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A characterization of plane strain

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A new characterization of plane strain is given that holds even under weak regularity hypotheses on the deformation.

1. Introduction

In this paper we present a new characterization of plane strain that holds even under weak regularity hypotheses.

Let $\Omega \subset \mathbb{R}^n$, $n > 1$, be a bounded, connected, strongly Lipschitz, open set. A mapping $y \in W^{1,1}(\Omega; \mathbb{R}^n)$ is a *plane strain* (with respect to x_n) if it has the form

$$y(x) = Q(z_1(x), \ldots, z_{n-1}(x), \lambda x_n + \mu), \quad \text{a.e. } x \in \Omega, \tag{1.1}$$

where $Q \in \mathrm{SO}(n)$, $\lambda, \mu \in \mathbb{R}$, and $z_{i,n} = 0$ for $1 \leqslant i \leqslant n-1$. Thus $z_1, \ldots, z_{n-1}$ have representatives that are functions only of $x_1, \ldots, x_{n-1}$ in any open subset of Ω which is convex in the x_n direction. In a plane strain, apart from the rotation Q, the planes perpendicular to the x_n direction all experience the same plane deformations, while the lines in the x_n direction experience a given pure stretch in that direction.

If y is a plane strain then

$$Dy(x) = Q\begin{bmatrix} z_{1,1} & \cdots & z_{1,n-1} & 0 \\ \vdots & & \vdots & \vdots \\ z_{n-1,1} & \cdots & z_{n-1,n-1} & 0 \\ 0 & \cdots & 0 & \lambda \end{bmatrix}, \tag{1.2}$$

and so

$$Dy(x)^{\mathrm{T}} Dy(x)\, e_n = \lambda^2 e_n, \quad \text{a.e. } x \in \Omega. \tag{1.3}$$

Furthermore, we have formally that

$$\frac{\partial}{\partial x_n} \det Dy(x) = 0. \tag{1.4}$$

The aim of this paper is to show that the necessary conditions (1.3), (1.4) are also *sufficient* for y to be a plane strain. We consider the case $\det Dy(x) > 0$ which is of interest in continuum mechanics. In fact the example

$$y(x_1, x_2) = (0, \theta(x_2)), \quad \theta'(x_2) = \operatorname{sgn}(x_2) \tag{1.5}$$

shows that if $\det Dy(x) = 0$ then conditions (1.3), (1.4) are not sufficient for y to be a plane strain. With $\det Dy(x) > 0$ both (1.3) and (1.4) are restrictions on the strain matrix $Dy(x)^{\mathrm{T}} Dy(x)$. Condition (1.3) says that there exists a constant principal axis of strain e_n with a corresponding constant principal stretch, while (1.4) says that the specific volume is locally independent of x_n.

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The need for such a result arose in a study of the macroscopic deformations that can be obtained by compatibly mixing two variants of martensite (the *two-well* problem). In this case the variants are specified by the 3×3 matrices

$$\mathrm{SO}(3)\,S^{+} \cup \mathrm{SO}(3)\,S^{-}, \quad S^{\pm} = 1 \pm \delta e_3 \otimes e_1, \tag{1.6}$$

where $\delta > 0$ is constant, and the macroscopic deformation gradient Dy satisfies (Ball & James 1990)

$$Dy(x)^{\mathrm{T}}\, Dy(x)\, e_2 = e_2, \quad \det Dy(x) = 1, \tag{1.7}$$

as well as certain other constraints. It follows from our result that y is a plane strain with respect to x_2.

2. Invertibility

We make the following hypotheses:

(A 1) $y \in W^{1,p}(\Omega;\mathbb{R}^n)$ for some $p > n$,

(A 2) $\det Dy(x) > 0$ for a.e. $x \in \Omega$, and

$$y|_{\partial\Omega} = y_0|_{\partial\Omega} \tag{2.1}$$

for some mapping $y_0 \in C(\bar{\Omega};\mathbb{R}^n)$ which is $1-1$ in Ω.

In (2.1) and below we choose the representative of y that is Hölder continuous in $\bar{\Omega}$; this representative exists thanks to (A 1) and Morrey (1966, p. 83). The assumption (A 2) is made to ensure that y is invertible almost everywhere; in fact we have the following result.

