LAMINAR ELASTIC COMPOSITES WITH CRYSTALLOGRAPHIC SYMMETRY*

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Abstract. Francfort and Murat [*Arch. Rational Mech. Anal.*, 94 (1986), pp. 307-334] derived an explicit formula for the effective elasticity tensor of a multiply layered composite made from two isotropic elastic materials in prescribed proportion. For multiply layered composites with crystallographic symmetry, it is shown that these formulae can be represented as a group average over the crystallographic group. The special case of cubically symmetric elastic composites made by multiple layering is considered. This article determines precisely the set of elasticity tensors that correspond to these composites. Extremal property of laminar composites is then used (see Avellaneda [*SIAM J. Appl. Math.*, 47 (1987), pp. 1216-1228]) to obtain new optimal bounds on the effective shear moduli for elastic composites with cubic symmetry.

Key words. finite-rank laminates, group average, optimal bounds

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1. Introduction. An effective elastic medium represents the limiting behavior of a heterogeneous elastic material as the ratio ε between microscopic and macroscopic scales tends to zero. Mathematically, the limiting behavior of the heterogeneous medium as ε goes to zero is described by the theory of H-convergence [5], [16], [17] or G-convergence [19], [20], [24].

Let $\mathbb{C}_1$ and $\mathbb{C}_2$ be the elasticity tensors for two isotropic elastic materials with shear and bulk moduli μ_1, μ_2 and κ_1, κ_2, respectively. Throughout the paper we will suppose there exist positive constants $\alpha < \beta$ for which $\alpha \mathbb{I} \leq \mathbb{C}_s$ and $|\mathbb{C}_{s,ijkl}| \leq \beta$, for $s = 1$, 2. A composite constructed with these two components is defined by the characteristic function χ of material 1 in $\mathbb{R}^3$. The elasticity tensor for the composite is given by

$$(1.1) \qquad \mathbb{C}(x) = \mathbb{C}_1 \chi(x) + \mathbb{C}_2 (1 - \chi(x)).$$

A composite body for which the heterogeneities are of the size ε relative to the domain $\Omega \subseteq \mathbb{R}^3$ occupied by the body is described by the rapidly oscillating characteristic function $\chi^\varepsilon(x)$ of material 1. We consider the family of composites with elasticity tensors

$$(1.2) \qquad \mathbb{C}^\varepsilon(x) = \mathbb{C}_1 \chi^\varepsilon(x) + \mathbb{C}_2 (1 - \chi^\varepsilon)(x))$$

such that

$$(1.3) \qquad \chi^\varepsilon \to \theta(x) \quad \text{in } L^\infty(\Omega) \text{ weak } *$$

as ε tends to zero. The limit θ is interpreted as the local average volume fraction of material 1 in the effective material.

THEOREM 1.1 (cf. [5]). *A subsequence of tensors* $\mathbb{C}^{\varepsilon'}$ *of the sequence* $\mathbb{C}^{\varepsilon}$ *given by* (1.2) *and* (1.3) *H-converges to the tensor* $\mathbb{C}^*$ *with* $\|\mathbb{C}^*_{ijkl}\|_{L^\infty(\Omega)} \leqq \beta$ *and* $\mathbb{C}^* \geqq \alpha \mathbb{I}$ *almost everywhere in* Ω, *i.e., for any* f *in* $H^{-1}(\Omega)^3$ *the unique solution* $u^{\varepsilon'}$ *in* $H_0^1(\Omega)^3$ *of*

$$(1.4) \qquad \begin{aligned} E^{\varepsilon'} &= \tfrac{1}{2}(\nabla u^{\varepsilon'} + (\nabla u^{\varepsilon'})^T) \\ \operatorname{div} \mathbb{C}^{\varepsilon'} E^{\varepsilon'} &= f \end{aligned}$$

satisfies

$$u^{\varepsilon'} \to u^* \quad \text{weakly in } H_0^1(\Omega)^3$$

as ε' *goes to zero, where* $u^* \in H_0^1(\Omega)^3$ *is the unique solution of*

$$(1.5) \qquad \operatorname{div} \mathbb{C}^* E^* = f.$$

The resulting fourth-order tensor $\mathbb{C}^*$ is a symmetric linear transformation on the space of symmetric 3×3 matrices.

DEFINITION 1.2. The tensor $\mathbb{C}^*$ delivered by Theorem 1.1 is called the effective elasticity tensor of the two-phase composite defined by (1.2) and (1.3).

Incompressible elastic composites are constructed using two isotropic incompressible elastic components characterized by the shear moduli μ_1, μ_2 only. The analogue of the H-convergence Theorem 1.1 for incompressible elasticity is stated in [11] and the effective elasticity $\mathbb{C}^*$ is a symmetric linear transformation on the space of symmetric 3×3 trace-free matrices.

Effective material can be anisotropic and can exhibit crystallographic symmetries. The easiest materials to characterize are the isotropic ones, because the effective tensor contains only two parameters. The characterization becomes more difficult when fewer assumptions on the symmetry are made.

In general there are no explicit formulas for the effective tensor $\mathbb{C}^*$; however, when the H-convergent sequence (1.2) involves laminar geometries (e.g., χ^ε is a function of $x \cdot n$ where n is a fixed unit vector in $\mathbb{R}^3$) then $\mathbb{C}^*$ is given by an algebraic formula. The resulting materials are referred to as laminates. Algebraic formulas for the effective tensors of laminates were introduced by Tartar [21] for the case of heat conductivity. Francfort and Murat [5] applied the ideas of Tartar and obtained formulae for the effective elasticity of laminates. Formulae for the effective elasticity of incompressible elastic laminates are given in [10], [11].

The ultimate goal is to characterize the set of all effective elasticity tensors arising from H-converging sequences of composites of two isotropic components taken in specified proportion in $\mathbb{R}^3$.

In this article we characterize the set of all effective tensors with cubic symmetry arising from H-converging sequences of incompressible and compressible elastic laminates (see Corollary 5.8 and Corollary 5.12). From the comparison theorem of Avellaneda (Theorem 1 of [1]) and our knowledge of the set of all cubic laminates, we derive new optimal bounds on the effective shear moduli for cubically symmetric composites. These bounds correlate the shear moduli (see Theorem 5.13 of § 5.C) and improve the shear moduli bounds given in [1].

Many bounds characterizing the set of effective elasticity tensors have been derived [1], [2], [5], [8]–[15], [23]. Recently Kohn and Milton [14] derived a set of bounds on the effective elasticity tensor that follows from the Hashin–Shtrikman variational principle (equations 6.61 and 6.62 of [14]). The extent to which these bounds completely characterize the set of effective tensors remains largely uninvestigated. One of the bounds included in [14] is a bound on the eigenvalues of the effective tensor. For the special case of incompressible elasticity, bounds on the eigenvalues of the effective

tensor are also derived independently in [11]. In two dimensions it is shown in [11] that eigenvalue bounds completely describe the set of effective elasticity tensors. For the three-dimensional incompressible case with $\theta = \text{const.}$ it is known [11] that the eigenvalues $\lambda_1, \cdots, \lambda_5$ of the effective tensor lie in the convex region of $\mathbb{R}^5$ given by the intersection of Paul's bounds (cf. [18])

$$(1.6) \qquad 2(\mu_1^{-1}\theta + \mu_2^{-1}(1-\theta))^{-1} \leqq \lambda_i \leqq 2(\mu_1\theta + \mu_2(1-\theta)), \qquad i = 1, \cdots, 5$$

and the eigenvalue bounds

$$(1.7) \qquad \sum_{i=1}^{5} \frac{1}{\lambda_i - 2\mu_1} \leqq \frac{1}{2(\mu_1^{-1}\theta + \mu_2^{-1}(1-\theta))^{-1} - 2\mu_1} + \frac{3/2}{2(\mu_1\theta + \mu_2(1-\theta)) - 2\mu_1}$$

$$(1.8) \qquad \sum_{i=1}^{5} \frac{1}{2\mu_2 - \lambda_i} \leqq \frac{1}{2\mu_2 - 2(\mu_1^{-1}\theta + \mu_2^{-1}(1-\theta))^{-1}} + \frac{3/2}{2\mu_2 - 2(\mu_1\theta + \mu_2(1-\theta))}.$$

These restrictions do not presuppose any particular symmetry of the effective medium. To understand how well the eigenvalue bounds characterize the effective tensor in three dimensions it is of immediate interest to know whether all $(\lambda_1, \cdots, \lambda_5)$, consistent with (1.7) and (1.8) and (1.6) are attainable by laminar composites. As of this writing the answer to this question is open. In fact, in three dimensions the set of effective tensors is not characterized by restrictions on eigenvalues only. We give a simple mathematical statement of this problem in § 5, Problem 5.1. However, if we limit our attention to the class of effective composites with cubic symmetry, then $\lambda_1 = \lambda_2 = \lambda_3 = \alpha$, $\lambda_4 = \lambda_5 = \gamma$ (see Remark 5.4) and (1.7) and (1.8) become

$$(1.9) \qquad \frac{3}{\alpha - 2\mu_1} + \frac{2}{\gamma - 2\mu_1} \leqq \frac{1}{2(\mu_1^{-1}\theta + \mu_2^{-1}(1-\theta))^{-1} - 2\mu_1} + \frac{3/2}{2(\mu_1\theta + (1-\theta)\mu_2) - 2\mu_1}$$

$$(1.10) \qquad \frac{3}{2\mu_2 - \alpha} + \frac{2}{2\mu_2 - \gamma} \leqq \frac{1}{2\mu_2 - 2(\mu_1^{-1}\theta + \mu_2^{-1}(1-\theta))^{-1}} + \frac{3/2}{2\mu_2 - 2(\mu_1\theta + u_2(1-\theta))}.$$

We show in § 5.A that not all points on (1.9) and (1.10) are attainable by finite rank laminates with cubic symmetry. This result is in sharp contrast to the two-dimensional anisotropic case where every point on the eigenvalue bound analogous to (1.7) and (1.8) is achieved by a rank-2 laminate [11].

