $y'(X) \geq \hat{\alpha}$ on a set of positive measure, so (5.31) implies that

$$\frac{dW}{du}(\beta^1) \leq c \leq \frac{dW}{du}(\alpha^1). \tag{5.39}$$

The inequalities (5.39) assure that every solution of (5.31) may be parameterized by c:

$$y'(X) = \begin{cases} \pi_\alpha(c) & X \in S_\alpha \\ \pi_\beta(c) & X \in S_\beta. \end{cases} \tag{5.40}$$

S_α and S_β are disjoint, measurable subsets of $[0, L]$. Let the measures of S_α and S_β be denoted by m_α and m_β, respectively. By virtue of (5.40), the condition $y(L)=l$ becomes

$$\pi_\alpha(c)\, m_\alpha + \pi_\beta(c)\, m_\beta = l, \tag{5.41}$$

in which

$$m_\alpha + m_\beta = L. \tag{5.42}$$

The equations (5.41) and (5.42) comprise a linear system of equations for m_α and m_β, to which must be adjoined the inequalities $m_\alpha \geqq 0$ and $m_\beta \geqq 0$. The constants l and L are given, whereas the constant c may be adjusted within the limits imposed by (5.39). The determinant of the system is $\pi_\beta(c) - \pi_\alpha(c)$ which can never be zero, so the system has the unique solution,

$$\begin{aligned} m_\alpha &= \frac{1}{\pi_\beta(c) - \pi_\alpha(c)} (\pi_\beta(c)\, L - l), \\ m_\beta &= \frac{1}{\pi_\beta(c) - \pi_\alpha(c)} (l - \pi_\alpha(c)\, L). \end{aligned} \tag{5.43}$$

The measures m_α and m_β are non-negative if and only if

$$\pi_\alpha(c)\, L \leqq l \leqq \pi_\beta(c)\, L. \tag{5.44}$$

Still regarding l as assigned gives (5.44) as a condition on c such that (5.43) corresponds to a solution of the problem. Since $\max_c \pi_\alpha(c) \leqq \min_c \pi_\beta(c)$, (5.44) is satisfied by a closed interval of values of c. That interval reduces to a single point if $l = \hat{\beta} L$ or $l = \hat{\alpha} L$, but, otherwise, it has a non-empty interior. If it happens that $\alpha^1 L \leqq l \leqq \beta^1 L$, that interval will be the entire set defined by (5.39). To each c in this interval corresponds a solution given by

$$y(X) = \int_0^X \{\pi_\alpha(c)\, \chi_{S_\alpha} + \pi_\beta(c)\, \chi_{S_\beta}\}\, dX. \tag{5.45}$$

The measures of S_α and S_β satisfy (5.43).

In Case 3, non-uniqueness arises from two sources. First, any value of c in the appropriate interval produces a solution. Once c has been chosen, the measures of the sets S_α and S_β, but not the sets themselves, are fixed by the equations (5.43). Cases 1, 2 and 3, above, deliver all solutions to the problem (5.31).

We now treat the case of an inhomogeneous bar pulled by a gravitational body force. The equation of equilibrium may be written

$$\frac{\partial W}{\partial u}(y'(X), X) = \int_X^L \rho(Y)\, g\, dY + c \equiv B(X, c); \tag{5.46}$$

$B(X, c)$ denotes the right hand side, and c can be adjusted to secure existence. Solutions will exist only if l has certain values, and only if the body force is sufficiently weak. Recall that $\rho(Y)$ is a positive, measurable, bounded function which is essentially bounded away from zero. We assume that

$$\operatorname*{ess\,sup}_{X\in[0,L]} \frac{\partial W}{\partial u}(\alpha_X, X) - \int_X^L \rho(Y)\,g\,dY < \operatorname*{ess\,inf}_{X\in[0,L]} \frac{\partial W}{\partial u}(\beta_X, X) - \int_X^L \rho(Y)\,g\,dY. \tag{5.47}$$

Let σ_1 and σ_2 be the values of the left hand side and of the right hand side of (5.47), respectively. A necessary condition that there be a solution of (5.46) is that $c\in[\sigma_1, \sigma_2]$.

Rather than prescribe the length of the bar beforehand, we shall describe the set of all solutions and, then, examine the possible lengths which can be produced. Let $c\in[\sigma_1, \sigma_2]$ and define

$$\begin{aligned} \mathscr{S}_\alpha(c) &= \left\{ X \,|\, B(X,c) < \frac{\partial W}{\partial u}(\alpha_X^1, X) \right\}, \\ \mathscr{S}_\beta(c) &= \left\{ X \,|\, B(X,c) > \frac{\partial W}{\partial u}(\beta_X^1, X) \right\}. \end{aligned} \tag{5.48}$$

It follows from (5.47), (5.48) and the definition of the constitutive function that

$$\mathscr{S}_\alpha(c) \cup \mathscr{S}_\beta(c) = [0, L]. \tag{5.49}$$

Following the notation already introduced, we define

$$S(c) \equiv \mathscr{S}_\alpha(c) \cap \mathscr{S}_\beta(c). \tag{5.50}$$

On $\mathscr{S}_\alpha$ the inverse π_α may be used, and on $\mathscr{S}_\beta$ the inverse π_β may be used. There is an ambiguous choice of inverse on the set $S(c)$. $S(c)$ is divided into two arbitrary, disjoint measurable subsets,

$$S(c) = S_\alpha \cup S_\beta, \tag{5.51}$$

and the solution corresponding to this choice is given by

$$y'(X) = \begin{cases} \pi_\alpha(B(X,c), X) & X\in(\mathscr{S}_\alpha - S)\cup S_\alpha \\ \pi_\beta(B(X,c), X) & X\in(\mathscr{S}_\beta - S)\cup S_\beta. \end{cases} \tag{5.52}$$

If we forget momentarily the end condition $y(L) = l$, equation (5.52) delivers all solutions of (5.46). Therefore each end position which belongs to a solution is produced by the formula

$$\int_{(\mathscr{S}_\alpha - S)\cup S_\alpha} (B(Y,c), Y)\,dY + \int_{(\mathscr{S}_\beta - S)\cup S_\beta} (B(Y,c), Y)\,dY = l, \tag{5.53}$$

for some choice of c, and for some partition of $S(c)$ into sets S_α and S_β. To characterize the set of all end positions which correspond to solutions of the problem, we examine (5.53). First, we note that $\pi_\alpha(\mu, X) < \pi_\beta(\nu, X)$ no matter what be the values μ and ν, as long as they are chosen to lie in the appropriate domains. If $\sigma_1 \leqq \bar{c} < \bar{\bar{c}} \leqq \sigma_2$, then $\mathscr{S}_\alpha(\bar{c}) \supset \mathscr{S}_\alpha(\bar{\bar{c}})$ and $\mathscr{S}_\beta(\bar{c}) \subset \mathscr{S}_\beta(\bar{\bar{c}})$. Hence, for any

$$c\in[\sigma_1, \sigma_2], \quad \mathscr{S}_\beta(\sigma_2) \supset \mathscr{S}_\beta(c) \supset (\mathscr{S}_\beta(c) - S(c)) \cup S_\beta.$$

It follows that the maximum value of (5.53) is given by

$$l_{\max}=\int_{\mathscr{S}_\alpha(\sigma_2)-S(\sigma_2)}\pi_\alpha(B(Y,\sigma_2),Y)\,dY+\int_{\mathscr{S}_\beta(\sigma_2)}\pi_\beta(Y,\sigma_2)\,Y)\,dY. \tag{5.54}$$

By similar reasoning,

$$l_{\min}=\int_{\mathscr{S}_\alpha(\sigma_1)}\pi_\alpha(B(Y,\sigma_1),Y)\,dY+\int_{\mathscr{S}_\beta(\sigma_1)-S(\sigma_1)}\pi_\beta(B(Y,\sigma_1),Y)\,dY. \tag{5.55}$$