Theorem 2.1. *If* (A 1), (A 2) *hold, then*

(i) $y(\bar{\Omega}) = y_0(\bar{\Omega}) = \overline{y_0(\Omega)}$;

(ii) *y maps measurable sets in $\bar{\Omega}$ to measurable sets in $\overline{y_0(\Omega)}$, and the change of variables formula*

$$\int_A f(y(x)) \det Dy(x)\, \mathrm{d}x = \int_{y(A)} f(v)\, \mathrm{d}v \tag{2.2}$$

holds for any measurable subset $A \subset \bar{\Omega}$ and any measurable function $f:\mathbb{R}^n \to \mathbb{R}$, provided only that one of the integrals in (2.2) *exists*;

(iii) *y is $1-1$ almost everywhere, that is the sets*

$$S = \{v \in \overline{y_0(\Omega)} : y^{-1}(v) \text{ contains more than one point}\}$$

and $y^{-1}(S)$ are of measure zero;

(iv) *If $v \in y_0(\Omega)$ then $y^{-1}(v)$ is a continuum contained in Ω, while if $v \in \partial y_0(\Omega)$ then each connected component of $y^{-1}(v)$ intersects $\partial\Omega$*;

(v) *Let $\tilde{x} : \overline{y_0(\Omega)} \to \bar{\Omega}$ with $\tilde{x}(v) \in y^{-1}(v)$ for each $v \in \overline{y_0(\Omega)}$. Then*

$$\tilde{x}(y(x)) = x \quad \textit{for } \text{a.e. } x \in \bar{\Omega}, \tag{2.3}$$

and $\tilde{x} \in W^{1,1}(y_0(\Omega);\mathbb{R}^n)$ *with* $D\tilde{x}(v) = Dy(\tilde{x}(v))^{-1}$ a.e. $v \in y_0(\Omega)$.

Proof. Parts (i)–(iv) are the statement of Ball (1981, Theorem 1); the only extra remark is that $\operatorname{meas} y^{-1}(S) = 0$, which follows from (2.2) with $f = 1$ and $A = y^{-1}(S)$.

Part (v) is essentially due to Šverák (1988, Theorem 8); for the reader's convenience we give a proof based on his ideas. We first note that, since y maps measurable sets to measurable sets, $\tilde{x}$ is measurable. Hence, by (iii), $Dy(\tilde{x}(\cdot))$ is also measurable. Also, (2.3) follows from (iii). Next, since

$$\frac{\partial}{\partial x_\gamma} (\operatorname{adj} Dy)_{\gamma i} = 0,$$

the identity
$$\int_\Omega (\operatorname{adj} Dy)_{\alpha i}\,\phi\,\mathrm{d}x = -\int_\Omega x_\alpha (\operatorname{adj} Dy)_{\gamma i}\frac{\partial\phi}{\partial x_\gamma}\,\mathrm{d}x \tag{2.4}$$
holds for any $y \in C^\infty(\Omega;\mathbb{R}^n)$, $\phi\in C_0^\infty(\Omega)$. Since $p>n$, by approximation (2.4) holds also for $y\in W^{1,p}(\Omega,\mathbb{R}^n)$, $\phi\in W_0^{1,p}(\Omega)$. Now let $\psi\in C_0^\infty(y_0(\Omega))$ and define $\phi=\psi(y(\cdot))$. Since y is continuous on $\bar{\Omega}$, and
$$\frac{\partial\phi(x)}{\partial x_\gamma} = \frac{\partial\psi(y(x))}{\partial y_j}\frac{\partial y_j(x)}{\partial x_\gamma}\text{ a.e.},$$
it is easily proved that $\phi\in W_0^{1,p}(\Omega)$.