In § 6 we give a simple proof of isotropy for the finite rank isotropic compressible elastic laminates introduced by Francfort and Murat [5]. We note that our proof applies to the incompressible case as well (see Theorem 6.1).

Lastly, if we consider incompressible elastic composites with isotropic effective elasticity then

$$\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = \mu^*$$

and (1.7) and (1.8) become

$$(1.11) \qquad \mu_l \leqq \mu^* \leqq \mu_u$$

where

$$(1.12) \qquad \mu_l = \mu_1 + \frac{1-\theta}{\dfrac{1}{\mu_2 - \mu_1} + \dfrac{2\theta}{5\mu_1}},$$

$$(1.13) \qquad \mu_u = \mu_2 + \frac{\theta}{\dfrac{1}{\mu_1 - \mu_2} + \dfrac{2(1-\theta)}{5\mu_2}}.$$

We first note that (1.12) and (1.13) may be obtained by passing to the limit $\kappa_i \to \infty$ in the Hashin–Shtrikman bounds on the shear moduli for macroscopically isotropic compressible elastic composites (cf. equations (0.2) and (0.3) of [5]). Secondly we remark that the equations of incompressible elasticity are identical to the equations of Stokes flow when surface tension is neglected. Thus the bounds (1.12) and (1.13) are identical to the Hashin–Shtrikman bounds on the effective viscosity for a statistically isotropic mixture of two Stokes fluids [7]. In § 6, Theorem 6.2, we show that (1.11) describes the set of effective shear moduli for all isotropic incompressible elastic composites.

In this paper small Greek letters α, β, θ, ε, $\cdots$ denote real numbers. Small Latin letters a, b, e, n, $v, \cdots$ denote vectors in $\mathbb{R}^3$. Capital Latin letters K, M denote second-order tensors on $\mathbb{R}^3$. Hollow Latin letters $\mathbb{C}, \mathbb{I}, \mathbb{P}, \cdots$ denote fourth order tensors on $\mathbb{R}^3$.

Throughout the paper the coefficients of all tensors are given with respect to a fixed orthonormal basis in $\mathbb{R}^3$, i.e., e_1, e_2, e_3. We define scalar product, tensor products, and contractions as follows:

- $a \cdot b$ is the scalar with value $a_i b_i$;
- $K : M$ is the scalar with value $K_{ij} M_{ij}$;
- Ka is the vector with components $(Ka)_i = K_{ij} a_j$;
- $\mathbb{C} K$ is the second-order tensor with components $(\mathbb{C} K)_{ij} = \mathbb{C}_{ijkl} K_{kl}$;
- $\mathbb{C} QQQQ$ is the fourth-order tensor with components

$$(\mathbb{C} QQQQ)_{ijkl} = \mathbb{C}_{mnop} Q_{im} Q_{jn} Q_{ok} Q_{pl};$$

- $a \otimes b$ is the second order tensor with components $(a \otimes b)_{ij} = a_i b_j$;
- $a \circledS b$ is the symmetric rank two tensor with components

$$(a \circledS b)_{ij} = \tfrac{1}{2}(a_i b_j + a_j b_i);$$

- $\mathrm{tr}\, \mathbb{C}$ is the scalar given either by $\mathbb{C}_{ijij}$ or by the sum of the eigenvalues of $\mathbb{C} : S_3 \to S_3$.

The following vector spaces of 3×3 matrices are used:

- S_3 is the space of symmetric 3×3 matrices;
- S_{diag} is the space of diagonal 3×3 matrices with zero trace;
- S_{off} is the space of symmetric 3×3 matrices with zero diagonal elements;
- S_t is the space of symmetric 3×3 matrices with zero trace;
- O_3 is the set of all orthogonal 3×3 matrices;
- O_3^+ is the set of all proper orthogonal 3×3 matrices;
- I denotes the identity matrix;
- $\mathbb{I}$ denotes the identity mapping on S_3;
- $\tilde{\mathbb{I}}$ is the restriction of $\mathbb{I}$ to S_t.

All elasticity tensors considered in this article are symmetric linear transformations on S_3 and therefore have the component symmetries

$$\mathbb{C}_{ijkl} = \mathbb{C}_{jikl} = \mathbb{C}_{klij}.$$

2. The effective elasticity tensors of finite rank laminates. Sequentially laminated composite materials, or for short, laminates, have been widely used to prove the attainability of bounds on effective tensors for equations of heat conductivity [21] and systems of equations of linear elasticity [5].

To define a rank-1 laminate we consider two isotropic elastic materials 1 and 2 with elasticity tensors defined by

$$(2.1) \qquad \mathbb{C}_i = 2\mu_i \mathbb{I} + (\kappa_i - \tfrac{2}{3}\mu_i) I \otimes I$$

where μ_i, κ_i, $i = 1, 2$, are the shear and bulk moduli, respectively, with $\mu_2 > \mu_1 > 0$, $\kappa_2 > \kappa_1 > 0$. Let $\chi^{\varepsilon,1} : \mathbb{R} \to \{0, 1\}$ be a sequence of step functions which converge to the constant ρ_1 in the $L^\infty(\mathbb{R})$ weak $*$ topology and let n_1 be a fixed unit vector in $\mathbb{R}^3$. Consider the sequence of laminar materials in $\mathbb{R}^3$ with layer direction n_1 and layer thickness ε with elasticity tensors given by

$$(2.2) \qquad \mathbb{C}^\varepsilon(x) = \mathbb{C}_1 \chi^{\varepsilon,1}(x \cdot n_1) + \mathbb{C}_2(1 - \chi^{\varepsilon,1}(x \cdot n_1)).$$

The homogenization result of Francfort and Murat [5, Cor. 4.1] states that the H-limit of the sequence (2.2) defines the effective elasticity tensor $\mathbb{C}^*$ given by

$$(2.3) \qquad \rho_2(\mathbb{C}_1^* - \mathbb{C}_1)^{-1} - (\mathbb{C}_2 - \mathbb{C}_1)^{-1} + \rho_1 \mathbf{T}^1(n_1)$$

where $\rho_1 + \rho_2 = 1$, and by definition

$$(2.4) \qquad \mathbf{T}^1(n_1) = \frac{3/2}{3\kappa_1 + 4\mu_1} \mathbb{R}(n_1) + \frac{1}{2\mu_1}(\mathbb{P}(n_1) - \mathbb{R}(n_1)),$$

and

$$(2.5) \qquad \begin{aligned} \mathbb{P}(v)M &= 2(Mv) \circledS v \\ \mathbb{R}(v)M &= 2(M : v \otimes v)v \otimes v, \end{aligned}$$

for all M in S_3 and unit vectors v in $\mathbb{R}^3$.

To construct a rank-2 laminate we consider the sequence of materials $\chi^{\varepsilon,2}$ with elasticity tensors given by

$$(2.6) \qquad \mathbb{C}_2^\varepsilon = \mathbb{C}_1 \chi^{\varepsilon,2}(x \cdot n_2) + \mathbb{C}_1^*(1 - \chi^{\varepsilon,2}(x \cdot n_2)),$$

where $\chi^{\varepsilon,2} : \mathbb{R} \to [0, 1]$ converges to $\tau_1 = \text{const.}$ in $L^\infty(\mathbb{R})$ weak $*$, n_2 is a fixed unit vector denoting another layer direction, and $\mathbb{C}_1^*$ is given by (2.3). From Corollary 4.1 of [5] the H-limit of the sequence (2.6) is the effective tensor $\mathbb{C}_2^*$ given by

$$(2.7) \qquad \tau_2(\mathbb{C}_2^* - \mathbb{C}_1)^{-1} = (\mathbb{C}_1^* - \mathbb{C}_1)^{-1} + \tau_1 \mathbf{T}^1(n_2),$$

where $\tau_1 + \tau_2 = 1$. We use (2.3) to expand $(\mathbb{C}_1^* - \mathbb{C}_1)^{-1}$ in (2.7) and obtain

$$\theta_2(\mathbb{C}_2^* - \mathbb{C}_1)^{-1} = (\mathbb{C}_2 - \mathbb{C}_1)^{-1} + \theta_1(\Xi_1 \mathbf{T}^1(n_1) + \Xi_2 \mathbf{T}^1(n_2))$$

where θ_1 is the total volume fraction of material 1 in the composite, $\theta_2 = 1 - \theta_1$, and $\Xi_1 + \Xi_2 = 1$ where $\Xi_1 = \rho_1/\theta_1$, $\Xi_2 = (\theta_1 - \rho_1)/\theta_1$. It is now apparent that Corollary 4.1 of [5] gives a convenient formula for the effective tensor for a multiply layered composite. Repeating the above process j times we arrive at the following definition.