It remains to ascertain whether any $l\in[l_{\min},l_{\max}]$ determines a solution. The definitions (5.48) of $\mathscr{S}_\alpha(c)$ and $\mathscr{S}_\beta(c)$ show that, given $\varepsilon>0$ and given $c\in[\sigma_1,\sigma_2]$ there is a $\delta>0$ such that if

$$|\hat{c}-c|<\delta, \quad \text{then} \begin{cases} m\{\mathscr{S}_\alpha(\hat{c})-\mathscr{S}_\alpha(c)\}<\varepsilon, \\ m\{\mathscr{S}_\beta(\hat{c})-\mathscr{S}_\beta(c)\}<\varepsilon, \end{cases} \tag{5.56}$$

in which m denotes Lebesgue measure. Since

$$\begin{aligned} S(\hat{c})-S(c) &= [\mathscr{S}_\alpha(\hat{c})\cap\mathscr{S}_\beta(c)]-[\mathscr{S}_\alpha(c)\cap\mathscr{S}_\beta(c)] \\ &= [(\mathscr{S}_\alpha(\hat{c})\cap\mathscr{S}_\beta(\hat{c}))-\mathscr{S}_\alpha(c)]\cup[(\mathscr{S}_\alpha(\hat{c})\cap\mathscr{S}_\beta(\hat{c}))-\mathscr{S}_\beta(c)] \\ &\subset [\mathscr{S}_\alpha(\hat{c})-\mathscr{S}_\alpha(c)]\cup[\mathscr{S}_\beta(\hat{c})-\mathscr{S}_\beta(c)], \end{aligned} \tag{5.57}$$

then whenever $|\hat{c}-c|<\delta$,

$$\max[m\{S(\hat{c})-S(c)\},m\{S(c)-S(\hat{c})\}]<2\varepsilon. \tag{5.58}$$

We shall use the notation $m\{|S(\hat{c})-S(c)|\}$ to denote the left hand side of (5.58). Let $R_\alpha(c)$ and $R_\beta(c)$ be a pair of one parameter families of disjoint, measurable sets such that $R_\alpha(c)\cup R_\beta(c)=[0,L]\,\forall c$. Suppose the families $R_\alpha(c)$ and $R_\beta(c)$ satisfy

$$\begin{gathered} |\hat{c}-c|<\delta\Rightarrow m\{|R_\alpha(\hat{c})-R_\alpha(c)|\}<\varepsilon \quad\text{and}\quad m\{|R_\beta(\hat{c})-R_\beta(c)|\}<\varepsilon, \\ R_\beta(\sigma_1)=\emptyset, \qquad R_\alpha(\sigma_2)=\emptyset. \end{gathered} \tag{5.59}$$

We shall choose

$$\begin{aligned} S_\alpha(c)&\equiv R_\alpha(c)\cap S(c), \\ S_\beta(c)&\equiv R_\beta(c)\cap S(c), \end{aligned}$$

so that $S_\alpha(\sigma_1)=S$ and $S_\beta(\sigma_2)=S$. With these choices of S_α and S_β, l becomes a function of c. Also

$$\begin{aligned} &[(\mathscr{S}_\alpha(\hat{c})-S(\hat{c}))\cup S_\alpha(\hat{c})]-[(\mathscr{S}_\alpha(c)-S(c))\cup S_\alpha(c)] \\ &\quad\subset[\mathscr{S}_\alpha(\hat{c})-\mathscr{S}_\alpha(c)]\cup[S(c)-S(\hat{c})]\cup[S_\alpha(\hat{c})-S_\alpha(c)]. \end{aligned} \tag{5.60}$$

It follows from (5.56), (5.58) and $(5.59)_1$ that

$$m\{|[(\mathscr{S}_\alpha(\hat{c})-S(\hat{c}))\cup S_\alpha(\hat{c})]-[(\mathscr{S}_\alpha(c)-S(c))\cup S_\alpha(c)]|\}<6\,\varepsilon. \tag{5.61}$$

The construction of S_α and S_β just outlined insures that $l(c)$ is a continuous function on $[\sigma_1,\sigma_2]$, which assumes its extremes at σ_1 and σ_2. Therefore, every value of $c\in[\sigma_1,\sigma_2]$ produces a solution. Typically, many solutions correspond to each value of c.

If the end position is regarded as disposable, the general proof of existence for the soft device may be used for the hard device. A continuous dependence theorem, like the one that has just been demonstrated in the special case of a gravitational body force, appears difficult.

6. Existence of Metastable Solutions

Having collected equilibrium solutions in the preceding section, we now select the metastable ones. Suppose $y(X)$ is one of these equilibrium solutions. We assume, as before, that $y'(X)$ is contained in the domain of the α- or β-branch. The set of values of v which makes the excess function $\mathscr{E}(v,y'(X);X)$ positive can be found in Section 4. The α- and β-branches were defined on open intervals; hence, given any $y(X)$ with $y'(X)$ belonging to the α-branch or the β-branch, there is a positive, measurable function ε_X such that

$$\mathscr{E}(v,y'(X);X)\geqq 0 \qquad \text{for } |v-y'(X)|<\varepsilon_X. \tag{6.1}$$

If the body force potential and loading device do not promote instability, a notion that will be made precise shortly, $y(X)$ will be proved infinitesimally stable. If, additionally, $\operatorname*{ess\,inf}_{X\in[0,L]}\varepsilon_X>0$, $y(X)$ will be proved metastable.

However, the placement $y(X)$ may be metastable under weaker conditions. Certain well behaved constitutive functions, for example, may insure the metastability of a placement when, otherwise, the body force and loading device would tend to deny it. The classical methods of JACOBI will be used to explore sufficient conditions for metastability.

It is convenient to introduce **excess functions** for the body force and loading device, by analogy to the Weierstrass excess function:

$$\begin{aligned} \mathscr{B}(z,y;X) &\equiv P(z,X)-P(y,X)-(z-y)\frac{\partial P}{\partial y}(y,X),\\ \mathscr{L}(z,y) &\equiv \int_y^z \sigma_0(s)\,ds-(z-y)\,\sigma_0(y). \end{aligned} \tag{6.2}$$

Suppose $y(X)$ is a solution of the equation of equilibrium (5.1) for the soft device. Let $z\in\mathscr{F}$. Then the difference $E_S[z]-E_S[y]$ is given by

$$\begin{aligned} E_S[z]-E_S[y] = \int_0^L \{&\mathscr{E}(z'(X),y'(X);X)\\ &+\mathscr{B}(z(X),y(X);X)\}\,dX-\mathscr{L}(z(L),y(L)). \end{aligned} \tag{6.3}$$

The integration by parts involved in this calulation is justified for the functions considered. If w and y belong to $\mathscr{F}$, $w(L)=y(L)$, and y is a solution of the equilibrium equation (5.30) for the hard device, then it follows by similar reasoning that

$$E_H[w]-E_H[y]=\int_0^L \{\mathscr{E}(w'(X),y'(X);X)+\mathscr{B}(w(X),y(X);X)\}\,dX. \tag{6.4}$$

Strong sufficient conditions for infinitesimal stability are now evident; suppose that the inequalities

$$\mathscr{B}(z(X),y(X);X)\geqq 0 \quad \text{and} \quad \mathscr{L}(z(L),y(L))\leqq 0 \tag{6.5}$$

hold for all z satisfying $|z'(X)-y'(X)|<\varepsilon_X$ a.e. Then from (6.3) and (6.4), y is infinitesimally stable in the hard or soft device, according to which equilibrium equation it solves. Of course, the first inequality in (6.5) need only hold for such z that satisfy the end condition, in the case of the hard device. If, in addition to (6.5), both of the conditions

(i) $\operatorname*{ess\,inf}_{X\in[0,L]} \varepsilon_X=\varepsilon>0$

and

(ii) $y'(X)-\varepsilon\geqq\alpha_X, \quad y'(X)+\varepsilon\leqq\beta_X$

are fulfilled, then y is metastable.