Applying (2.2) with $f(v) = |Dy(\tilde{x}(v))^{-1}|$ we deduce using (2.3) that
$$\int_{y_0(\Omega)} |Dy(\tilde{x}(v))^{-1}|\,\mathrm{d}v = \int_\Omega |\operatorname{adj} Dy(x)|\,\mathrm{d}x < \infty. \tag{2.5}$$
In particular, $Dy(\tilde{x}(\cdot))^{-1}\psi(\cdot)$ is integrable over $y_0(\Omega)$. Thus from (2.4) and (2.2)
$$\begin{aligned}\int_{y_0(\Omega)} (Dy(\tilde{x}(v))^{-1})_{\alpha i}\,\psi(v)\,\mathrm{d}v &= -\int_\Omega x_\alpha\frac{\partial\psi}{\partial y_i}(y(x))\det Dy(x)\,\mathrm{d}x\\ &= -\int_{y_0(\Omega)} \tilde{x}_\alpha(v)\frac{\partial\psi}{\partial y_i}(v)\,\mathrm{d}v.\end{aligned}$$
Together with (2.5) this proves (*v*). □

3. Main result

Theorem 3.1. *Assume* (A 1), (A 2). *Then y is a plane strain if and only if* (1.3) *holds for some constant $\lambda\neq 0$ and* (1.4) *holds in Ω in the sense of distributions.*

Proof. Necessity. The calculation leading to (1.2), (1.3) given above is rigorous, while $\det Dy(x)>0$ a.e. implies $\lambda\neq 0$. Let $\phi\in C_0^\infty(\Omega)$. Mollifying $z_1,\dots,z_{n-1}$ we obtain sequences $z_1^{(j)},\dots,z_{n-1}^{(j)}$ of smooth functions converging to $z_1,\dots,z_{n-1}$ respectively in $W^{1,p}(E)$, for some open set E containing $\operatorname{supp}\phi$, and $z_{1,n}^{(j)}=\dots=z_{n-1,n}^{(j)}=0$ in E. Thus
$$\int_\Omega \phi_{,n}\det Dy\,\mathrm{d}x = \lim_{j\to\infty}\int_E \phi_{,n}\,\lambda\frac{\partial(z_1^{(j)},\dots,z_{n-1}^{(j)})}{\partial(x_1,\dots,x_{n-1})}\,\mathrm{d}x = 0,$$
and so (1.4) holds in $\mathscr{D}'(\Omega)$.

Sufficiency. Let $\tilde{x}$ be the inverse of y given by Theorem 2.1 (iii), (v). We first show that $\tilde{x}_n$ is harmonic in $y_0(\Omega)$. Let $\psi\in C_0^\infty(y_0(\Omega))$. Then since $\tilde{x}\in W^{1,1}(y_0(\Omega))$, $\tilde{x}_{n,i}\psi_{,i}\in L^1(y_0(\Omega))$ and from (2.2)
$$\int_{y_0(\Omega)} \tilde{x}_{n,i}(v)\,\psi_{,i}(v)\,\mathrm{d}v = \int_\Omega (Dy(x))^{-1}_{ni}\,\psi_{,i}(y(x))\det Dy(x)\,\mathrm{d}x. \tag{3.1}$$
From (1.3) we have
$$y_{i,n}(x) = \lambda^2 (Dy(x))^{-1}_{ni}\quad\text{for a.e. } x\in\Omega. \tag{3.2}$$
Let $\phi=\psi(y(\cdot))$, so that (cf. the proof of Theorem 2.1) $\phi\in W_0^{1,p}(\Omega)$. From (3.2) we deduce that
$$\lambda^2(Dy(x))^{-1}_{ni}\,\psi_{,i}(y(x)) = \phi_{,n}(x)\text{ a.e.},$$

and so by (3.1), $\phi_{,n} \det Dy \in L^1(\Omega)$ and

$$\lambda^2 \int_{y_0(\Omega)} \tilde{x}_{n,i}(v)\, \psi_{,i}(v)\, \mathrm{d}v = \int_\Omega \phi_{,n}(x) \det Dy(x)\, \mathrm{d}x. \tag{3.3}$$

We now use the following lemmas.

Lemma 3.2. *Let $g \in L^1(\Omega)$, $g_{,n} = 0$ in $\mathscr{D}'(\Omega)$. Then g has a representative which is constant on every open interval in the set*

$$\Omega(x') \overset{\text{def}}{=} \{x_n \in \mathbb{R} : (x', x_n) \in \Omega\}$$

for every $x' \in \mathbb{R}^{n-1}$.