DEFINITION 2.1. Let the j unit vectors $n_1, \cdots, n_j$, the scalar θ_1, and the elasticity tensors $\mathbb{C}_1$ and $\mathbb{C}_2$ be given. Let $\theta_2 = 1 - \theta_1$, $0 \leq \Xi_i \leq 1$, and $\sum_{i=1}^{j} \Xi_i = 1$. The effective elasticity tensor $\mathbb{C}_j^*$ of rank-j laminate with core material 2 is

$$(2.8) \qquad \theta_2(\mathbb{C}_j^* - \mathbb{C}_1)^{-1} = (\mathbb{C}_2 - \mathbb{C}_1)^{-1} + \theta_1 \sum_{i=1}^{j} \Xi_i \mathbf{T}^1(n_i).$$

Remark 2.2. If we choose material 1 as the core material, then the effective elasticity of the resulting rank-j laminate is given by

$$(2.9) \qquad \theta_1(\mathbb{C}_j^* - \mathbb{C}_2)^{-1} = (\mathbb{C}_1 - \mathbb{C}_2)^{-1} + \theta_2 \sum_{i=1}^{j} \Xi_i \mathbf{T}^2(n_i),$$

where

$$(2.10) \qquad \mathbf{T}^2(n) = \frac{3/2}{3\kappa_2 + 4\mu_2} \mathbb{R}(n) + \frac{1}{2\mu_2}(\mathbb{P}(n) - \mathbb{R}(n)).$$

Remark 2.3. Incompressible, isotropic, elastic materials 1 and 2 are characterized by their shear moduli μ_i, $i = 1, 2$, only. The analogues of (2.8) and (2.9) for incompressible rank-j laminates are

$$(2.11) \qquad \theta_2(\mathbb{C}_j^* - \mathbb{C}_1)^{-1} = (\mathbb{C}_2 - \mathbb{C}_1)^{-1} + \frac{\theta_1}{2\mu_1} \sum_{i=1}^{j} \Xi_i \mathbf{T}(n_i),$$

and

$$(2.12) \qquad \theta_1(\mathbb{C}_j^* - \mathbb{C}_2)^{-1} = (\mathbb{C}_1 - \mathbb{C}_2)^{-1} + \frac{\theta_2}{2\mu_2} \sum_{i=1}^{j} \Xi_i \mathbf{T}(n_i),$$

where

$$(2.13) \qquad \mathbf{T}(n) = \mathbb{P}(n) - \mathbb{R}(n)$$

see [10], [11]. Equations (2.11) and (2.12) may be obtained by passing to the limit $\kappa_i \to \infty$, $i = 1, 2$ in (2.8) and (2.9).

3. Symmetries, transformations and spectral properties of degenerate compliance tensors. We investigate the algebraic properties of the degenerate compliance tensors $\mathbf{T}^1$, $\mathbf{T}^2$, and $\mathbf{T}$ given in (2.4), (2.10), and (2.13). It is evident from (2.4), (2.10), and (2.13) that the algebraic and spectral properties of the degenerate compliance tensors follows from those of the tensors $\mathbb{P}$ and $\mathbb{R}$ given by (2.5).

It follows directly from (2.5) that both tensors have the symmetries

$$(3.1) \qquad M_1 : \mathbb{R}(v) M_2 = M_2 : \mathbb{R}(v) M_1, \qquad M_1 : \mathbb{P}(v) M_2 = M_2 : \mathbb{P}(v) M_1,$$

and

$$(3.2) \qquad \mathbb{R}(v) M = (\mathbb{R}(v) M)^T, \qquad \mathbb{P}(v) M = (\mathbb{P}(v) M)^T$$

for all unit vectors v in $\mathbb{R}^3$. It follows also from (2.5) that $\mathbb{P}(v)$ and $\mathbb{R}(v)$ are even in v:

$$(3.3) \qquad \mathbb{P}(-v) = \mathbb{P}(v),$$

$$(3.4) \qquad \mathbb{R}(-v) = \mathbb{R}(v)$$

for all unit vectors v in $\mathbb{R}^3$. In general, an orthogonal transformation of v transforms the tensors according to the rule

$$(3.5) \qquad \mathbb{P}(Qv) = \mathbb{P}(v) QQQQ,$$

$$(3.6) \qquad \mathbb{R}(Qv) = \mathbb{R}(v) QQQQ.$$

DEFINITION 3.1. *Let* $v \in \mathbb{R}^3$ *be a unit vector. The matrix representation of a subgroup of* O_3 *that has* v *as a fixed point is defined by* G_v:

$$G_v = \{Q \in O_3 : Qv = v\}.$$

DEFINITION 3.2. *If* G *is a matrix representation of a subgroup of* O_3 *then the subspace* T *of* S_3 *is said to be invariant under* G *if and only if* $QKQ^T \in T$ *for every* $Q \in G$ *and every* $K \in T$.

In the following we establish the spectral properties of the tensors $\mathbb{P}$ and $\mathbb{R}$. It is easily seen from (2.5) that $\frac{1}{2}\mathbb{R}(v)$ is a projection onto the one-dimensional subspace of S_3 spanned by $v \otimes v$. The spectral properties of $\mathbb{P}(v)$ are given by Lemma 3.2.

LEMMA 3.2. *Let* a, b, *and* v *form an orthonormal basis of* $\mathbb{R}^3$. *If* U *and* V *are subspaces of* S_3 *defined by*

$$(3.7) \qquad U = \text{span}\,\{v \otimes v\}, \qquad V = \text{span}\,\{\sqrt{2}\,v \circledS a, \sqrt{2}\,v \circledS b\}$$

then the degenerate compliance tensor $\mathbb{P}(v)$ given in (2.5) is expressible as a linear combination of two orthogonal projections $\mathbb{P}_U(v): S_3 \to U$ and $\mathbb{P}_V(v): S_3 \to V$:

$$(3.8) \qquad\qquad \mathbb{P}(v) = 2\mathbb{P}_U(v) + \mathbb{P}_V(v).$$

Moreover, the subspaces U and V are invariant under G_v.

Proof. It is evident from Definition 3.2 that U is invariant under G_v. We now demonstrate that V is also invariant under G_v. The matrices

$$(3.9) \qquad \sqrt{2}\, v \circledS a, \quad \sqrt{2}\, v \circledS b, \quad \sqrt{2}\, a \circledS b, \quad a \otimes b, \quad b \otimes b, \quad v \otimes v$$

form an orthonormal basis in S_3. Direct calculation shows that for all $Q \in G_v$ the matrices

$$Q(\sqrt{2}\, v \circledS a)Q^T \quad \text{and} \quad Q(\sqrt{2}\, v \circledS b)Q^T$$

remain orthogonal to the last four matrices in (3.9). Therefore, according to Definition 3.2 V is invariant under G_v. To obtain the spectral decomposition (3.8) of the tensor $\mathbb{P}(v)$ we operate $\mathbb{P}(v)$ on each matrix in the basis (3.9) and observe that

$$a \otimes a, \quad b \otimes b, \quad \text{and} \quad \sqrt{2}(a \circledS b)$$

span the null space of $\mathbb{P}(v)$ and that

$$\mathbb{P}(v)A = A \quad \text{for } A \text{ in } V$$

and

$$\mathbb{P}(v)A = 2A \quad \text{for } A \text{ in } U.$$

Remark 3.3. For future reference we note that it easily follows from (2.13) and (3.8) that the sum of eigenvalues for the tensor $\mathbf{T}(v)$ is 2, i.e.,

$$(3.10) \qquad\qquad \operatorname{tr} \mathbf{T}(v) = 2$$

and from (2.4) and (2.10) that

$$(3.11) \qquad\qquad \operatorname{tr} \mathbf{T}^i = \frac{3}{3\kappa_i + 4\mu_i} + \frac{1}{\mu_i}, \quad i = 1, 2.$$

4. Finite rank laminates with crystallographic symmetry. Let G be a matrix representation of a subgroup of O_3^+. We say that a material has a G-group symmetry if its elasticity tensor $\mathbb{C}$ is invariant under G, i.e.,

$$(4.1) \qquad\qquad \mathbb{C} = \mathbb{C}QQQQ,$$

for all Q in G.

LEMMA 4.1. *If a fourth order-tensor $\mathbb{C}$ with the symmetries $\mathbb{C}_{ijkl} = \mathbb{C}_{jikl} = \mathbb{C}_{klij}$ is invariant under G then its eigenspaces are invariant under G, and $\mathbb{C}$ has the spectral decomposition*

$$(4.2) \qquad\qquad \mathbb{C} = \sum_{i=1}^{j} \lambda_i \mathbb{D}_i,$$

where $1 \leq j \leq 6$ and $\mathbb{D}_i$ are orthogonal projections onto the eigenspaces of $\mathbb{C}$.

Proof. From the index symmetries of $\mathbb{C}$ it follows that $\mathbb{C}$ is a symmetric linear transformation on the vector space S_3. Therefore, the eigenspaces of $\mathbb{C}$ are pairwise orthogonal. From the invariance of $\mathbb{C}$ under the group G it is evident that if M is in the eigenspace E_λ then so is QMQ^T for Q in G, hence the lemma follows.

Remark 4.2. The inverse of $\mathbb{C}$ given by (4.2) is written

$$(4.3) \qquad\qquad \mathbb{C}^{-1} = \sum_{i=1}^{j} \lambda_i^{-1} \mathbb{D}_i.$$

In the following we work with elastic materials possessing cubic and icosahedral symmetries.