Body forces and loading devices which fail to satisfy (6.5) are not exceptional, although the common examples of gravitational body force and dead loading device satisfy (6.5) with equality. If a loading device satisfies $(6.5)_2$ with strict inequality, a reversal in sign of the function $\sigma_0(s)$, *i.e.* a reversal in the direction of the end load produces one which causes $(6.5)_2$ to fail. Likewise, the body force produced by rotating the bar with constant spin about an axis perpendicular to the bar, which passes through the point $X=0$, will never satisfy $(6.5)_1$. However, the bar may be metastable under the influence of such body forces and loading devices, as we shall show presently.

Of course, we cannot weaken the Weierstrass condition, $\mathscr{E}(z'(X),y'(X):X)\geqq 0$, no matter what body force and loading device we choose since it is a necessary condition for metastability. In fact, it must be slightly strengthened in order to incorporate a larger class of body forces and loading devices than allowed by (6.5). We shall say that a placement $y(X)$ **promotes stability** if there is a continuous function $s(X)>0$, such that

$$\frac{\dfrac{\partial W}{\partial u}(y'(X)+\delta,X)-\dfrac{\partial W}{\partial u}(y'(X),X)}{\delta}\geqq s(X) \quad \text{a.e.,} \tag{6.6}$$

whenever $\delta\in[-\varepsilon,\varepsilon]$. Here, ε is a constant independent of the choice of X. It follows by integration of (6.6) with respect to δ that if y promotes stability, then

$$\mathscr{E}(y'(X)+\delta,y'(X);X)\geqq s(X)\frac{\delta^2}{2}, \qquad \delta\in[-\varepsilon,\varepsilon], \qquad \text{a.e.} \tag{6.7}$$

The geometrical meaning of (6.6) is plain: in particular, the deformation $y'(X)$ cannot equal α_X^1 or β_X^1, except on a set of measure zero, if y promotes stability.

The following main theorems on metastability show that an equilibrium solution $y(X)$ which promotes stability can be metastable under a larger class of body forces and loading devices allowed by (6.5).

Theorem 2. *Let $\varepsilon = \text{const.} > 0$ be given. Suppose $y \in \mathscr{F}$ is a solution of the equation of equilibrium for the soft device such that*

$$y'(X) + \varepsilon < \beta_X \quad and \quad y'(X) - \varepsilon > \alpha_X \quad \text{a.e.} \tag{6.8}$$

Assume y promotes stability, i.e. *there is a continuous, positive function $s(X)$ for which*

$$\frac{\dfrac{\partial W}{\partial u}(y'(X)+\delta, X) - \dfrac{\partial W}{\partial u}(y'(X), X)}{\delta} \geqq s(X) \cdot \forall \delta \in [-\varepsilon, \varepsilon]. \tag{6.6$_r$}$$

Suppose, in addition, that there is a continuous function $p(X)$ and a constant k such that

$$\frac{\sigma_0(y(L)+\mu) - \sigma_0(y(L))}{\mu} \leqq k \tag{6.9}$$

and

$$\frac{\dfrac{\partial P}{\partial y}(y(X)+\tau, X) - \dfrac{\partial P}{\partial y}(y(X), X)}{\tau} \geqq p(X) \tag{6.10}$$

whenever $\mu \in [-\varepsilon L, \varepsilon L]$ and $\tau \in [-\varepsilon X, \varepsilon X]$. Then if $|z'(X) - y'(X)| < \varepsilon$,

$$\mathscr{E}(z'(X), y'(X); X) \geqq \frac{s(X)}{2}(z'(X) - y'(X))^2, \tag{6.7$_r$}$$

$$\mathscr{L}(z(L), y(L)) \leqq \frac{k}{2}(z(L) - y(L))^2, \tag{6.11}$$

$$\mathscr{B}(z(X), y(X); X) \geqq \frac{p(X)}{2}(z(X) - y(X))^2. \tag{6.12}$$

(i) *Now suppose only that* $(6.7)_r$, (6.11) *and* (6.12) *are satisfied for some given continuous functions $s(X) > 0$ and $p(X)$, for some given constant k, and for every function $z(X)$ such that $|z'(X) - y'(X)| < \varepsilon$* a.e. *Let $h(X) = z(X) - y(X)$, $X \in [0, L]$, and suppose $|h'(X)| < \varepsilon$* a.e.
Then

$$E_S[z] - E_S[y] \geqq \frac{1}{2}\int_0^L \{s(X)\,h'^2 - 2k\,h\,h' + p(X)\,h^2\}\,dX. \tag{6.13}$$

(ii) *Now assume only that* (6.13) *holds for continuous functions $s(X) > 0$, $p(X)$ and a constant k, and for $h(X) \equiv z(X) - y(X)$, $|h'(X)| < \varepsilon$* a.e. *Let*

$$\min_{[0,L]} s(X) \equiv S, \qquad \min_{[0,L]} p(X) \equiv P. \tag{6.14}$$

Then the placement y is metastable in the soft device if

$$\left.\begin{array}{l} P>0 \\ \\ P=0 \\ \\ P<0 \end{array}\right\} \text{ and } \left\{\begin{array}{l} k<\sqrt{SP}\,\dfrac{\left(\exp 2L\sqrt{\dfrac{P}{S}}\right)+1}{\left(\exp 2L\sqrt{\dfrac{P}{S}}\right)-1}, \\ k<S/L, \\ k<\dfrac{-\sqrt{-SP}}{\tan\left(-\sqrt{\dfrac{-P}{S}}L\right)}. \end{array}\right. \tag{6.15}$$

For the hard device we have the corresponding theorem:

Theorem 3. *Let $\varepsilon>0$ be given. Suppose y is a solution of the equation of equilibrium for the hard device which satisfies* (6.8) *and promotes stability. Assume* (6.12) *is satisfied for a continuous function $p(X)$ and for $|z'(X)-y'(X)|<\varepsilon$* a.e. *Let*

$$P\equiv\min_{[0,L]} p(X). \tag{6.16}$$

Then the placement y is metastable in the hard device if

$$P>0 \quad \text{or} \quad \left\{\begin{array}{l} P<0 \quad \text{and} \\ L\sqrt{\dfrac{-P}{S}}<\pi. \end{array}\right. \tag{6.17}$$

Before constructing the proofs of these theorems, we note that they allow a larger class of body forces and loading devices than allowed by (6.5). This becomes evident when (6.11) and (6.12) are combined with (6.15); (6.15) allows $k=0$ and $p\equiv 0$, but some positive values of k and some negative values of $p(X)$ are included as well. It will be clear from the proofs of these theorems how the bounds (6.15) can be improved in special cases by numerical methods.

The results of these theorems can be interpreted directly from (6.14), (6.15) and (6.17). One result of interest concerns the dependence on L. Suppose the sufficient conditions for metastability outlined in Theorem 2 are satisfied for one constant L. Let $\hat{L}$ be less than L, and consider the bar interval $[0,\hat{L}]$. Suppose a new soft loading device applied at $y(\hat{L})$ is given by

$$\hat{\sigma}_0(s)=\sigma_0(s+y(L)-y(\hat{L})). \tag{6.18}$$

Also, let the constitutive function, body force potential and function space be restricted to the interval $[0,\hat{L}]$. *Then* (6.15) *shows that $y(X)$, $X\in[0,\hat{L}]$ is metastable in the soft loading device defined by* (6.18). For the hard loading device, if the bar is restricted to the interval $[0,\hat{L}]$, and the position $y(\hat{L})$ is held fixed for all possible competitors, then *if* (6.17) *holds for the bar of length L, it will also hold for the bar of length $\hat{L}$.* We now turn to the proofs of these theorems.

Proof of Theorem 2. Let $y \in \mathscr{F}$ be an equilibrium solution for the soft device which satisfies (6.8). Suppose $z \in \mathscr{F}$ satisfies $|z'(X) - y'(X)| < \varepsilon$ a.e. Assume $(6.6)_r$, (6.9) and (6.10) hold for continuous functions $s(X) > 0$, $p(X)$ and a constant k. By the same argument that produced (6.7) from (6.6), it follows immediately that (6.9) implies (6.11) and (6.10) implies (6.12).

