ANNALES DE L'I. H. P., SECTION C

R. D. JAMES

S. J. SPECTOR

Remarks on $W^{1,p}$-quasiconvexity, interpenetration of matter, and function spaces for elasticity

Annales de l'I. H. P., section C, tome 9, n° 3 (1992), p. 263-280.

<http://www.numdam.org/item?id=AIHPC_1992__9_3_263_0>

Ann. Inst. Henri Poincaré,

Vol. 9, n° 3, 1992, p. 263-280. *Analyse non linéaire*

Remarks on $W^{1,p}$-quasiconvexity, interpenetration of matter, and function spaces for elasticity

by

R. D. JAMES

Department of Aerospace Engineering and Mechanics,
107 Akerman Hall, University of Minnesota,
Minneapolis, MN 55455, U.S.A.

and

S. J. SPECTOR

Department of Mathematics, Southern Illinois University,
Carbondale, IL 62901, U.S.A

ABSTRACT. — We show that, under mild hypotheses on the elastic energy function, the minimizer of the energy in the space $\{\mathbf{f} \in W^{1,p} : \det \nabla \mathbf{f} > 0, 1 \leq p < n\}$ of a nonlinear elastic ball subject to the severe compressive boundary conditions $\mathbf{f}(\mathbf{x}) = \lambda \mathbf{x}, \lambda \ll 1$ will not be the expected uniform compression $\mathbf{f}(\mathbf{x}) = \lambda \mathbf{x}$. To show this, we construct competitors in this space that reduce the energy but interpenetrate matter.

We also prove that the $W^{1,p}$-quasiconvexity condition of Ball and Murat [1984] is a necessary condition for a local minimum in a setting that includes nonlinear elasticity. This theorem is well suited to analyses of the formation of voids in nonlinear elastic materials. Our analysis illustrates the delicacy of the choice of function space for nonlinear elasticity.

RÉSUMÉ. — On montre que, sous des hypothèses faibles pour le potentiel d'énergie élastique, le minimum énergétique dans l'espace $\{\mathbf{f} \in W^{1,p} : \det \nabla \mathbf{f} > 0, 1 \leq p < n\}$ pour une sphère élastique soumise à une

---

Classification A.M.S. : 73 G 05, 49 K 20.

compression très forte à sa surface ($\mathbf{f}(\mathbf{x})=\lambda\,\mathbf{x}$, $\lambda \ll 1$) n'est pas une compression uniforme $\mathbf{f}(\mathbf{x})=\lambda\mathbf{x}$. A cet effet on construit dans cet espace des champs de déplacement à plus faible énergie. Ces champs ne respectent cependant pas la non interpénétrabilité de la matière.

Nous démontrons également que la condition de $W^{1,p}$-quasi-convexité de Ball et Murat [1984] est une condition nécessaire d'existence d'un minimum local, ce dans un contexte qui inclue notamment l'élasticité non linéaire. Notre théorème est particulièrement bien adapté à l'analyse de la formation de cavités dans des matériaux non linéairement élastiques. Notre analyse illustre le caractère critique du bon choix d'espace fonctionnel en élasticité non linéaire.

1. INTRODUCTION

The aim of this paper is two-fold. First, we point out certain difficulties that arise when spaces of the type

$$\{\mathbf{f}\in W^{1,p}(\Omega,\mathbb{R}^n)\cap L^\infty(\Omega,\mathbb{R}^n):\det\nabla\mathbf{f}>0\text{ a. e., }1\leq p<n\} \tag{1.1}$$

are used as the basic function spaces for the theory of nonlinear elasticity. These difficulties are illustrated by several examples which show that in such spaces and with mild restrictions on the energy function, a ball of material under severe compressive boundary conditions of the form

$$\mathbf{f}(\mathbf{x})=\lambda\,\mathbf{x},\qquad |\mathbf{x}|=1,\qquad \lambda\ll 1, \tag{1.2}$$

will not suffer an expected uniform contraction, $\mathbf{f}(\mathbf{x})=\lambda\,\mathbf{x}$, $|\mathbf{x}|<1$, but rather will reduce its energy by interpenetrating matter, that is, by failing to be one-to-one. These examples are foreshadowed by a theorem of Ball and Murat [1984, Theorem 4.1 (iii)] to the effect that for $1\leq p<n$, $W(\nabla\mathbf{f})=\delta(\det\nabla\mathbf{f})$ is $W^{1,p}$-quasiconvex ([1]) at every $\nabla\mathbf{f}\in\mathbb{R}^{n^2}$ if and only if δ is constant.

The second purpose of this paper is to prove (using elementary methods) a new $W^{1,p}$-quasiconvexity theorem (*See* Sect. 4 for a precise statement). The form of this theorem is ideally suited to analyses of the formation of voids in nonlinear elastic materials. We use this theorem in a forthcoming paper (James and Spector [1991]) to show that under physically reasonable

([1]) The notion of $W^{1,p}$-quasiconvexity was introduced by Ball and Murat [1984].

hypotheses on the energy function, some of the radial solutions found in the literature for the formation of spherical voids are in fact unstable relative to the formation of *filamentary voids*.