Let O denote the matrix representation of the cubic group for a cubically symmetric material with crystal axes e_1, e_2, e_3. This subgroup of O_3^+ has 24 elements and is the symmetry group of the cube with face normals e_1, e_2, e_3, (cf. [4]). It follows from Lemma 4.1 that any elasticity tensor $\mathbb{C}$ with O group symmetry has the spectral decomposition of the form

$$(4.4) \qquad \mathbb{C} = \lambda_1 \mathbb{D}_{\text{off}} + \lambda_2 \mathbb{D}_{\text{diag}} + \lambda_3 \mathbb{D}_I,$$

where $\mathbb{D}_{\text{off}}$ is the projection onto the three-dimensional subspace S_{off}, $\mathbb{D}_{\text{diag}}$ is the projection onto the two-dimensional subspace S_{diag}, and $\mathbb{D}_I$ is the projection onto the identity matrix. The formula (4.4) follows easily from Table 16.4 of [22]. We note that the eigenvalues λ_1, λ_2, and λ_3 are usually written $\lambda_1 = 2\mu''$, $\lambda_2 = 2\mu'$, $\lambda_3 = 3\kappa$ where μ'', μ' are the Lamé moduli and κ is the bulk modulus.

We introduce the icosahedral group. This subgroup of O_3^+ has 60 elements and is the symmetry group of the regular icosahedron [4], (see Fig. 1). This group has the 12 axes of 5-fold symmetry $v_1, \cdots, v_{12}$ given by

$$\tau = \frac{\sqrt{5}+1}{2}$$

$$(4.5) \qquad \begin{aligned} v_1 &= (0, \tau, 1), & v_5 &= (1, 0, \tau), & v_9 &= (\tau, 1, 0), \\ v_2 &= (0, \tau, -1), & v_6 &= (1, 0, -\tau), & v_{10} &= (\tau, -1, 0) \\ v_3 &= (0, -\tau, 1), & v_7 &= (-1, 0, \tau), & v_{11} &= (-\tau, 1, 0) \\ v_4 &= (0, -\tau, -1), & v_8 &= (-1, 0, -\tau), & v_{12} &= (-\tau, -1, 0). \end{aligned}$$

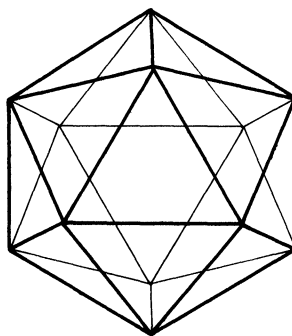We denote the matrix representation of the symmetry group for the icosahedron given above by Y.

FIG. 1. *The regular icosahedron.*

LEMMA 4.3. *The spectral decomposition of elasticity tensors $\mathbb{C}$ with Y group symmetry is of the form*

$$(4.6) \qquad \mathbb{C} = 2\mu(\mathbb{D}_{\text{off}} + \mathbb{D}_{\text{diag}}) + 3K\mathbb{D}_I.$$

Remark 4.4 (cf. [6]). An isotropic elasticity tensor $\mathbb{C}$ is a tensor of the form (4.5), or equivalently (2.1). Thus, Lemma 4.3 states that an elastic material with icosahedral symmetry is an isotropic material.

Proof of Lemma 4.3. Consider a cube with face normals e_1, e_2, e_3, and consider a tetrahedron with 3-fold rotation axes being the vertices of the cube given by $(1, 1, 1)$, $(-1, 1, 1)$, $(1, 1, -1)$, and $(1, -1, 1)$. It is evident that any rotation leaving the tetrahedron invariant simply permutes the axes e_1, e_2, e_3. It then follows from (4.5) that any rotation leaving the tetrahedron invariant simply permutes the symmetry axes of the icosahedron. Thus the tetrahedral group is a subgroup of the icosahedral group. The energy of a material with tetrahedral symmetry is given by

$$(4.7) \quad \mathbb{C}E:E = (C_{11} - C_{44})(\text{tr } E)^2 + (C_{12} - 2C_{11})(E_{11}E_{22} + E_{11}E_{33} + E_{22}E_{33}) + C_{44}E:E,$$

where C_{11}, C_{12}, and C_{44} are the components of the elasticity matrix for a material with tetrahedral symmetry given in Table 16.4 of [22]. The invariance property of the tensor $\mathbb{C}$ under Y is equivalent to

$$(4.8) \qquad \qquad \mathbb{C}E:E = \mathbb{C}QEQ^T:QEQ^T$$

for all $E \in S_3$ and for all $Q \in Y$. Imposing (4.8) for $E = e_1 \otimes e_1$ and Q in Y given by

$$Q = \text{Rotation of } \frac{2\pi}{5} \text{ about the symmetry axis } (0, \tau, 1)$$

we find that

$$Qe_1 = \cos\frac{2\pi}{5}e_1 + \gamma e_2 - \tau\gamma e_3$$

where $\tau = (\cos^2 2\pi/5)/(1+\tau^2)$, and it follows from (4.7) that

$$(C_{12} - 2C_{11}) = 0.$$

Hence from (4.7) we conclude

$$\mathbb{C}E:E = (C_{11} - C_{44})(\text{tr } E)^2 + C_{44}E:E$$

and therefore $\mathbb{C}$ is of the form (2.1). According to Remark 4.4, $\mathbb{C}$ is an isotropic tensor.

In the following we establish the relation between the symmetries of effective elasticity tensors of rank-j laminates given in (2.8), (2.9), (2.11), (2.12) and convex sums of degenerate compliance tensors.

Remark 4.5. From Lemma 4.1 and Remark 4.2 we observe that the rank-j laminate $\mathbb{C}_j^*$ given in (2.8) possesses a G-group symmetry if and only if the convex sum $\sum_{i=1}^j \Xi_i \mathbf{T}^1(n_i)$, $\sum_{i=1}^j \Xi_i = 1$ possesses G-group symmetry. Similar statements hold for laminates given by (2.9), (2.11), and (2.12).

We now show how to construct a rank-ν laminate that is invariant under a symmetry group of order ν.

DEFINITION 4.6. Let $G = \{Q_1, \cdots, Q_\nu\}$ be the matrix representation of a finite subgroup of O_3^+. The group average $\mathbb{C}_G$ of the fourth-order tensor $\mathbb{C}$ is given by

$$\mathbb{C}_G = \frac{1}{\nu}\sum_{\gamma=1}^{\nu}\mathbb{C}Q_\gamma Q_\gamma Q_\gamma Q_\gamma.$$

The fundamental property of $\mathbb{C}_G$ is that it is invariant under G, i.e.,

$$(4.9) \qquad \qquad \mathbb{C}_G QQQQ = \mathbb{C}_G,$$

for all $Q \in G$.

LEMMA 4.7. *Let* $G = \{Q, \cdots, Q_\nu\}$ *be a matrix representation of a finite subgroup of* O_3^+ *and* $\mathbf{T}^1(v)$ *be the tensor defined in* (2.4). *If* $\Xi_i = 1/\nu$ *and* $n_i = Q_i v$ *for* $1 \leq i \leq \nu$ *then*

$$(4.10) \qquad \qquad \sum_{i=1}^{\nu} \Xi_i \mathbf{T}^1(n_i) = \mathbf{T}^1(v)_G.$$

Proof. Equations (3.5) and (3.6) imply that

$$(4.11) \qquad\qquad \mathbf{T}^1(Qv) = \mathbf{T}^1(v)QQQQ,$$

thus the lemma follows from our choice of vectors n_i. $\square$

Lemma 4.7 and Remark 4.5 imply that we can construct a laminate invariant under any finite crystallographic symmetry group. We note, however, that because of the special form of $\mathbf{T}^1$ these laminates might have greater symmetry than that represented by the group G used in the group average. We do not pursue this issue further here.

5.A. Attainability of the eigenvalue bounds by laminates with cubic symmetry. We are interested in where all points on the boundary of the region described by (1.6), (1.7), and (1.8) are realized by the eigenvalues of incompressible laminates defined in (2.11) and (2.12).

In the following we formulate the attainability problem. We then completely describe the set of incompressible elastic laminates with cubic symmetry and describe the subset of points on the boundary of the region (1.6), (1.7), and (1.8) attained by these materials. Introducing the variable α_i given by

$$(5.1) \qquad\qquad \alpha_i = \frac{2\theta_2\mu_1}{\theta_1}\left(\frac{1}{\lambda_i - 2\mu_1} - \frac{1}{2\theta_2(\mu_2 - \mu_1)}\right), \quad 1 \leq i \leq 5,$$

Paul's bounds (1.6) become

$$(5.2) \qquad\qquad 0 \leq \alpha_i \leq 1$$

and the lower eigenvalue bound (1.7) becomes

$$(5.3) \qquad\qquad \sum_{i=1}^{5} \alpha_i \leq 2.$$

Similarly for

$$(5.4) \qquad\qquad \beta_i = \frac{2\theta_1\mu_2}{\theta_2}\left(\frac{1}{\lambda_i - 2\mu_2} - \frac{1}{2\theta_1(\mu_2 - \mu_1)}\right), \quad 1 \leq i \leq 5,$$

Paul's bounds are

$$(5.5) \qquad\qquad 0 \leq \beta_i \leq 1,$$

and the upper eigenvalue bound (1.8) is written

$$(5.6) \qquad\qquad \sum_{i=1}^{5} \beta_i \leq 2.$$

We claim that under the change of variable (5.1) the set of all points on (1.7) satisfying Paul's bounds is given by a region $\Delta_\alpha \subseteq \mathbb{R}^5$ where

$$(5.7) \qquad\qquad \Delta_\alpha = \left\{(\alpha_1, \cdots, \alpha_5) \,\middle|\, 0 \leq \alpha_i \leq 1, \sum_{i=1}^{5} \alpha_i = 2\right\}.$$

To see this, we take the trace of the laminate tensor given in (2.11) and get

$$(5.8) \qquad\qquad \theta_2\left(\sum_{i=1}^{5} \frac{1}{\lambda_i - 2\mu_1}\right) = \frac{5}{2}(\mu_2 - \mu_1)^{-1} + \frac{\theta_1}{2\mu_1}\sum_{i=1}^{j} \Xi_i \operatorname{tr} \mathbf{T}(n_i)$$

where $\lambda_1, \cdots, \lambda_5$ are the five eigenvalues of the effective tensor. From Remark 3.6, equation (5.8) reduces to

$$(5.9) \qquad\qquad \theta_2 \sum_{i=1}^{5} \frac{1}{\lambda_i - 2\mu_1} = \frac{5}{2}(\mu_2 - \mu_1)^{-1} + \frac{\theta_1}{\mu_1}.$$

An elementary computation shows that (5.9) is the same as the equality in (1.7). Hence the eigenvalues λ_i of the laminate tensor naturally attain the lower bound (1.7).