Proof. Mollifying g it is easily shown that for any open cube $Q \subset\subset \Omega$ with edges parallel to the axes there is a representative independent of x_n in Q. The argument is completed by covering

$$\Omega_j \overset{\text{def}}{=} \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > 1/j\}$$

by a finite family of such cubes, for each j. □

Lemma 3.3. *Let $\phi \in W_0^{1,1}(\Omega)$. Then $\phi(x', \cdot) \in W_0^{1,1}(\Omega(x'))$ for* a.e. *$x' \in \mathbb{R}^{n-1}$ with $\Omega(x')$ non-empty.*

Proof. Let $\phi^{(j)} \in C_0^\infty(\Omega)$, $\phi^{(j)} \to \phi$ in $W^{1,1}(\Omega)$. Then

$$\lim_{j\to\infty} \int_\Omega [|\phi^{(j)} - \phi| + |D\phi^{(j)} - D\phi|]\, \mathrm{d}x = \lim_{j\to\infty} \int_{\mathbb{R}^{n-1}} \int_{\Omega(x')} [|\phi^{(j)} - \phi| + |D\phi^{(j)} - D\phi|]\, \mathrm{d}x_n\, \mathrm{d}x' = 0.$$

The proof is completed by choosing a representative of ϕ in $W^{1,1}(\Omega(x'))$ for a.e. x', extracting a subsequence such that the inner integral converges to zero for a.e. x', and noting that $\phi^{(j)}(x', \cdot) \in C_0^\infty(\Omega(x'))$. □

Lemma 3.4. *Let $g \in L^1(\Omega)$, $g_{,n} = 0$ in $\mathscr{D}'(\Omega)$, $\phi \in W_0^{1,1}(\Omega)$ and $\phi_{,n} g \in L^1(\Omega)$. Then*

$$\int_\Omega \phi_{,n}\, g\, \mathrm{d}x = 0.$$

Proof. We have that

$$\int_\Omega \phi_{,n}\, g\, \mathrm{d}x = \int_{\mathbb{R}^{n-1}} \int_{\Omega(x')} \phi_{,n}(x', x_n)\, g(x', x_n)\, \mathrm{d}x_n\, \mathrm{d}x'.$$

Pick x' such that the conclusion of Lemma 3.3 holds. Now $\Omega(x')$ is a union of countably many maximal disjoint open intervals. For any such interval I we have that $\phi(x', \cdot) \in W_0^{1,1}(I)$, while by Lemma 3.2 we have $g(x', x_n)$ independent of x_n. Hence

$$\int_I \phi_{,n}(x', x_n)\, g(x', x_n)\, \mathrm{d}x_n = 0,$$

and the result follows. □

Lemma 3.5. *Let $\Omega' \subset \mathbb{R}^n$ be a bounded domain, u be harmonic in Ω' and $|Du| =$* const. a.e. *in Ω'. Then Du is constant* a.e. *in Ω'.*

Proof. Since u is harmonic, $u \in C^\infty(\Omega')$. The result follows from the identity

$$u_{,ij}u_{,ij} = \tfrac{1}{2}(u_{,i}u_{,i})_{,jj} - u_{,i}u_{,jji}. \qquad \square$$

Continuation of proof of Theorem 3.1. Applying Lemma 3.4 with $g = \det Dy$, we deduce from (1.4), (3.3) that

$$\int_{y_0(\Omega)} \tilde{x}_{n,i}\psi_{,i}\,\mathrm{d}v = 0 \quad \text{for all} \quad \psi \in C_0^\infty(y_0(\Omega)),$$

so that $\tilde{x}_n$ is harmonic in $y_0(\Omega)$. Furthermore, from (3.2) and Theorem 2.1 (iii), (v)

$$\tilde{x}_{n,i}(v)\,\tilde{x}_{n,i}(v) = \lambda^{-2} \quad \text{a.e. } v \in y_0(\Omega).$$

Thus by Lemma 3.5, $D\tilde{x}_n(v) = a$ a.e. in $y_0(\Omega)$ for some $a \in \mathbb{R}^n$ with $|a| = \lambda^{-1}$. Therefore

$$x_n = a \cdot y(x) + k \quad \text{a.e. } x \in \Omega, \tag{3.4}$$

for some constant k.