Now assume only that $(6.7)_r$, (6.11) and (6.12) hold. When substituted into (6.3) these inequalities yield

$$E_S[z] - E_S[y] \geqq \frac{1}{2}\int_0^L \{s(X)\,h'^2 - 2k\,h\,h' + p(X)\,h^2\}\,dX. \tag{6.19}$$

We add the term $(r(X)\,h^2)'$ to the integrand in (6.19), while insuring that the continuously differentiable function $r(X)$ satisfies $r(L) = 0$. Since the integral on the right hand side of (6.19) will thereby remain unchanged, we still have the estimate

$$\begin{aligned} E_S[z] - E_S[y] &\geqq \frac{1}{2}\int_0^L \{s(X)\,h'^2 - 2\,k\,h\,h' + p(X)\,h^2 + (r(X)\,h^2)'\}\,dX \\ &= \frac{1}{2}\int_0^L \{s(X)\,h'^2 + 2(r(X) - k)\,h\,h' + (p(X) + r'(X))\,h^2\}\,dX. \end{aligned} \tag{6.20}$$

The integrand is a perfect square if

$$(r(X) - k)^2 = s(X)\,(p(X) + r'(X)), \tag{6.21}$$

in which case

$$E_S[z] - E_S[y] \geqq \frac{1}{2}\int_0^L s(X)\left[h' + \frac{r(X) - k}{s(X)}\,h\right]^2 dX \geqq 0. \tag{6.22}$$

If equation (6.21) has a continuously differentiable solution $r(X)$, which is defined on $[0, L]$ and vanishes at $X = L$, then, by (6.22), $y(X)$ is metastable.

Let $u(X)$ be defined by the transformation

$$u(X) = u_0 \exp\left\{\int_0^X \frac{k - r(Y)}{s(Y)}\,dY\right\}, \tag{6.23}$$

u_0 being a nonzero constant. Since $s(Y)$ is positive and continuous if $r(X)$ is continuous, then $u(X)$ is continuously differentiable on $[0, L]$. In that case (6.23) implies that

$$-s(X)\,\frac{u'(X)}{u(X)} = r(X) - k. \tag{6.24}$$

Suppose $r(X)$ is a continuously differentiable solution of (6.21). Then, by the identity (6.24), $(s\,u') \in C^1_{[0,L]}$, $u(X) \neq 0$, and u satisfies the equation

$$(s(X)\,u')' - p(X)\,u = 0, \tag{6.25}$$

and the end condition

$$s(L)\frac{u'(L)}{u(L)}=k. \tag{6.26}$$

Conversely, if $(s(X)u')\in C^1_{[0,L]}$, $u(X)\neq 0$, u satisfies (6.25) and (6.26), and r is defined by (6.24), then $r(X)$ is a continuously differentiable solution of (6.21) satisfying $r(L)=0$.

Therefore the problem reduces to a determination of the zeros of solutions of (6.25) and (6.26). If (6.25) and (6.26) have a solution which does not vanish on $[0,L]$, then the argument leading to (6.22) is justified, so y is metastable. The Sturm comparison theorems [25, Chapter XI, Theorems 3.1 and 3.2] relate the existence of zeros on $[0,L]$ to the values of the functions $p(X)$ and $s(X)$, and the value of the constant k. When these functions are known explicitly, a numerical procedure might well be more useful [26, Chapter 5, p. 122].

The terminology of HARTMAN [25] is adopted. Let $S(X)$, $P(X)$ be continuous functions on $\mathbb{R}^1$. The equation

$$(SU')'-PU=0 \tag{6.27}$$

is called a *Sturm majorant* for (6.25) if

$$s(X)\geqq S(X)>0 \quad \text{and} \quad p(X)\geqq P(X). \tag{6.28}$$

We shall denote by K the value $S(L)\dfrac{U'(L)}{U(L)}$, calculated for a solution of (6.27), by analogy with (6.26). We shall always assume K and k are finite constants, so that U and u do not vanish at $X=L$. Suppose that

$$k\leqq K \tag{6.29}$$

and that (6.27) is a Sturm majorant for (6.25). Then by Sturm's first comparison theorem* it follows that if U does not have a zero on $[0,L]$, then u does not have a zero on $[0,L]$. Therefore if U does not have a zero on $[0,L]$, then y is metastable.

Furthermore, if (6.27) is a *strict* Sturm majorant $\big(P(X)>p(X)$, or $s(X)>S(X)>0$ and $P(X)\neq 0$ for some $X\in[0,L]\big)$, or if (6.29) holds with strict inequality, then the non-existence of zeros of U on $(0,L]$ implies the non-existence of zeros of u on $(0,L]$.

Now suppose (6.14) is satisfied, so that (6.27) is a Sturm majorant for (6.25) and (6.27) has constant coefficients. Suppose that $k\leqq K$. Consider first the case $P>0$. The Sturm majorant (6.27) has solutions

$$U(X)=U_1\cosh\sqrt{\frac{P}{S}}(X-L)+U_2\sinh\sqrt{\frac{P}{S}}(X-L). \tag{6.30}$$

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* HARTMAN's statement of the theorem may be adapted to the present situation by the change of variables $X\to L-X$.

Among them, the solutions

$$U(X)=U_1\left\{\cosh\sqrt{\frac{P}{S}}(X-L)+\frac{K}{\sqrt{SP}}\sinh\sqrt{\frac{P}{S}}(X-L)\right\} \tag{6.31}$$

satisfy $S\dfrac{U'(L)}{U(L)}=K$. Let $U_1 \neq 0$. Then $U(X)$ has no zero in $[0,L]$ (*i.e.* y is metastable) if

$$K<\sqrt{SP}\left\{\frac{e^{2\sqrt{\frac{P}{S}}L}+1}{e^{2\sqrt{\frac{P}{S}}L}-1}\right\}. \tag{6.32}$$

Obviously, for $k \leqq 0$ these sufficient conditions for stability are satisfied. Some positive values of k also permit y to be metastable.

In the case $P=0$, the solutions of (6.27) which satisfy the end condition are given by

$$U=U_1\left\{(X-L)+\frac{S}{K}\right\}. \tag{6.33}$$

Those solutions have no zeros in $[0,L]$ if

$$K<\frac{S}{L}. \tag{6.34}$$

Again, some positive values of k are allowed.

Finally, if $P<0$, the solutions

$$U=U_1\left\{\cos\sqrt{\frac{-P}{S}}(X-L)+\frac{K}{\sqrt{-SP}}\sin\sqrt{\frac{-P}{S}}(X-L)\right\} \tag{6.35}$$

satisfy (6.27) and the end condition. A *necessary condition* that $U(X)$ does not have a zero on $[0,L]$ is that

$$L\sqrt{\frac{-P}{S}}<\pi. \tag{6.36}$$

Suppose (6.36) holds. Then the interval $[0,L]$ does not contain a zero of U if

$$K<\frac{-\sqrt{-SP}}{\tan\left(-\sqrt{\frac{-P}{S}}L\right)}. \tag{6.37}$$

Here K is always less than zero if $\frac{\pi}{2}<L\sqrt{\frac{-P}{S}}<\pi$. However, if $0<L\sqrt{\frac{-P}{S}}<\frac{\pi}{2}$, then some positive values of K are admitted by (6.37).

Proof of Theorem 3. The procedure for the hard device is a classical one which is outlined in many books on the calculus of variations [27, for example]. We begin with (6.19) and complete the square as in (6.20). The calculation therein is justified with $r(X)$ unrestricted by end conditions. The transformation (6.23) remains valid, so the problem reduces to the determination of the zeros of a solution u of

$$(s(X)\,u')' - p(X)\,u = 0. \tag{6.38}$$

Again, a numerical procedure may be used to locate these zeros in particular problems. For rough bounds on the positions of the zeros, STURM's first comparison theorem, in conjunction with a Sturm majorant having constant coefficients, delivers the classical results. Those results may be extracted from the arguments put forth for the soft device. In particular, if $S>0$ and $P \geqq 0$ (using the notation of (6.27) for the Sturm majorant), the placement y is metastable; this may be seen from the original argument summarized by (6.4) and (6.12), or from (6.36) with $U_2 = 0$. If $P<0$, the estimate

$$L\sqrt{\frac{-P}{S}} < \pi, \tag{6.36}_r$$

which also appears as the necessary condition (6.36), delivers the sufficient conditions.