Under the change of variables given by (5.1), equation (5.8) reads

$$
(5.10) \qquad \sum_{i=1}^{5} \alpha_i = \mathrm{tr}\left(\sum_{i=1}^{j} \Xi_i \mathbf{T}(n_i)\right)
$$

where, from (5.1) and (5.9), $\sum_{i=1}^{5} \alpha_i = 2$.

Therefore, the question of attainability by laminates of all points on (1.7) subject to Paul's bounds is given by Problem 5.1.

PROBLEM 5.1. Attainability of the eigenvalue bounds by laminates. Given a point $(\alpha_1, \cdots, \alpha_5)$ in Δ_α find an integer $j \geqq 1$, a point $(\Xi_1, \cdots, \Xi_j)$ in

$$
\Delta_\Xi = \left\{ (\Xi_1, \cdots, \Xi_j) \,\middle|\, 0 \leqq \Xi_i \leqq 1, \; \sum_{i=1}^{j} \Xi_i = 1 \right\},
$$

and unit vectors $n_1, \cdots, n_j$ in $\mathbb{R}^3$, such that the tensor

$$
(5.11) \qquad \mathbf{T}^* = \sum_{i=1}^{j} \Xi_i \mathbf{T}(n_i)
$$

has $\alpha_1, \cdots, \alpha_5$ as eigenvalues.

For the case of the upper bounds we take the trace of (2.12) to find that all laminates made using (2.12) automatically satisfy (1.8) with equality. Moreover the question of attainability of all points on (1.8) by laminates reduces under (5.4) to Problem 5.1.

Remark 5.2. As of this writing the set of points in Δ_α attained by laminates is unknown.

However, in the following we solve an easier problem. We give a complete characterization of the set of laminates of the form (2.11) and (2.12) with cubic symmetry, and describe the set of points in the region Δ_α of Problem 5.1 realized by this class.

Remark 5.3. It is evident from (2.11) and (2.12) that the complete characterization of the set of effective elasticity tensors of laminates reduces to the characterization of the set of all tensors of the form $\mathbf{T}^*$ given by (5.11).

Remark 5.4. For the case of incompressible elasticity the elasticity tensor has the trace $\mathbb{C}_{ijkl} = 0$. Thus it follows from (4.4) that the spectral decomposition of an elasticity tensor for incompressible elasticity invariant under the cubic group O is given by

$$
(5.12) \qquad \mathbb{C} = \lambda_1 \mathbb{D}_{\mathrm{off}} + \lambda_2 \mathbb{D}_{\mathrm{diag}}.
$$

Under the restriction of cubic symmetry the region Δ_α given by (5.7) reduces to $\tilde{\Delta}_{\alpha,\gamma} \subseteq \mathbb{R}^5$

$$
(5.13) \qquad \tilde{\Delta}_{\alpha,\gamma} = ((\alpha_1, \cdots, \alpha_5) \,|\, \alpha_1 = \alpha_2 = \alpha_3 = \alpha, \; \alpha_4 = \alpha_5 = \gamma, \; 3\alpha + 2\gamma = 2, \; 0 \leqq \gamma \leqq 1).
$$

Thus it follows from Remark 4.5 that if we limit ourselves to materials with cubic symmetry, Problem 5.1 becomes Problem 5.6.

PROBLEM 5.6. Attainability of the eigenvalue bounds by laminates with cubic symmetry. Given a point $(\alpha_1, \cdots, \alpha_5)$ in $\tilde{\Delta}_{\alpha,\gamma}$ find an integer $j \geqq 1$, a point $(\Xi_1, \cdots, \Xi_j)$ in the simplex Δ_Ξ, and unit vectors $n_1, \cdots, n_j$ in $\mathbb{R}^3$ such that the tensor $\mathbf{T}^*$ given by (5.11) has cubic symmetry and α and γ as eigenvalues.

The solution to Problem 5.6 is given by Theorem 5.7.

THEOREM 5.7. *All tensors* $\mathbf{T}^*$ *of the form* (5.11) *with cubic symmetry are of the form*

$$
(5.14) \qquad \mathbf{T}^* = \tilde{\alpha} \mathbb{D}_{\mathrm{off}} + \tilde{\gamma} \mathbb{D}_{\mathrm{diag}},
$$

where the eigenvalues $\tilde{\alpha}$ and $\tilde{\gamma}$ lie on the line segment given by

$$(5.15) \qquad 3\tilde{\alpha} + 2\tilde{\gamma} = 2 \quad \text{for} \quad 0 \leqq \tilde{\gamma} \leqq \tfrac{2}{3}.$$

Moreover, there exist unit vectors n_1 to n_j such that there exist tensors $\mathbf{T}^* = \sum_{i=1}^{j} \Xi_i \mathbf{T}(n_i)$ with eigenvalues $\tilde{\alpha}$, $\tilde{\gamma}$ satisfying (5.15) for every $\tilde{\gamma}$ in $0 \leqq \tilde{\gamma} \leqq \tfrac{2}{3}$. Finally, it is evident from (5.15) and (5.13) that Problem 5.6 has no solution for points in $\tilde{\Delta}_{\alpha,\gamma}$ for which $\tfrac{2}{3} < \tilde{\gamma} \leqq 1$.

It follows immediately from Theorem 5.7 that the complete characterization of incompressible elastic laminates with cubic symmetry is given by Corollary 5.8.

COROLLARY 5.8. *The set of all effective tensors $\mathbb{C}^*$ for finite rank incompressible laminates with cubic symmetry made using isotropic core material 2 and isotropic layer material 1 as in (2.11) is the set of tensors $\mathbb{C}^*$ given by*

$$(\mathbb{C}^* - \mu_1 \tilde{\mathbb{I}})^{-1} = \left(\frac{1}{\theta_2(\mu_2 - \mu_1)} + \frac{\theta_1}{2\mu_1\theta_2} \tilde{\alpha} \right) \mathbb{D}_{\text{off}} + \left(\frac{1}{\theta_2(\mu_2 - \mu_1)} + \frac{\theta_1}{2\mu_1\theta_2} \tilde{\gamma} \right) \mathbb{D}_{\text{diag}}$$

where $(\tilde{\gamma}, \tilde{\alpha})$ is any point on the line segment given by (5.15).

Remark 5.9. An analogous statement holds for finite rank laminates made using material 1 as the core and layering with material 2.

Proof of Theorem 5.7. We start by showing that the set of tensors $\mathbf{T}^*$ with cubic symmetry is not empty. It follows directly from Lemma 4.7 that for the choice of $\Xi_j = \tfrac{1}{24}$, $n_i = Q_i v$, $1 \leqq i \leqq 24$, $Q_i \in O$ in (5.11) we obtain

$$\mathbf{T}^* = \mathbf{T}(v)_O.$$

Thus the set of tensors $\mathbf{T}^*$ with cubic symmetry is not empty. From Remark 5.4 we have $\mathbf{T}(v)_O$ is of the form

$$(5.16) \qquad \mathbf{T}(v)_O = \alpha(v)\mathbb{D}_{\text{off}} + \gamma(v)\mathbb{D}_{\text{diag}},$$

where $\alpha(v)$ and $\gamma(v)$ are the eigenvalues of $\mathbf{T}(v)_O$. In general, if $\mathbf{T}^*$ has cubic symmetry we have

$$(5.17) \qquad \mathbf{T}^* = \tilde{\alpha}\mathbb{D}_{\text{off}} + \tilde{\gamma}\mathbb{D}_{\text{diag}},$$

where $\tilde{\alpha}$ is an eigenvalue of multiplicity 3 and $\tilde{\gamma}$ is of multiplicity 2. We appeal to Remark 3.6 and take the trace of (5.17) to obtain

$$(5.18) \qquad 2 = 3\tilde{\alpha} + 2\tilde{\gamma}.$$

To complete the proof we obtain bounds on $\tilde{\gamma}$. First we note that any convex combination of tensors of the form $\mathbf{T}(v)_O$ is also a tensor of the form $\mathbf{T}^*$.

To obtain tight bounds on $\tilde{\gamma}$ we show that any tensor $\mathbf{T}^*$ with cubic symmetry is equivalent to a convex combination of tensors $\mathbf{T}(v)_O$.