Let $Q \in \mathrm{SO}(n)$ satisfy $Qe_n = a/|a|$. Then

$$Q^{\mathrm{T}}y(x)\cdot e_n = \lambda(x_n - k) \quad \text{a.e. } x \in \Omega, \tag{3.5}$$

while for $i \neq n$, using (3.2), (3.4),

$$\begin{aligned}(Q^{\mathrm{T}}y(x)\cdot e_i)_{,n} &= Q_{ji}\,y_{j,n}(x) \\ &= Q_{ji}\,\lambda^2\tilde{x}_{n,j}(y(x)) \\ &= \lambda^2 Q_{ji}\,a_j = 0 \quad \text{a.e. } x \in \Omega.\end{aligned}$$

Thus y has the form (1.1) with $\mu = -k\lambda$. $\square$

Remark 3.6. The key point in the proof of sufficiency in Theorem 3.1 is to show that $\tilde{x}_n$ is harmonic. This is reminiscent of the technique of Reshetnyak (1967). For the convenience of those readers content with proving the result for smooth diffeomorphisms y we give a quick proof of this fact. In fact, from (3.2) and

$$\frac{\partial}{\partial y_i}(\operatorname{adj} Dx)_{\mathrm{in}} = 0$$

we deduce that

$$\begin{aligned}\lambda^2\tilde{x}_{n,ii} &= \frac{\partial}{\partial y_i}y_{i,n} \\ &= \frac{\partial}{\partial y_i}[(\operatorname{adj} Dx)_{\mathrm{in}}\det Dy] \\ &= (\operatorname{adj} Dx)_{\mathrm{in}}\frac{\partial}{\partial y_i}(\det Dy) \\ &= \frac{1}{\det Dy}y_{i,n}\frac{\partial}{\partial y_i}(\det Dy) \\ &= (\ln\det Dy)_{,n} = 0,\end{aligned}$$

as required.

Remark 3.7. Theorem 3.1 remains valid if (A 1), (A 2) are replaced by the hypotheses of Šverák (1988, §5), namely that Ω is of class C^∞ and

$$\text{(A 1')} \quad y \in \mathscr{A}_{p,q}(\Omega) \overset{\text{def}}{=} \{z \in W^{1,p}(\Omega;\mathbb{R}^n) : \operatorname{adj} Dz \in L^q(\Omega; M^{n\times n})\}$$

for some $p > n-1$, $q \geqslant p/(p-1)$.

$$\text{(A 2')} \quad \det Dy(x) > 0 \quad \text{for a.e. } x \in \Omega, \text{ and}$$

$$y|_{\partial\Omega} = y_0|_{\partial\Omega}$$

for some $y_0 \in \mathscr{A}_{p,q}(\Omega_0)$, where Ω_0 is a bounded open subset of $\mathbb{R}^n$ containing $\bar{\Omega}$ and y_0 is a homeomorphism of Ω_0 onto $y_0(\Omega_0)$ with $\det Dy_0(x) > 0$ a.e. in Ω_0 and $y_0|_{\partial\Omega} \in \mathscr{A}_{p,q}(\partial\Omega)$. (For the analogous definition of $\mathscr{A}_{p,q}(\partial\Omega)$ see Šverák (1988, p. 109).)

While (A 1′) is weaker than (A 1), (A 2′) is stronger than (A 2). The proof follows the same pattern, with Theorem 2.1 (v) being replaced by Šverák (1988, Theorem 8). In the proof of Šverák (1988, Theorem 8) and to apply Lemma 3.4, it is necessary to show that $\phi = \psi(y(\cdot)) \in W_0^{1,1}(\Omega)$ if $\psi \in C_0^\infty(y_0(\Omega))$. This follows by considering the mapping $\bar{y}: \Omega_0 \to \mathbb{R}^n$ given by

$$\bar{y}(x) = \begin{cases} y(x) & \text{if} \quad x \in \Omega, \\ y_0(x) & \text{if} \quad x \in \Omega_0 \backslash \Omega, \end{cases}$$