The sufficient conditions for metastability cannot generally be made necessary, because if a placement $z \in \mathscr{F}$ produces equality in (6.6) when δ is given the value $z'(X) - y'(X)$, that z will not generally produce equality in (6.8) and (6.9) when μ is given the value $z(X) - y(X)$.

7. Distribution of the Phases

The methods of Section 6 do not lend themselves to the study of the absolute or neutral stability of equilibrium solutions, since they involve essentially local arguments. Furthermore, the distribution of the phases in the bar played no direct role in the stability analysis given there. For example, in treating the soft device and gravitational body force, the strong sufficient conditions for metastability (6.5) were satisfied. However, the study of the corresponding equilibrium equation in Section 5 showed that an infinite variety of solutions was possible; some of those solutions could have an infinite number of subintervals of the bar arranged consecutively in the phases $\alpha\beta\alpha\beta\alpha\beta\ldots$.

Arguments of this section which use finite stability will show that in several important special cases the absolutely stable solutions contain only one or two phase boundaries.

a. Soft Device

The simplest case of a dead loaded, homogeneous bar with null body force shall be treated first. This problem has been solved by ERICKSEN [4]. As in the discussion of Section 5a (see equations (5.2) to (5.5)), we assume that σ_0 belongs to the range of $\frac{dW}{du}(\cdot)$. Two possibilities arise; either $\frac{dW}{du}(\cdot)$ has a unique inverse at σ_0 or it does not. In the former case (5.3) applies, and the unique equilibrium solution is the homogeneous placement $y(X)=\gamma X$. If γ belongs to the *open* set (α,β), then this placement is absolutely stable. That is, from the summary in Section 4, the Weierstrass condition is satisfied with strict inequality. Hence, by equation (6.3),

$$E_S[z]-E_S[y]>0, \qquad z \neq y, \qquad z\in\mathscr{F}. \tag{7.1}$$

If $\frac{dW}{du}(\cdot)$ has a double valued inverse at σ_0, the equilibrium solution is given by equation (5.5):

$$y(X)=\int_{[0,X]\cap S_\alpha} \mu\, dX+\int_{[0,X]\cap S_\beta} \nu\, dX. \tag{7.2}$$

Here, S_α and S_β are disjoint measurable subsets of $[0,L]$, whose union is $[0,L]$, and μ and ν are constants on the α- and β-branches, respectively, which correspond to the stress σ_0. If μ and ν belong to the open set (α,β), then any solution of the form (7.2) is metastable. The total energy of y is

$$\begin{aligned} E_S[y]&=\int_{S_\alpha}(W(\mu)-\sigma_0\mu)\,dX+\int_{S_\beta}(W(\nu)-\sigma_0\nu)\,dX\\ &=\big(W(\mu)-\sigma_0\mu\big)m(S_\mu)+W(\nu)-\sigma_0\nu\big)m(S_\nu). \end{aligned} \tag{7.3}$$

We seek absolutely stable solutions which correspond to the dead load σ_0. In doing so, we encounter three possibilities:

(i) $\frac{dW}{du}(\alpha)<\sigma_0<\sigma^*$. In this case

$$\mathscr{E}(\nu,\mu)=W(\nu)-W(\mu)-\sigma_0(\nu-\mu)>0.$$

It follows from (7.3) that $E_S[y]$ is minimized by the choice

$$S_\alpha=[0,L], \qquad S_\beta=\emptyset. \tag{7.4}$$

For this choice of S_α and S_β, y is absolutely stable. The bar is homogeneously deformed in the α-phase.

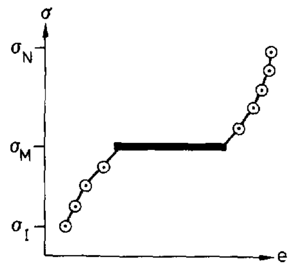
(ii) $\sigma^*<\sigma_0<\frac{dW}{du}(\beta)$. Here $\mathscr{E}(\mu,\nu)>0$, so $E_S[y]$ has its minimum value when

$$S_\alpha=\emptyset, \qquad S_\beta=[0,L]. \tag{7.5}$$

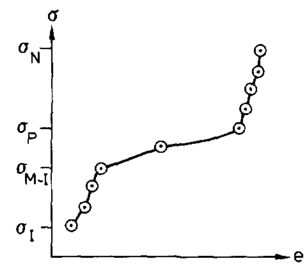
This choice makes y absolutely stable; the bar is homogeneously deformed in the β-phase.

(iii) $\sigma_0=\sigma^*$. Since $\mathscr{E}(\mu,\nu)=0$, all measurable choices of S_α and S_β, in which $m(S_\alpha)+m(S_\beta)=L$, yield the same value of the energy. Each solution of this kind is neutrally stable. The two phases may be distributed in any way whatsoever in the bar.

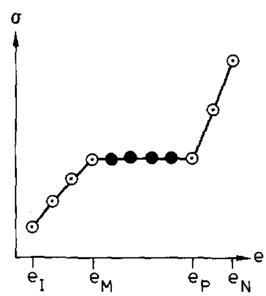
The result summarized in (iii) is not indicative of the results that shall be deduced when a body force or inhomogeneity is present. The contrast can be



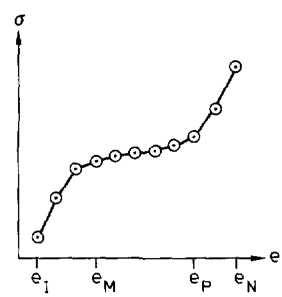
a) Dead load, no body force, homogeneous bar.



b) Dead load, gravitational body force, inhomogeneous bar (measure $\{X|B(X,\sigma_0)=\sigma_X^*\}=0$).



c) Hard device, no body force, homogeneous bar.



d) Hard device, gravitational body force, inhomogeneous bar (measure $\{X|B(X,c)=\sigma_X^*\}=0$).

Fig. 2. Typical stress-extension curves plotted from theory. Wherever possible, the values have been calculated for absolutely stable solutions. If no absolutely stable solution exists for the data given, the values are plotted for a neutrally stable solution corresponding to that data. Dots or dark regions mark neutrally stable solutions. Open circles mark absolutely stable solutions. For the soft device $e \equiv \frac{y(L)}{L}$ and for the hard device $e \equiv \frac{l}{L}$.

developed most clearly if we imagine that the bar is subject to a simple tension test, in which the corresponding placements are calculated from theory. Hence, we assume that the bar is successively dead loaded with an increasing sequence of stresses $\sigma_0 < \sigma_1 < \sigma_2 \ldots < \sigma_N$, and that at each stress $\sigma_K (K = 1, 2 \ldots, N)$ the bar equilibrates in the most stable placement. Theory predicts that if $\sigma_K < \sigma^*$ the bar will be homogeneously deformed in the α-phase; if $\sigma_K > \sigma^*$ the bar will be homogeneously deformed in the β-phase. If $\sigma_M = \sigma^*$ for some M, then the distribution of the phases will be indeterminate for a corresponding equilibrium solution. At this value of stress large regions of the bar could spontaneously jump from one phase to another, as far as static theory is concerned. The end position could change drastically during such a shift of phase. A typical stress-extension curve calculated from theory has been reproduced in Figure 2a.