LEMMA 5.10. *If a tensor $\mathbf{T}^*$ is of the form (5.11) with cubic symmetry and spectrum as in (5.17) then $\mathbf{T}^*$ has the same spectral decomposition as $\mathbf{T}_O^*$, i.e.,*

$$(5.19) \qquad \mathbf{T}_O^* = \sum_{i=1}^{j} \Xi_i \mathbf{T}(n_i)_O = \tilde{\alpha}\mathbb{D}_{\text{off}} + \tilde{\gamma}\mathbb{D}_{\text{diag}}.$$

From (5.19) and (5.16) we observe that

$$(5.20) \qquad \tilde{\alpha} = \sum_{i=1}^{j} \Xi_i \alpha(n_i),$$

and

$$(5.21) \qquad \tilde{\gamma} = \sum_{i=1}^{j} \Xi_i \gamma(n_i).$$

Therefore we obtain bounds on $\tilde{\gamma}$ by finding the extrema of $\gamma(v)$ over all unit vectors v in $\mathbb{R}^3$. We choose the trace-free 3×3 diagonal matrix $M = I - 3e_3 \otimes e_3$ and use (5.16) to write

$$(5.22) \qquad \mathbf{T}(v)_O M : M = \gamma(v) \mathbb{D}_{\text{diag}} M : M.$$

Without loss of generality we consider the projection on diagonal trace-free matrices given by

$$(5.23) \qquad \mathbb{D}_{\text{diag}} = \begin{pmatrix} 1/\sqrt{2} & & \\ & -1/\sqrt{2} & \\ & & 0 \end{pmatrix} \otimes \begin{pmatrix} 1/\sqrt{2} & & \\ & -1/\sqrt{2} & \\ & & 0 \end{pmatrix}$$
$$+ \begin{pmatrix} 1/\sqrt{6} & & \\ & 1/\sqrt{6} & \\ & & -2/\sqrt{6} \end{pmatrix} \otimes \begin{pmatrix} 1/\sqrt{6} & & \\ & 1/\sqrt{6} & \\ & & -2/\sqrt{6} \end{pmatrix}$$

and obtain

$$(5.24) \qquad \mathbb{D}_{\text{diag}} M : M = 6.$$

We expand the left-hand side of (5.22) noting that e_3 is a face normal to the cube centered at the origin with face normals e_1, e_2, e_3 and obtain

$$(5.25) \qquad \begin{aligned} \mathbf{T}(v)_O M : M &= \frac{1}{3} \sum_{i=1}^{3} \mathbf{T}(v)(I - 3e_i \otimes e_i) : (I - 3e_i \otimes e_i) \\ &= 12\{v_1^2 v_2^2 + v_1^2 v_3^2 + v_2^2 v_3^2\}. \end{aligned}$$

Equations (5.22), (5.24), and (5.25) yield

$$(5.26) \qquad \gamma(v) = 2\{v_1^2 v_2^2 + v_1^2 v_3^2 + v_2^2 v_3^2\}.$$

An easy calculation shows that the range of $\gamma(v)$ is the closed interval $[0, \frac{2}{3}]$ with

$$(5.27) \qquad \gamma(v) = \frac{2}{3} \quad \text{for} \quad v = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)^T$$

and

$$(5.28) \qquad \gamma(v) = 0 \quad \text{for} \quad v = e_i, \quad i = 1, 2, 3.$$

It is evident from (5.21) that $0 \leq \tilde{\gamma} \leq \frac{2}{3}$.

Moreover, let $\tilde{\alpha} = \alpha(v)$ and $\tilde{\gamma} = \gamma(v)$, where $\alpha(v)$ and $\gamma(v)$ are the eigenvalues of the cubically symmetric tensor given by (5.16). It is easily seen from (5.27), (5.28), and the continuity of $\gamma(v)$ that all materials with eigenvalues satisfying (5.15) for every $\tilde{\gamma}$ in $0 \leq \tilde{\gamma} \leq \frac{2}{3}$ can be constructed.

5.B. The complete characterization of the set of compressible elastic laminates with cubic symmetry. In this section we characterize the set of effective elasticity tensors for compressible elastic laminates given by (2.8) and (2.9) with cubic symmetry. It is evident from (2.8) and (2.9) that the problem of describing the set of effective elasticity tensors obtained from laminates reduces to characterizing the set of tensors of the form

$$(5.29) \qquad \overset{*}{\mathbf{T}}{}^l = \sum_{i=1}^{j} \Xi_i \mathbf{T}^l(n_i), \quad l = 1, 2$$

with cubic symmetry.

The set of all tensors of the form (5.29) with cubic symmetry is given by Theorem 5.11.

THEOREM 5.11. *All tensors* $\overset{*}{\mathbf{T}}{}^{l}$ *of the form* (5.29) *with cubic symmetry are of the form*

$$(5.30) \qquad \overset{*}{\mathbf{T}}{}^{l} = \tilde{\alpha}_{l}\mathbb{D}_{\text{off}} + \tilde{\gamma}_{l}\mathbb{D}_{\text{diag}} + \tilde{\delta}_{l}\mathbb{D}_{I}$$

where

$$(5.31) \qquad \tilde{\delta}^{l} = \frac{1}{3\kappa_{l} + 4\mu_{l}}, \quad l = 1, 2$$

and the eigenvalues $\tilde{\alpha}_{l}$ *and* $\tilde{\gamma}_{l}$ *lie on the line segment given by*

$$(5.32) \qquad 3\tilde{\alpha}_{l} + 2\tilde{\gamma}_{l} = \frac{2}{3\kappa_{l} + 4\mu_{l}} + \frac{1}{\mu_{l}}$$

for

$$(5.33) \qquad \frac{1}{3\kappa_{l} + 4\mu_{l}} \leqq \tilde{\gamma}_{l} \leqq \frac{1}{3\mu_{l}}, \quad l = 1, 2.$$

Moreover, there exist normals n_{1} *to* n_{j} *such that there exist tensors* $\overset{*}{\mathbf{T}}{}^{l} = \sum_{i=1}^{j} \Xi_{i}\mathbf{T}^{l}(n_{i})$ *with eigenvalues* $\tilde{\alpha}_{l}$, $\tilde{\gamma}_{l}$, $\tilde{\delta}_{l}$ *satisfying* (5.31) *and* (5.32) *for every* $\tilde{\gamma}_{l}$ *in the interval* (5.33).

It follows directly from Theorem 5.11 that the complete characterization of the set of effective tensors for compressible elastic laminates with cubic symmetry given by (2.8) and (2.9) is shown in Corollary 5.12.

COROLLARY 5.12. *The set of all effective tensors* $\mathbb{C}^{*}$ *for finite rank compressible elastic laminates with cubic symmetry constructed using isotropic core material 2 and isotropic layer material 1 as in* (2.8) *is the set of tensors given by*

$$(5.34) \qquad \begin{aligned} (\mathbb{C}^{*} - \mathbb{C}_{1})^{-1} &= \left(\frac{1}{\theta_{2}(\mu_{2} - \mu_{1})} + \frac{\theta_{1}}{\theta_{2}} \tilde{\alpha}_{1} \right) \mathbb{D}_{\text{off}} \\ &+ \left(\frac{1}{2\theta_{2}(\mu_{2} - \mu_{1})} + \frac{\theta_{1}}{\theta_{2}} \tilde{\gamma}_{1} \right) \mathbb{D}_{\text{diag}} \\ &+ \left(\frac{1}{3\theta_{2}(\kappa_{2} - \kappa_{1})} + \frac{\theta_{1}}{\theta_{2}} \left(\frac{1}{3\kappa_{1} + 4\mu_{1}} \right) \right) \mathbb{D}_{I} \end{aligned}$$

where $(\tilde{\alpha}_{1}, \tilde{\gamma}_{1})$ *is any point on the line segment given by* (5.32) *and* (5.33) *for* $l = 1$. *And the set of all effective elasticity tensors for finite rank compressible elastic laminates with cubic symmetry made using material 1 as the core and material 2 for the layers as in* (2.9) *is given by*

$$(5.35) \qquad \begin{aligned} (\mathbb{C}_{2} - \mathbb{C}^{*})^{-1} &= \left(\frac{1}{2\theta_{1}(\mu_{2} - \mu_{1})} - \frac{\theta_{2}}{\theta_{1}} \tilde{\alpha}_{2} \right) \mathbb{D}_{\text{off}} \\ &= \left(\frac{1}{2\theta_{1}(\mu_{2} - \mu_{1})} - \frac{\theta_{2}}{\theta_{1}} \tilde{\gamma}_{2} \right) \mathbb{D}_{\text{diag}} \\ &= \left(\frac{1}{3\theta_{1}(\kappa_{2} - \kappa_{1})} - \frac{\theta_{2}}{\theta_{1}} \left(\frac{1}{3\kappa_{2} + 4\mu_{2}} \right) \right) \mathbb{D}_{I} \end{aligned}$$

where $(\tilde{\alpha}_{2}, \tilde{\gamma}_{2})$ *is any point on the line segment given by* (5.32) *and* (5.33) *for* $l = 2$.

