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On motions which preserve ellipsoidal holes

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Let $\mathscr{E}$ be an ellipsoid in $\mathbb{R}^3$ contained in a region Ω. Suppose one body occupies the region $\Omega - \mathscr{E}$ in a certain stress-free reference configuration while a second body, the inclusion, occupies the region $\mathscr{E}$ in a stress-free reference configuration. Assume the inclusion is free to slip at $\partial\mathscr{E}$. Now suppose that by changing some variable such as the temperature, pressure, humidity, etc., we cause the trivial deformation $y(x) = x$ of the inclusion to become unstable relative to some other deformation. For example, the inclusion may be made out of such a material that if it were removed from the body, it would suddenly change shape to another stress-free configuration specified by a deformation $y = Fx$, $F^TF = C$, C being a fixed tensor characteristic of the material, at a certain temperature. However, with an appropriate material model for the surrounding body, we expect it will resist the transformation, and both body and inclusion will end up stressed.

In a recent paper, Mura and Furuhashi [1] find the following unexpected result within the context of infinitesimal deformations: certain homogeneous deformations of the ellipsoid which take it to a stress-free configuration also leave the surrounding body stress-free. These are essentially homogeneous, infinitesimal deformations which preserve ellipsoidal holes. In this paper, we find all *finite* homogeneous deformations and motions which preserve ellipsoidal holes.

1. Deformations which preserve ellipsoidal holes

Let an ellipsoid $\mathscr{E}$ in $\mathbb{R}^3$ be given by

$$\mathscr{E} = \{x \mid x \cdot Cx \leqslant 1\}, \tag{1.1}$$

C being a constant positive-definite tensor. We will say that a homogeneous deformation

$$y(x) = Fx, \tag{1.2}$$

$F = const.$, det $F > 0$, *preserves the ellipsoid* $\mathscr{E}$ is there is a symmetric tensor C' of the form RCR^T for some rotation R such that $\mathscr{E}' = y(\mathscr{E})$ can be written

$$\mathscr{E}' = \{y \mid y \cdot C'y \leqslant 1\}. \tag{1.3}$$

Note that $\mathscr{E}'$ uniquely determines C' by (1.3).

THEOREM. *Let the ellipsoid $\mathscr{E}$ be given by* (1.1) *and let U be the positive definite symmetric square root of C. The homogeneous deformation $y(x) = Fx$, $F = const.$,* det $F > 0$ *preserves the ellipsoid $\mathscr{E}$ if and only if there are rotations $\bar{R}$ and $\hat{R}$ such that*

$$F = \hat{R}U^{-1}\bar{R}U. \tag{1.4}$$

Proof. Without loss of generality, write F in the form

$$F = H^{-1}U, \tag{1.5}$$

$H = const.$, det $H > 0$. Let $x \in \mathscr{E}$ so that

$$x \cdot Cx \leqslant 1. \tag{1.6}$$

The vector $y = Fx$ then satisfies

$$y \cdot F^{-T}CF^{-1}y \leqslant 1, \tag{1.7}$$

that is,

$$1 \geqslant y \cdot H^T U^{-1} C U^{-1} H y = y \cdot H^T H y. \tag{1.8}$$

Because of the uniqueness of the representation (1.3), there is a rotation $\hat{R}$ such that

$$H^T H = \hat{R} C \hat{R}^T. \tag{1.9}$$

By the polar decomposition theorem, there are rotations $\bar{R}$ and $\hat{R}$ such that

$$H = \bar{R}^T U \hat{R}^T. \tag{1.10}$$

Thus,

$$F = \hat{R}U^{-1}\bar{R}U. \tag{1.11}$$

The converse follows immediately by inserting (1.11) into (1.7). □

All homogeneous motions which preserve ellipsoids are obtained by allowing $\hat{R}$ and $\bar{R}$ to depend upon time. For these to be dynamically possible motions in a homogeneous elastic body with no body force, they must also be accelerationless.