We now treat the case of an inhomogeneous, dead loaded bar pulled by a gravitational body force. The relevant equilibrium equation, which we have analyzed in detail in Section 5, is written

$$\frac{\partial W}{\partial u}(y'(X), X) = \int_X^L \rho(Y)\, g\, dY + \sigma_0 \equiv B(X, \sigma_0). \tag{7.6}$$

Not all dead loads will produce solutions, so we consider only those that do, *i.e.* those that satisfy (5.7). Furthermore, we must strengthen that requirement by assuming that for some $\varepsilon > 0$,

$$\frac{\partial W}{\partial u}(\alpha_X, X) + \varepsilon \leqq B(X, \sigma_0) \leqq \frac{\partial W}{\partial u}(\beta_X, X) - \varepsilon. \tag{7.7}$$

Then the solutions are given by

$$y'(X) = \begin{cases} \pi_\alpha(B(X, \sigma_0), X), & X \in (\mathscr{S}_\alpha - S) \cup S_\alpha, \\ \pi_\beta(B(X, \sigma_0), X), & X \in (\mathscr{S}_\beta - S) \cup S_\beta. \end{cases} \tag{7.8}$$

Recall that S is the set of X on which either inverse can be used, $S = S_\alpha \cup S_\beta$ is an arbitrary, disjoint partition of S into measurable sets. Again, we seek absolutely stable placements. The definitions (6.2) show that for a gravitational body force potential and a dead loading device, the corresponding excess functions vanish. Therefore, if y is given by (7.8) and $z \in \mathscr{F}$, then

$$\begin{aligned} E_S[z] - E_S[y] = &\int_{(\mathscr{S}_\alpha - S) \cup S_\alpha} \mathscr{E}(z'(X), \pi_\alpha(B(X, \sigma_0), X); X)\, dX \\ &+ \int_{(\mathscr{S}_\beta - S) \cup S_\beta} \mathscr{E}(z'(X), \pi_\beta(B(X, \sigma_0), X); X)\, dX. \end{aligned} \tag{7.9}$$

Referring to Section 4, we see clearly that both integrands of (7.9) are non-negative if S_α and S_β are chosen as the sets

$$\begin{aligned} S_\alpha &= \{X \mid \pi_\alpha(B(X, \sigma_0), X) \leqq \alpha_X^*\} \cap S, \\ S_\beta &= \{X \mid \pi_\beta(B(X, \sigma_0), X) > \beta_X^*\} \cap S. \end{aligned} \tag{7.10}$$

The equality sign in $(7.10)_1$ can be switched to $(7.10)_2$; in fact the set $\{X \mid \pi_\alpha(B(X, \sigma_0), X) = \alpha_X^*\}$ can be partitioned in any way whatsoever among S_α and S_β while preserving the non-negativity of (7.8). Neutral stability will always occur if $\{X \mid \pi_\alpha(B(X, \sigma_0), X) = \alpha_X^*\}$ has positive measure. A necessary and sufficient condition for the existence of an absolutely stable placement is that the measure of this set vanish.

The results are best illustrated by some special examples. Suppose the bar is homogeneous, but a non-trivial gravitational body force acts upon it. Then, from (7.6), $B(X, \sigma_0)$ is a strictly decreasing function of X. If $B(0, \sigma_0) < \sigma^*$, then the absolutely stable solution is determined by the choice $\mathscr{S}_\beta = S_\beta = S = \emptyset$; the bar is (inhomogeneously) deformed in the α-phase. If $B(L, \sigma_0) > \sigma^*$, then the choice $\mathscr{S}_\alpha = S_\alpha = S = \emptyset$ produces the absolutely stable solution; the bar is deformed in the β-phase. Otherwise, $B(X, \sigma_0) = \sigma^*$ has the *unique* solution X^*. It follows from (7.9) that the absolutely stable solution is obtained from (7.8) by the choice

$$\begin{aligned} (\mathscr{S}_\beta - S) \cup S_\beta &= [0, X^*], \\ (\mathscr{S}_\alpha - S) \cup S_\alpha &= [X^*, L]. \end{aligned} \tag{7.11}$$

In summary, there is an absolutely stable solution $y(X)$, $X \in [0, L]$; on $[0, X^*]$ the bar is in the β-phase, and on $[X^*, L]$ the bar is in the α-phase. The deformation $y'(X)$ has a jump discontinuity at X^*; otherwise $y'(X)$ is absolutely continuous. The position of the phase boundary X^* is the unique solution of the equation $B(X, \sigma_0) = \sigma^*$. As the assigned dead load σ_0 increases, X^* increases, strictly and continuously.

In comparison with the dead loaded homogeneous bar, whose solutions at $\sigma_0 = \sigma^*$ could be extremely rough, here the slightest gravitational body force makes the absolutely stable solution piecewise differentiable.

The absolutely stable solutions corresponding to a simple tension test for the homogeneous bar will be set forth. Let $\sigma_0 < \sigma_1 < \sigma_2 < \ldots < \sigma_N$ be a sequence of stresses, each satisfying the condition (7.7). Keeping in mind that $B(X, \sigma)$ is a strictly increasing function of σ, we have the following possibilities:

(i) $B(0, \sigma_k) < \sigma^*$, $k = 0, 1, \ldots, M-1$.

The solutions are

$$y_k(X) = \int_0^X \pi_\alpha(B(Y, \sigma_k))\, dY, \tag{7.12}$$

which correspond to end positions

$$l(\sigma_k) = \int_0^L \pi_\alpha(B(Y, \sigma_k))\, dY. \tag{7.13}$$

The function $l(\cdot)$ is a continuous, strictly increasing function of σ_k.

(ii) $B(0, \sigma_k) \geqq \sigma^*$ and $B(L, \sigma_k) \leqq \sigma^*$,

$$k = M,\ M+1, \ldots, P.$$

The solutions are given by

$$y_k(X) = \int_0^{X^*(\sigma_k)} \pi_\beta(B(Y, \sigma_k))\, dY + \int_{X^*(\sigma_k)}^X \pi_\alpha(B(Y, \alpha_k))\, dY, \tag{7.14}$$

if $X > X^*(\sigma_k)$, and are given by

$$y_k(X) = \int_0^X \pi_\beta(B(Y, \sigma_k))\, dY, \tag{7.15}$$

if $X < X^*(\sigma_k)$. The corresponding end positions are

$$l(\sigma_k) = \int_0^{X^*(\sigma_k)} \pi_\beta(B(Y, \sigma_k))\, dY + \int_{X^*(\sigma_k)}^L \pi_\alpha(B(Y, \sigma_k))\, dY. \tag{7.16}$$

The function $l(\cdot)$ is strictly increasing. $y'(X)$ suffers a single discontinuity in $[0, L]$.

(iii) $B(L,\sigma_k)>\sigma^*$, $k=P+1, P+2, \ldots N$.
The solutions are

$$y_k(X)=\int_0^X \pi_\beta(B(Y,\sigma_k))\,dY, \tag{7.17}$$

with end positions

$$l_k(\sigma_k)=\int_0^L \pi_\beta(B(Y,\sigma_k))\,dY. \tag{7.18}$$

Let σ_0 and σ_L be the values of stress which solve $X^*(\sigma)=0$ and $X^*(\sigma)=L$, respectively, *i.e.* the two values of stress at which the phase boundary just enters and just leaves the bar. It is interesting to note that if we were to assume $W(u)$ twice continuously differentiable, then the function $l(\sigma)$, defined as the composite of the functions (7.13), (7.16), and (7.18), would have a discontinuous derivative at σ_0 and σ_L. Figure 2b depicts a typical stress-extension curve calculated from (7.13), (7.16), and (7.18).

By comparing Figures 2a and 2b, it is clear that the character of solutions, in terms of absolute *vs.* neutral stability and in terms of the distribution of the phases, do not exhibit continuous dependence on the body force in the following sense: As $g\to 0$, the graph in 2b tends uniformly to the graph in 2a, the central portion in 2b approaching the horizontal. However, for any $g>0$, there are absolutely stable solutions, which contain a single phase boundary. At $g=0$ a great number of configurations have the same energy, and the distribution of phases among them may differ drastically.