Proof of Theorem 5.11. The proof of Theorem 5.11 follows the proof of Theorem 5.7, so we provide only a sketch of the proof. Arguing as before, we have that the set of tensors $\overset{*}{\mathbf{T}}{}^{l}$ with cubic symmetry is not empty and

$$(5.36) \qquad \overset{*}{\mathbf{T}}{}^{l} = \mathbf{T}^{l}(v)_{O}$$

for the choice $\Xi_j = \frac{1}{24}$, $n_i = Q_i v$, $1 \leqq i \leqq 24$, $Q_i \in O$ in (5.29). In general, when $\overset{*}{\mathbf{T}}{}^l$ has cubic symmetry it follows from Remark 5.4 that $\mathbf{T}^l$ is of the form

$$(5.37) \qquad \overset{*}{\mathbf{T}}{}^l = \tilde{\alpha}_l \mathbb{D}_{\text{off}} + \tilde{\gamma}_l \mathbb{D}_{\text{diag}} + \tilde{\delta}l \mathbb{D}_I.$$

Taking the trace of (5.37) and appealing to Remark 3.6 we obtain

$$(5.38) \qquad \frac{3}{3\kappa_l + 4\mu_l} + \frac{1}{\mu_l} = 3\tilde{\alpha}_l + 2\tilde{\gamma}_l + \tilde{\delta}_l.$$

A straightforward calculation shows that

$$\overset{*}{\mathbf{T}}{}^l I : I = \frac{3}{3\kappa_l + 4\mu_l};$$

therefore from (5.37) we obtain

$$(5.39) \qquad \tilde{\delta}_l = \frac{1}{3\kappa_l + 4\mu_l}$$

and (5.38) becomes

$$(5.40) \qquad 3\tilde{\alpha}_l + 2\tilde{\gamma}_l = \frac{2}{3\kappa_l + 4\mu_l} + \frac{1}{\mu_l}.$$

To conclude the proof we obtain bounds on $\tilde{\gamma}_l$. In analogy to the incompressible case we have that any tensor $\overset{*}{\mathbf{T}}{}^l$ with cubic symmetry can be written as a convex combination of tensors $\mathbf{T}^l(v)_O$. Indeed

$$\overset{*}{\mathbf{T}}{}^l = \overset{*}{\mathbf{T}}{}^l_O = \sum_{i=1}^{j} \Xi_i \mathbf{T}^l(n_i)_O.$$

It follows that bounds on $\tilde{\gamma}_l$ are obtained by finding the extrema of $\tilde{\gamma}_l(v)$ over all unit vectors v in $\mathbb{R}$ where $\tilde{\gamma}_l(v)$ and $\tilde{\alpha}_l(v)$ are eigenvalues of

$$(5.41) \qquad \mathbf{T}^l(v)_O = \tilde{\alpha}_l(v) \mathbb{D}_{\text{off}} + \tilde{\gamma}_l(v) \mathbb{D}_{\text{diag}} + \tilde{\delta}_l \mathbb{D}_I.$$

We choose the trace-free 3×3 diagonal matrix $M = I - 3e_3 \otimes e_3$ and compute $\mathbf{T}^l(v)_O M : M$ to obtain

$$(5.42) \qquad \tilde{\gamma}_l(v) = \frac{1}{3\kappa_l + 4\mu_l} + \left(\frac{1}{\mu_l} - \frac{3}{3\kappa_l + 4\mu_l} \right)(v_1^2 v_2^2 + v_1^2 v_3^2 + v_2^2 v_3^2)$$

for $l = 1, 2$. Lastly, a straightforward computation shows that

$$(5.43) \qquad \frac{1}{3\kappa_l + 4\mu_l} \leqq \tilde{\gamma}_l(v) \leqq \frac{1}{3\mu_l}, \qquad l = 1, 2$$

for all unit vectors in $\mathbb{R}^3$. The lower bound in (5.43) is attained if v is parallel to any coordinate axis and the upper bound is attained for

$$v = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right)^T.$$

Attainability of every point in (5.43) by $\tilde{\gamma}(v)$ follows from the continuity of $\tilde{\gamma}(v)$.

5.C. Optimal bounds correlating the shear moduli for elastic two-phase composites with cubic symmetry. The effective elasticity of a cubically symmetric composite is written

$$\mathbb{C} = 2\mu'' \mathbb{D}_{\text{off}} + 2\mu' \mathbb{D}_{\text{diag}} + 3\kappa \mathbb{D}_I$$

where μ'' and μ' are the Lamé shear moduli and κ is the bulk modulus. Bounds on the bulk modulus κ were derived by Hashin and Shtrikman [8] for isotropic composites and, subsequently, Kantor and Bergman [9] observed that these bounds apply to the quantity $\kappa = \frac{1}{9}\mathbb{C}I : I$ for composites without any symmetry. The bounds are given by

$$(5.44) \qquad \kappa_L = \kappa_1 + \frac{\theta_2}{(\kappa_2 - \kappa_1)^{-1} + 3\theta_1(3\kappa_1 + 4\mu_1)^{-1}}$$

$$(5.45) \qquad \kappa_U = \kappa_2 + \frac{\theta_1}{(\kappa_1 - \kappa_2)^{-1} + 3\theta_2(3\kappa_2 + 4\mu_2)^{-1}}$$

and

$$\kappa_L \leqq \kappa \leqq \kappa_U.$$

It is immediate that all laminates given by (5.34) have bulk moduli equal to κ_L. Similarly all laminates given by (5.35) have bulk moduli given by κ_U. We take μ'', μ', and κ as three coordinate axes and observe that the set of laminates given by (5.34) and (5.35) are pieces of curve on the $\kappa = \kappa_L$ and $\kappa = \kappa_U$ planes, respectively (see Fig. 2). Taking the trace of (5.34) it is easily seen that the curve segment on the $\kappa = \kappa_L$ plane is given by

$$(5.46) \qquad \frac{3}{2\mu'' - 2\mu_1} + \frac{2}{2\mu' - 2\mu_1} = \frac{5}{2\theta_2(\mu_2 - \mu_1)} + \frac{\theta_1}{\theta_2}\left\{\frac{2}{3\kappa_1 + 4\mu_1} + \frac{1}{\mu_1}\right\}$$

for μ' in the interval

$$(5.47) \qquad \frac{1}{\mu_2 - \mu_1} + \frac{2\theta_1}{3\kappa_1 + 4\mu_1} \leqq \frac{\theta_2}{\mu' - \mu_1} \leqq \frac{1}{\mu_2 - \mu_1} + \frac{2\theta_1}{3\mu_1}.$$

Similarly the curve segment on the $\kappa = \kappa_U$ plane corresponding to (5.35) is

$$(5.48) \qquad \frac{3}{2\mu_2 - 2\mu''} + \frac{2}{2\mu_2 - 2\mu'} = \frac{5}{2\theta_1(\mu_2 - \mu_1)} - \frac{\theta_2}{\theta_1}\left(\frac{2}{3\kappa_2 + 4\mu_2} + \frac{1}{\mu_2}\right)$$
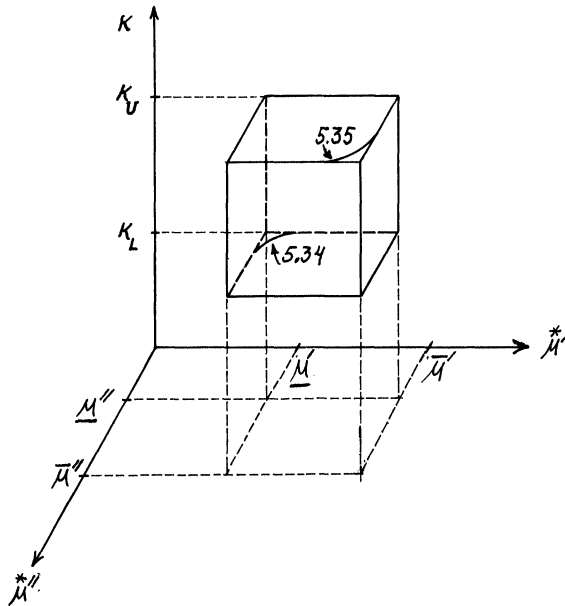

FIG. 2. *Representation of set of cubic laminates in κ, μ', μ'' space.*

for μ' in the interval

$$(5.49) \qquad \frac{1}{\mu_2 - \mu_1} - 2\theta_2 \left(\frac{2}{3} \frac{1}{\mu_2} - \frac{1}{3\kappa_2 + 4\mu_2} \right) \leq \frac{\theta_1}{\mu_2 - \mu'} \leq \frac{1}{\mu_2 - \mu_1} - \frac{2\theta_2}{3\kappa_2 + 4\mu_2}.$$

Recently Avellaneda [1] obtained upper and lower bounds on the effective shear moduli μ', μ'' for cubic composites; they are given by

$$(5.50) \qquad \underline{\mu}'' \leq \overset{*}{\mu}'' \leq \bar{\mu}''$$

where

$$\bar{\mu}'' = \mu_2 - \theta_1 \left(\frac{1}{\mu_1 - \mu_2} - \theta_2 \frac{6\kappa_2 + 20\mu_2}{9\mu_2(3\kappa_2 + 4\mu_2)} \right)^{-1}$$

$$\underline{\mu}'' = \mu_1 + \theta_2 \left(\frac{1}{\mu_1 - \mu_2} + \frac{2\theta_1}{3\mu_1} \right)^{-1}$$

and

$$(5.51) \qquad \underline{\mu}' \leq \overset{*}{\mu}' \leq \bar{\mu}'$$

where

$$\bar{\mu}' = \mu_2 - \theta_1 \left(\frac{1}{\mu_2 - \mu_1} - \frac{2\theta_2}{(3\kappa_2 + 4\mu_2)} \right)^{-1}$$

$$\underline{\mu}' = \mu_1 + \theta_2 \left(\frac{1}{\mu_2 - \mu_1} + \frac{2\theta_1}{3\mu_1} \right)^{-1}.$$

In [1] it was shown that there exist cubic laminates which attain the end points of our curves (5.46) and (5.48). Also, Avellaneda [1] has shown that finite rank laminates are in a sense extremal in the set of all composites. Indeed, given any effective elasticity $\mathbb{C}^*$, with crystallographic symmetry $\mathscr{G}$ made from isotropic components κ_i, μ_i ($i = 1, 2$) in the volume fractions θ_i ($i = 1, 2$), there then exist finite rank laminates $\mathbb{C}_L$, $\mathbb{C}_U$, made from the same components and volume fractions in the symmetry class $\mathscr{G}'$, where $\mathscr{G} \subseteq \mathscr{G}' \subseteq O_3$ such that

$$(5.52) \qquad \mathbb{C}_L \leq \overset{*}{\mathbb{C}} \leq \mathbb{C}_U.$$

We note that our analysis shows that $\mathscr{G}' = \mathscr{G}$ for the cubic case. It follows immediately from (5.52) that our curves (5.46) and (5.48) yield new bounds on the shear moduli. We summarize our results in the following theorem.