2. Properties

All homogeneous deformations which preserve ellipsoids are isochoric. That is,

$$\det F = (\det U)^{-1}(\det U) = 1. \tag{2.1}$$

Generally, a six parameter family of deformations preserve ellipsoids, although three of these parameters are associated with rigid rotation. However, only rigid motions preserve spheres (see Mura [2]). That is, if $U = \alpha 1$, then

$$F = \hat{R}\bar{R}. \tag{2.2}$$

We can derive the conditions on $\hat{R}$ and $\overline{R}$ which reduce $y = Fx$ to a rigid motion by writing

$$1 = F^T F = U\overline{R}^T U^{-1} \hat{R}^T \hat{R} U^{-1} \overline{R} U = U R^{-T} U^{-2} \overline{R} U, \tag{2.3}$$

which yields

$$C = \overline{R}^T C \overline{R}. \tag{2.4}$$

Hence, a motion $y = Fx$ with F given by (1.4) is rigid if and only if the rotation $\overline{R}$ leaves the eigenspaces of C invariant.

This result suggests one kind of linearization. If C has distinct eigenvalues, (2.4) is satisfied by only $\overline{R} = 1$ and $\overline{R} = -1 + 2\, e \otimes e$, e being an eigenvector of C. Putting aside the 180° rotation, we can linearize about $\overline{R} = 1$ by writing

$$\overline{R} \cong 1 + W, \tag{2.5}$$

where W is skew. For simplicity put $\hat{R} = 1$. Then, approximately,

$$F \cong 1 + U^{-1} W U. \tag{2.6}$$

Thus, the small strain and rotation tensors are

$$\begin{aligned} \hat{E} &= \tfrac{1}{2}(U^{-1} W U - U W U^{-1}), \\ \hat{W} &= \tfrac{1}{2}(U^{-1} W U - U W U^{-1}). \end{aligned} \tag{2.7}$$

This is a disguised form of the expressions given by Mura [2, equation 36]. To see the connection, note first that (2.7) yields zero normal strains in the orthonormal basis (e_i) of principal axes of $\mathscr{E}$:

$$e_i \cdot \hat{E} e_i = 0 \text{ (no sum)}. \tag{2.8}$$

(The analagous equation with say $F^T F$ replacing $\hat{E}$ does not hold for the finite deformations which preserve ellipsoids. In particular, $\mathrm{tr} F^T F$ can be made arbitrarily large for the finite ones, and the principal stretches can be made to take on any values consistent with $\det F = 1$. See §3 for details.) Let (λ_i) be the eigenvalues of U. From (2.7), the shear strains are

$$\epsilon_{ij} = e_i \cdot \hat{E} e_j = \tfrac{1}{2}(e_i \cdot W e_j) \left(\frac{(\lambda_j)}{(\lambda_i)} - \frac{(\lambda_i)}{(\lambda_j)} \right), \tag{2.9}$$

while the components of infinitesimal rotation are

$$\omega_{ij} = e_i \cdot \hat{W} e_j = \tfrac{1}{2}(e_i \cdot W e_j) \left(\frac{(\lambda_j)}{(\lambda_i)} + \frac{(\lambda_i)}{(\lambda_j)} \right), \quad (i \neq j). \tag{2.10}$$

If we eliminate $e_i \cdot W e_j$ between (2.9) and (2.10) and rearrange, we get

$$\omega_{ij} = \epsilon_{ij} \left[\frac{\lambda_j^2 + \lambda_i^2}{\lambda_j^2 - \lambda_i^2} \right], \tag{2.11}$$

which is equation (36) of the paper by Mura [2], if we account for the notational changes $\omega_{ij} \to -\omega_{ij}$, $\lambda_i \to a_i^{-1}$.