It is perhaps evident to the reader how (7.8) and the results of Section 4 may be used to construct absolutely stable solutions and stress-extension curves for a dead loaded bar acted upon by a gravitational body force. However, because of its intrinsic interest to experiments, I shall treat the example which corresponds to the most common simple tension experiment. Usually in such experiments, the bar has an hourglass shape to accomodate the application of grips at the ends; that shape is commonly produced by cutting the specimen out of a bar, which is homogeneous along its length. To represent this kind of bar, we assume

(i) $\sigma^*_\cdot\in C^1_{[0,L]}$, $\rho(\cdot)\in C^0_{[0,L]}$,

(ii) for some $A\in(0,L)$, σ^*. and $\rho(\cdot)$ are strictly decreasing on $[0,A]$ and strictly increasing on $[A,L]$. (7.19)

These assumptions are *not* sufficient that the equation $\sigma^*_X-B(X,\sigma_0)=0$ have either zero, one or two solutions. However, they do imply that $\sigma^*_X-B(X,\sigma_0)$ is strictly increasing on $[A,L]$, so that at most one zero of $\sigma^*_X-B(X,\sigma_0)$ can occur on $[A,L]$. Let $[\bar{A},L]$ be the largest interval on which $\sigma^*_X-B(X,\sigma_0)$ is strictly increasing. The constant $\bar{A}$ will depend upon g but not upon σ_0. In addition to (i) and (ii) above, we assume that

$$\text{(iii)}\quad \bar{A}>0 \quad\text{and}\quad \frac{d}{dX}\sigma^*_X+\rho(X)g<0 \quad\text{on}\quad (0,\bar{A}). \tag{7.20}$$

Condition (iii) may be regarded as an assumption of weak body force; it implies that $\sigma_X^* - B(X, \sigma_0) = 0$ has zero, one, or two solutions. Assumption (i) shows that $\bar{A} < A$, and that $\bar{A}$ tends to A as $g \to 0$. For sufficiently small values of σ_0 $\left(\sigma_0 < \sigma_{\bar{A}}^* - \int_{\bar{A}}^{L} \rho(Y) g \, dY\right)$, no solutions will exist, and the absolutely stable solution will be produced by the choice $S_\beta = \emptyset$; the bar is deformed in the α-phase. The β-phase will first appear in an absolutely stable solution at the value of stress $\sigma_0 = \sigma_{\bar{A}}^* - \int_{\bar{A}}^{L} \rho(Y) g \, dY$. If σ_0 is increased slightly, two phase boundaries, say at $\bar{A}_1$ and $\bar{A}_2$, will spread outward from $\bar{A}$ toward the ends of the bar, $\bar{A}_1$ and $\bar{A}_2$ being the unique solutions of $\sigma_X^* - B(X, \sigma_0) = 0$. The absolutely stable solution will be given by

$$
\begin{aligned}
(\mathscr{S}_\beta - S) \cup S_\beta &= [\bar{A}_1, \bar{A}_2], \\
(\mathscr{S}_\alpha - S) \cup S_\alpha &= [0, \bar{A}_1) \cup (\bar{A}_2, L].
\end{aligned}
\tag{7.21}
$$

By increasing σ_0, the β-phase will eventually fill the entire bar. The corresponding stress-extension curve will have the qualitative features of Figure 2b.

b. Hard Device

The homogeneous bar in the absence of a body force will be treated first. The equilibrium equation has been discussed in Section 5b. We assume the necessary condition for existence, $\alpha L < l < \beta L$, holds. From Sections 5b.1, 5b.2 and 5b.3, we may deduce the following:

1. $\hat{\beta} L < l < \beta L$ or $\alpha L < l < \hat{\alpha} L$. The only solution is the homogeneous placement $y(X) = \frac{l}{L} X$. This solution is absolutely stable. That is, from the definitions (5.35) of $\hat{\alpha}$ and $\hat{\beta}$, it follows that $\hat{\alpha} < \alpha^*$, $\hat{\beta} > \beta^*$. Section 4 shows that $\mathscr{E}\left(z'(X), \frac{l}{L}\right) > 0$, unless $z'(X) = \frac{l}{L}$. The estimate (6.4), $E[z] - E[y] = \int_0^L \mathscr{E}\left(z'(X), \frac{l}{L}\right) dX$, then implies that $y(X) = \frac{l}{L} X$ is absolutely stable.

2. $\hat{\alpha} L \leq l \leq \hat{\beta} L$. The analysis of (5b.3) shows that the set of possible solutions compatible with a fixed value of l can be parameterized by a constant c, which belongs to a certain closed interval.

α. $l = \hat{\beta} L$. The closed interval of possible values for c reduces to the point $\frac{\partial W}{\partial u}(\alpha^1)$. Therefore, $y'(X) = \alpha^1$ or $y'(X) = \hat{\beta}$. From Section 4, y is not metastable if $y'(X) = \alpha^1$ on a set of positive measure. Hence the only metastable solution, $y(X) = \hat{\beta} X$, is absolutely stable.

β. $l = \hat{\alpha} L$. By similar reasoning as in 2α above, the only metastable solution is $y(X) = \hat{\alpha} X$, and it is absolutely stable.

γ. $\hat{\alpha}L < l < \alpha^* L$. The constant c may belong to the interval

$$\left[\frac{\partial W}{\partial u}(\hat{\alpha}), \frac{\partial W}{\partial u}\left(\frac{l}{L}\right)\right], \qquad \text{so } c < \sigma^*.$$

Therefore

$$E[z] - E[y] = \int_{S_\alpha} \mathscr{E}(z'(X), \pi_\alpha(c))\, dX + \int_{S_\beta} \mathscr{E}(z'(X), \pi_\beta(c))\, dX \tag{7.22}$$

is positive for all $z \in \mathscr{F}$, not identically equal to y, satisfying the end condition $z(L) = l$, if $S_\beta = \emptyset$. From (5.43), $S_\beta = \emptyset$ if and only if $l = \pi_\alpha(c)L$, which has a unique solution $\bar{c}$. Therefore the absolutely stable solution is $y(X) = \frac{l}{L} X$.

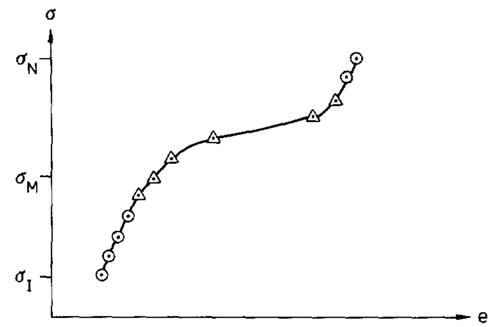
δ. $\beta^* L < l < \hat{\beta} L$. Similar reasoning as in part 2γ applies. The absolutely stable solution is $y(X) = \frac{l}{L} X$.

ε. $\alpha^* L \leqq l \leqq \beta^* L$. Here the constant c always belongs to an interval which contains σ^*. The choice $c = \sigma^*$ always makes the right hand side of (7.22) non-negative. If $l = \alpha^* L$, equations (5.43) show that $S_\beta = \emptyset$. Similarly, if $l = \beta^* L$, $S_\alpha = \emptyset$. Therefore, both of the solutions $y(X) = \alpha^* X$ and $y(X) = \beta^* X$, which correspond to end positions $\alpha^* L$ and $\beta^* L$, are absolutely stable. On the other hand, if $\alpha^* L < l < \beta^* L$, the choice $c = \sigma^*$ implies that the measures of both S_α and S_β are positive. Since those sets may be redistributed in any way whatsoever in the bar, as long as the redistribution is consistent with the condition (5.43) on the measures, then the solutions $y(X)$ corresponding to $c = \sigma^*$ are neutrally stable. No absolutely stable solutions exist when $\alpha^* L < l < \beta^* L$.