THEOREM 5.13. *All composites with cubic symmetry have bulk moduli and shear moduli lying inside the region given by*

$$\kappa_L \leq \overset{*}{\kappa} \leq \kappa_U$$

where for all $\overset{*}{\kappa}$ in $[\kappa_L, \kappa_U]$ the shear moduli $\overset{*}{\mu}'$, $\overset{*}{\mu}''$ lie in the intersection of

$$\underline{\mu}'' \leq \overset{*}{\mu}'' \leq \bar{\mu}''$$

$$\underline{\mu}' \leq \overset{*}{\mu}' \leq \bar{\mu}'$$

and

$$(5.53) \qquad \frac{3}{2\overset{*}{\mu}'' - 2\mu_1} + \frac{2}{2\overset{*}{\mu}' - 2\mu_1} \leq \frac{5}{2\theta_2(\mu_2 - \mu_1)} + \frac{\theta_1}{\theta_2} \left(\frac{2}{3\kappa_1 + 4\mu_1} + \frac{1}{\mu_1} \right)$$

$$(5.54) \qquad \frac{3}{2\mu_2 - 2\overset{*}{\mu}''} + \frac{2}{2\mu_2 - 2\overset{*}{\mu}'} \leqq \frac{5}{2\theta_1(\mu_2 - \mu_1)} - \frac{\theta_2}{\theta_1}\left(\frac{2}{3\kappa_2 + 4\mu_2} + \frac{1}{\mu_2}\right).$$

(*See Fig.* 3.)

Remark 5.14. We see that (5.53) and (5.54) give bounds that correlate the two shear moduli.

Remark 5.15. The endpoints of the curves (5.46) and (5.48) lie on the bounds given by (5.50) and (5.51) (see Fig. 3).

6. Isotropic elastic laminates and optimal bounds. In their elegant paper, Francfort and Murat [5] construct a laminated material with layer directions specified by the six northern hemisphere vertices of regular icosahedron. If we then compute each component of the effective tensor, relative to a basis in S_3, it is found after a tedious computation that the resulting tensor is isotropic. This procedure yields a pair of isotropic materials that attain the Hashin–Shtrikman upper and lower bounds [5] on the shear and bulk moduli.

In the first part of this section we present a simple proof of isotropy for the laminates introduced by Francfort and Murat. Our proof is based on the observation that the formula of Francfort and Murat is really a group average over the icosahedral group. In the second part we characterize the set of effective shear moduli μ^* for isotropic homogenized composites of two incompressible elastic materials taken in prescribed proportion.

THEOREM 6.1. *Elastic compressible or incompressible rank-6 laminates obtained from equations (2.8), (2.9) or (2.11), (2.12) by choosing layer directions n_i, $1 \leqq i \leqq 6$ associated with the six northern hemisphere vertices of the regular icosahedron have isotropic effective elasticity tensors.*

Proof. It follows from Remark 4.5 that the effective tensors defined by (2.8), (2.9), (2.11), (2.12) are isotropic if and only if the sum of degenerate compliance tensors appearing in those formulas is isotropic. To fix ideas we will construct an isotropic tensor of the form $\sum_{i=1}^{j} \Xi_j \mathbf{T}^2(n_i)$ and note that our method applies to convex combinations of the tensors $\mathbf{T}(n_i)$ and $\mathbf{T}^1(n_i)$ as well.

Consider a regular icosahedron oriented so that one of its twelve vertices is aligned in the direction of the north pole. For any rotation Q in the group Y of rotations that leave the icosahedron invariant, we observe that Q acting on any vertex rotates it into
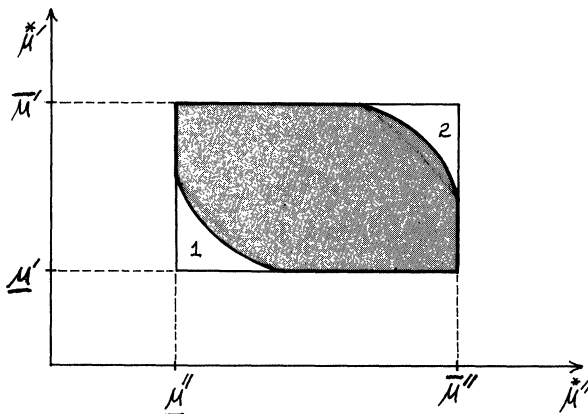

FIG. 3. *The set of all shear moduli for cubically symmetric 2-phase composites lie in the shaded region. Curve 1 is given by (5.46) and curve 2 is given by (5.48).*

another vertex of the icosahedron. We choose $v_1, v_2, \cdots, v_{12}$ to be the twelve vertices of the regular icosahedron and form

$$(6.1) \qquad \overset{*}{\mathbf{T}}^2 = \sum_{i=1}^{12} \frac{1}{12} \mathbf{T}^2(v_i).$$

From (3.3), (3.4), (3.5), (3.6), and (2.10) we see for any rotation Q that

$$(6.2) \qquad \mathbf{T}^2(v_i) QQQQ = \mathbf{T}^2(Qv_i)$$

and

$$(6.3) \qquad \mathbf{T}^2(-v_i) = \mathbf{T}^2(v_i).$$

Hence we observe that multiplying (6.1) by any rotation $QQQQ$ for Q in Y simply permutes the indices in the sum (6.1). Therefore $\overset{*}{\mathbf{T}}^2$ is invariant under the icosahedral group and is isotropic by Lemma 4.3. Lastly we observe that all vertices in the northern hemisphere are antipodal to vertices in the southern hemisphere, so from (6.3), $\overset{*}{\mathbf{T}}^2$ reduces to

$$\overset{*}{\mathbf{T}}^2 = \sum_{i=1}^{6} \frac{1}{6} \mathbf{T}(v_i)$$

where v_i, $i = 1, \cdots, 6$, are the northern hemisphere vertices of the icosahedron. $\qquad \square$

It is now possible to characterize the set of effective shear moduli for isotropic homogenized mixtures of incompressible elastic materials taken in prescribed proportions in three dimensions.

THEOREM 6.2. *The set of all effective shear moduli μ^* for isotropic, homogenized mixtures of two isotropic, incompressible elastic components with moduli $\mu_1 < \mu_2$ taken in the proportions θ_1 and θ_2 is the closed interval $[\mu_l, \mu_u]$ where*

$$(6.4) \qquad \mu_l = \mu_1 + \theta_2 \left(\frac{1}{\mu_2 - \mu_1} + \frac{2\theta_2}{5\mu_1} \right)^{-1}$$

and

$$(6.5) \qquad \mu_u = \mu_2 + \theta_1 \left(\frac{1}{\mu_1 - \mu_2} + \frac{2\theta_2}{5\mu_2} \right)^{-1}.$$

Proof. We begin by constructing extremal materials with shear moduli that attain the bounds μ_l and μ_u.

A straightforward calculation shows that the isotropic incompressible laminates obtained from equations (2.11) and (2.12) using layer directions and volume fractions given in Theorem 6.1 have shear moduli μ_l and μ_u, respectively. To construct laminates with effective shear moduli μ^* between μ_l and μ_u, we refer to Corollary 4.5 of [11] and form a finite rank laminate with core μ_u and layers μ_l with the overalll volume fraction of μ_l given by p. The resulting tensor $\mathbb{C}^*$,

$$(6.6) \qquad (\mathbb{C}^* - 2\mu_l)^{-1} = \frac{(\mu_u - \mu_l)^{-1}}{2(1-p)} \tilde{\mathbb{I}} + \frac{p}{2\mu_l} \sum_{i=1}^{j} \Xi_i \mathbf{T}(n_i)$$

is the effective elasticity for a mixture with material 1 and 2 in proportions θ_1 and θ_2.

Upon choosing the convex combination $\sum_{i=1}^{j} \Xi_i \mathbf{T}(n_i)$ exactly as in Theorem 6.1, (6.6) yields

$$(6.7) \qquad (\mathbb{C}^* - 2\mu_l)^{-1} = \frac{(\mu_u - \mu_l)^{-1}}{2(1-p)} \tilde{\mathbb{I}} + \frac{p}{5\mu_l(1-p)} \tilde{\mathbb{I}}$$

where $\mathbb{C}^* = 2\mu^*\tilde{\mathbb{I}}$ and

$$(6.8) \qquad 2\mu^* = 2\mu_l + (1-p)\left(\frac{1}{2(\mu_u - \mu_l)} + \frac{5p}{2\mu_l}\right)^{-1}.$$

Thus as p ranges through $[0, 1]$ the effective shear modulus μ^* sweeps out the interval $[\mu_l, \mu_u]$.

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