3. Examples and conclusion

According to (1.9), the two ellipsoids $\mathscr{E}$ and $\mathscr{E}' = y(\mathscr{E})$ have the same principal axes if $\hat{R} = 1$. Since $\hat{R}$ is the first multiplier in the expression for F, and therefore only contributes a rigid rotation, we put $\hat{R} = 1$ from now on. Thus, $\mathscr{E}' = \mathscr{E}$.

For the first example, let (e_i) be an orthonormal basis in $\mathbb{R}^3$ and write

$$U = \sum \lambda_i e_i \otimes e_i, \quad \lambda_1 \leqslant \lambda_2 \leqslant \lambda_3. \tag{3.1}$$

Let $\bar{R}$ be a 90° rotation about e_2:

$$\bar{R}e_2 = e_2, \quad \bar{R}e_3 = e_1, \quad \bar{R}e_1 = -e_3. \tag{3.2}$$

With $\hat{R} = 1$, it follows that

$$F = \frac{\lambda_3}{\lambda_1} e_1 \otimes e_3 + e_2 \otimes e_2 - \frac{\lambda_1}{\lambda_3} e_3 \otimes e_1. \tag{3.3}$$

F has principal stretches $(\lambda_1/\lambda_3, 1, \lambda_3/\lambda_1)$ which can be made arbitrarily close to (1, 1, 1) by an appropriate choice of ellipsoid. However, even with stretches nearly equal to 1, the material in the ellipsoid experiences a large rotation since $Fe_3 \propto e_1$. This example may be significant because the crystallographic theory of martensitic transformations (see Wayman [4]) always delivers transformation strains which have one principal stretch equal to 1. It also should be noted that many martensitic transformations have transformation deformations which are approximately simple shears of $0 - 10\%$.

For the second example, let U be given by (3.1) and suppose $\bar{R}$ permutes the (e_i) in the following way:

$$\bar{R}e_3 = e_2, \quad \bar{R}e_2 = e_1, \quad \bar{R}e_1 = e_3. \tag{3.4}$$

Then,

$$F = \frac{\lambda_1}{\lambda_3} e_3 \otimes e_1 + \frac{\lambda_2}{\lambda_1} e_1 \otimes e_2 + \frac{\lambda_3}{\lambda_2} e_2 \otimes e_3, \tag{3.5}$$

so

$$F^T F = \left(\frac{\lambda_1}{\lambda_3}\right)^2 e_1 \otimes e_1 + \left(\frac{\lambda_2}{\lambda_1}\right)^2 e_2 \otimes e_2 + \left(\frac{\lambda_3}{\lambda_2}\right)^2 e_3 \otimes e_3. \tag{3.6}$$

Since the coefficients in (3.6) can be made into any three numbers whose product is 1 by choice of the (λ_i), the principal stretches can be made to have any three values whose product is 1 by the choice of $\mathscr{E}$. Furthermore, the principal axes of strain can be made arbitrary since the replacements

$$\begin{aligned} U &\to RUR^T \\ \bar{R} &\to R\bar{R}R^T \end{aligned}, \tag{3.7}$$

simply rotate the (e_i) in the expression (3.6). Thus, *every isochoric homogeneous deformation preserves some ellipsoid.*

These results might be useful for experimental studies of displacive phase transformations. Isochoric transformation strains are encountered fairly often; twinning transformations and martensitic transformations in shape-memory materials provide examples. The essential difficulty in experiments, especially in multiaxial studies, is gripping the specimen so as to provide, say, uniform loads on a face of the crystal. As an alternative, an ellipsoidal body of shape $\mathscr{E}$ could be coated with a nonbinding agent and then cast in a transparent block. If $\mathscr{E}$ is designed to match the stress-free transformation strain according to the calculations given above, the ellipsoidal body could be made to transform, say by changing the temperature – while confined by a relatively hard device. (The effects of the hardness of the loading device on transformation temperatures have been noted in [5].) Transformation should be obvious by the large rotations involved. The device might provide a convenient means of applying small deformations to either phase.

References

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