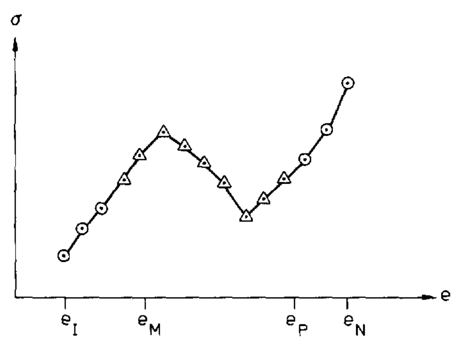
The best way to compare these results with the results for the soft device is by comparing the corresponding stress-extension curves. For the hard device an increasing sequence of lengths $\alpha L < l_1 < l_2 < \ldots < l_N < \beta L$ is prescribed and the stresses for the most stable solutions are plotted using the results just deduced. A typical stress-extension curve for the homogeneous bar held in a hard device, acted upon by zero body force, is given in Figure 2c.

We now briefly consider the problem of the inhomogeneous bar held by a hard device, and subject to a gravitational body force. The solutions of the equation of equilibrium have been given special treatment in Section 5b. There, the minimum and maximum possible values of l were defined by (5.54) and (5.55). Every $l \in [l_{\max}, l_{\min}]$ is the end position of some solution. We proved that fact by constructing a family of solutions depending on the parameter $c \in [\sigma_1, \sigma_2]$; in that construction the measurable sets $R_\alpha(c)$ and $R_\beta(c)$ were defined such that

$$\begin{aligned} &R_\alpha(\sigma_1) = \emptyset, \qquad R_\beta(\sigma_2) = \emptyset, \\ &R_\alpha \cup R_\beta = [0, L], \qquad R_\alpha \cap R_\beta = \emptyset, \\ &S_\alpha \equiv R_\alpha(c) \cap S(c), \\ &S_\beta \equiv R_\beta(c) \cap S(c). \end{aligned} \tag{7.23}$$



a. Dead load. Slight gravitational body force, or slight inhomogeneity. $e \equiv \frac{y(L)}{L}$.



b. Hard device. Slight gravitational body force, or slight inhomogeneity. $e \equiv \frac{l}{L}$.

Fig. 3. Typical stress-extension curves plotted from metastable solutions. The triangles represent metastable solutions. Open circles represent absolutely stable solutions.

Let us consider the following choice of $R_\alpha(c)$ and $R_\beta(c)$:

$$\begin{aligned} R_\alpha(c) &= \{X \in [0, L] \mid B(X, c) < \sigma_X^*\}, \\ R_\beta(c) &= \{X \in [0, L] \mid B(X, c) \geqq \sigma_X^*\}. \end{aligned} \tag{7.24}$$

These sets satisfy the continuity requirement $(5.59)_1$. If $\sigma_1 \leqq \hat{c} < c \leqq \sigma_2$, $R_\beta(\hat{c}) \subset R_\beta(c)$ and $R_\alpha(c) \subset R_\alpha(\hat{c})$. I assume, additionally, that

$$R_\beta(\sigma_1) = \emptyset, \qquad R_\alpha(\sigma_2) = \emptyset. \tag{7.25}$$

This assumption restricts the constitutive function and body force potential. It is a mild assumption which allows the full set $[l_{\min}, l_{\max}]$ to produce at least neutrally stable solutions. If (7.25) is not satisfied, a special, but not difficult, analysis can be used to find neutrally stable solutions for a smaller set of values of l.

In the definition (7.24), the set $\{X \mid B(X, c) = \sigma_X^*\}$ can be partitioned in any way whatsoever among R_α and R_β, as long as they remain measurable, and neutral stability will be preserved. A sufficient condition that a solution corresponding to a value of c be *absolutely* stable is that the set $\{X|B(X,c)=\sigma_X^*\}$ have measure zero. That this condition is not necessary follows from 2ε, for example.*

The summary of Section 4 and equation (6.4) imply that if y is an equilibrium solution, and if (7.24) and (7.25) hold, then

$$\begin{aligned} E_H[z]-E_H[y]= & \int_{(\mathscr{S}_\alpha - S)\cup S_\alpha} \mathscr{E}(z'(X), \pi_\alpha(B(X,c),X); X)\,dX \\ & + \int_{(\mathscr{S}_\beta - S)\cup S_\beta} \mathscr{E}(z'(X), \pi_\beta(B(X,c),X); X)\,dX \geqq 0, \end{aligned} \tag{7.26}$$

i.e. every $l\in[l_{\min}, l_{\max}]$ corresponds to a neutrally stable solution. Figure 2d is an example of a stress extension curve for this situation, in which $\{X|B(X,c)=\sigma_X^*\}$ has measure zero.

A great many metastable solutions are possible in this theory. However, a large class of them will produce stress-extension curves similar to those represented in Figure 3. Generally, for sufficiently small extensions or stresses, or for sufficiently large extensions or stresses, absolutely stable solutions will prevail. It is possible to produce a large class of metastable solutions for the hard device which lead to an abrupt falling of the stress-extension curve.

Appendix

We shall here sketch the proof of Theorem 1 of Section 5.**

Proof of Theorem 1. Let the region of the $X-y$ plane bounded by the lines $y=\hat{c}_1 X$ and $y=\hat{c}_2 X$ and the line $X=L$ be called $\mathscr{D}$ (see Figure 4). Assume R_α and R_β are disjoint, measurable sets, whose union equals $[0,L]$. Let a function $y_0(X)$ be defined on $[L, L+\varepsilon]$. This function can be extended to the interval $[L-\varepsilon, L+\varepsilon]$ by, first, inserting the function $y_0(\cdot+\varepsilon)$ into the right hand side of (5.1) and, then, by integrating (5.1) in the same manner as was done for the gravitational body force. As with the gravitational body force, there may be a set S of points in $[0,L]$ where there is an ambiguous choice of inverse; in that case, on $S\cap R_\alpha$ use the inverse of the α-branch and on $S\cap R_\beta$ use the inverse of the β-branch. This function, which we call $y_\varepsilon(X)$, is again translated to the left by an amount ε and inserted into the right hand side of (5.1), to extend its definition to the interval $[L-2\varepsilon, L+\varepsilon]$. Continuing in this fashion, the continuous function $y_\varepsilon(X)$ will cross the boundary of $\mathscr{D}$ at a point X^ε. By choosing the initial function $y_0(X)$ appropriately, it can be shown that for some choice of $y_0(X)$, the function $y_\varepsilon(X)$ will pass out of $\mathscr{D}$ at $X^\varepsilon=0$, so that $y(0)=0°$). The function $y_\varepsilon(X)$ constructed in this way will belong to $\mathscr{F}$. The family of functions $\{y_\varepsilon(X)\}$ is an

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* The absolutely stable solution $y=\beta^* X$ corresponding to the end position $l=\beta L$, and the condition $g=0$, has the property that $\{X|B(X,c)=\sigma_X^*\}=[0,L]$.

** Somewhat similar theorems are found in the literature (see KACZYŃSKI & OLECH [24]), although they apply only to equations of first order ((5.1) is an integrated equation of second order). The theorem also applies to the problem of the elastica with a non-convex energy function.

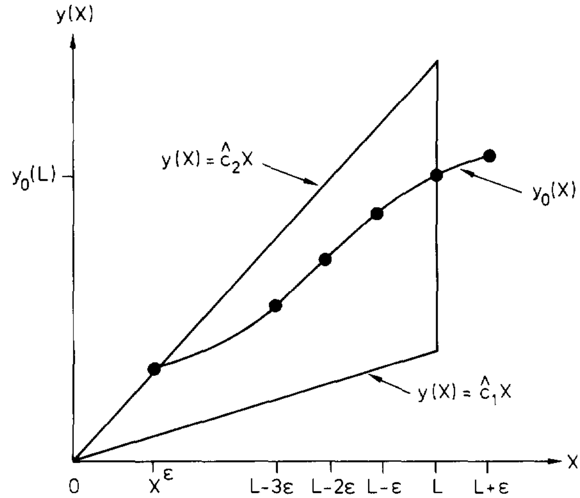

Fig. 4. Construction for the proof of existence for the soft device.

equicontinuous family, so there is a uniformly convergent subsequence $y_{\varepsilon_k} \to y$ as $k \to \infty$. The limit function y satisfies (5.1).

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