and noting that $\bar{\phi} = \psi(\bar{y}(\cdot)) \in W^{1,p}(\Omega_0)$ with $\bar{\phi}(x) = 0$ for a.e. $x \in \Omega_0\backslash\Omega$. If $\phi^{(j)} \in C^\infty(\Omega_0)$ with $\phi^{(j)} \to \bar{\phi}$ in $W^{1,p}(\Omega_0)$ then by trace theory applied to $W^{1,p}(\Omega)$ we have $\phi^{(j)} \to$ trace ϕ in $L^p(\partial\Omega)$. On the other hand, the sequence $\phi^{(j)}$ interlaced with zero converges to zero in $W^{1,p}(\Omega_0\backslash\bar{\Omega})$, so that by trace theory applied to $W^{1,p}(\Omega_0\backslash\bar{\Omega})$ it converges in $L^p(\partial\Omega)$ with limit zero. Hence trace $\phi = 0$ and hence by a standard result $\phi \in W_0^{1,p}(\Omega)$.

Remark 3.8. Ball (1977, p. 399) has remarked that the constraint of *inextensibility in a given direction* is not weakly continuous in $W^{1,p}(\Omega, \mathbb{R}^3)$ for any $p \geqslant 2$. This constraint is written in the form

$$e \cdot Dy(x)^{\mathrm{T}} Dy(x)\, e = 1, \tag{3.6}$$

where $e \in \mathbb{R}$, $|e| = 1$. This can be seen by noting that the left hand side of (3.6) is not a null lagrangian or, more directly, by observing that each member of the sequence $y^{(k)}(x) := k^{-1}y(kx)$, where

$$y(x) = \begin{cases} F^+x, & i < x\cdot e_2 \leqslant i + \frac{1}{2}, \\ F^-x, & i+\frac{1}{2} < x \cdot e_2 \leqslant i+1, \quad i = 1, 2, \ldots, \end{cases} \tag{3.7}$$
$$F^+ := e_1 \otimes e_1 + f^+ \otimes e_2,$$
$$F^- := e_1 \otimes e_1 + f^- \otimes e_2,$$
$$e_1, e_2 \text{ orthonormal},$$
$$|f^-| = 1, f^+ = (-1 + 2e_2 \otimes e_2) f^-$$

satisfies the constraint of inextensibility in the e_2 direction. However, $y^{(k)} \overset{*}{\rightharpoonup} z$ in $W^{1,\infty}(\mathbb{R}^2, \mathbb{R}^2)$ where $z(x) = (e_1 \otimes e_1 + (e_2 \cdot f^-)\, e_2 \otimes e_2)\, x$, and z does not satisfy the constraint of inextensibility in the e_2 direction if $f^- \neq \pm e_2$. On the other hand, the conditions (1.3) and (1.4) for a *plane strain* are weakly continuous in $W^{1,p}(\Omega, \mathbb{R}^n)$ for functions satisfying (A 1) and (A 2). This follows immediately by using Theorem 3.1.

It does not appear possible to obtain this result directly by applying weak continuity results to (1.3) and (1.4). In fact, any such proof would have to use something like $\det Dy^{(k)}(x) > 0$ a.e., as the following example shows. Let

$$y^{(k)}(x_1, x_2) := (0, \theta^{(k)}(x_2)),$$
$$\frac{\mathrm{d}}{\mathrm{d}x_2}\theta^{(k)}(x_2) = \begin{cases} 1, & x_2 > 0, \\ \chi^k(x_2), & x_2 \leqslant 0, \end{cases}$$

with χ^k periodic of period k^{-1} on $\mathbb{R}$ satisfying

$$\chi^k(x) = \begin{cases} +1 & \text{on} \quad [0, 1/2k), \\ -1 & \text{on} \quad [1/2k, 1/k). \end{cases}$$

Then, $y^{(k)}$ satisfies the conditions (1.3) and (1.4) but $y^{(k)} \overset{*}{\rightharpoonup} y = (0, \phi)$ in $W^{1,\infty}(\mathbb{R}^2, \mathbb{R}^2)$ where

$$\phi(x_2) = \begin{cases} x_2 & \text{for} \quad x_2 > 0, \\ 0 & \text{for} \quad x_2 \leqslant 0, \end{cases}$$

so that y does not satisfy (1.3).

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