The motivation for considering subsets of $W^{1,p}(\Omega, \mathbb{R}^n)$ for $1 \leqq p < n$ as function spaces for elasticity comes partly from a series of papers by Gent and his co-workers beginning with the fundamental paper of Gent and Lindley [1958]. Among other things, they showed that the formation of tiny voids in severely strained elastomers can be accurately predicted by a criterion based on a nonlinear elastic analysis of the radial deformation of a sphere containing a pre-existing void at the origin. Ball [1982] noticed that Gent and Lindley's criterion could essentially be obtained by minimizing the total energy of a ball (with no preexisting cavity) among *radial* deformations in the space (1.1). A rather complete picture of the radial case has emerged from papers of Stuart [1985], Horgan and Abeyaratne [1986] Podio-Guidugli, Vergara Caffarelli and Virga [1986], Sivaloganathan [1986 *a*, *b*], Antman and Negron-Marrero [1987], Pericak-Spector and Spector [1988], Marcellini [1989], Chou-Wang and Horgan [1989 *a*, *b*] and Horgan and Pence [1989 *a*, *b*]. However, a major open question that remains is what is an appropriate function space for three-dimensional elasticity that is consistent with the formation of voids and is also consistent with Gent and Lindley's criterion. The examples in this paper indicate the delicacy of this question. One possible direction of research is suggested in a recent paper of Giaquinta, Modica and Souček [1989, Section 7]; see also Müller [1988]. However, the function space of Giaquinta, Modica and Souček has the property that linear or sublinear growth of the energy is necessary for the formation of cavities, and even in that case it appears necessary to modify the expression for the energy in an *ad hoc* manner. Nevertheless, their idea of completing the set of smooth invertible functions in *some* norm seems to have the potential for ruling out the kinds of examples presented in this paper.

2. NOTATION

We let

$$\text{Lin} := \text{space of all linear transformations (tensors) from } \mathbb{R}^n \text{ into } \mathbb{R}^n$$

with norm

$$|\mathbf{H}| = [\operatorname{trace}(\mathbf{H}\mathbf{H}^T)]^{1/2},$$

where $\mathbf{H}^T$ denotes the transpose of $\mathbf{H}$. We write

$$\text{Lin}^{>} := \{ \mathbf{H} \in \text{Lin} : \det \mathbf{H} > 0 \},$$
$$\text{Lin}^{\leqq} := \{ \mathbf{H} \in \text{Lin} : \det \mathbf{H} \leqq 0 \},$$

where det denotes the determinant. Given two vectors $\mathbf{a}$, $\mathbf{b} \in \mathbb{R}^n$ we write $\mathbf{a} \otimes \mathbf{b}$ for the tensor product of $\mathbf{a}$ and $\mathbf{b}$; in components

$$(\mathbf{a} \otimes \mathbf{b})_{ij} = a_i b_j.$$

We write ∇ for the gradient operator in $\mathbb{R}^n$: for a vector field $\mathbf{u}$, $\nabla \mathbf{u}$ is the tensor field with components $(\nabla \mathbf{u})_{ij} = \partial u_i / \partial x_j$. Given any function $\Phi(\mathbf{a}, \mathbf{b}, \ldots, \mathbf{c})$ with vector or tensor arguments, we write, e. g., $\partial \Phi / \partial \mathbf{a}$ for the partial Frechet derivative with respect to $\mathbf{a}$ holding the remaining arguments fixed.

We call a bounded open region $\Omega \subset \mathbb{R}^n$ **regular** provided that $\partial \Omega$ has measure zero. For $1 \leqq p \leqq \infty$ we denote by $\| \cdot \|_{\Omega, p}$ the L^p-norm on Ω. Thus, if $\mathbf{f} : \Omega \to \mathbb{R}^n$ is (Lebesgue) measurable

$$\| \mathbf{f} \|_{\Omega, p} = \left\{ \begin{array}{c} \left[\int_\Omega | \mathbf{f}(\mathbf{x}) |^p \, d\mathbf{x} \right]^{1/p}, \ 1 \leqq p < \infty \\ \operatorname*{ess\,sup}_{\mathbf{x} \in \Omega} | \mathbf{f}(\mathbf{x}) |, \ p = \infty \end{array} \right\}.$$

We write $L^p(\Omega) = L^p(\Omega, \mathbb{R}^n)$ for the usual Banach space of functions (actually equivalence classes) with finite L^p-norm.

For $1 \leqq p < \infty$ we write $W^{1,p}(\Omega)$ for the usual Sobolev space (*cf.*, e.g., Adams [1975]) of $\mathbf{f} \in L^p(\Omega)$ whose *weak derivative* $\nabla \mathbf{f}$ is contained in $L^p(\Omega, \mathbb{R}^{n^2})$ and we define

$$W_0^{1,p}(\Omega) = \text{closure of } C_0^\infty(\Omega, \mathbb{R}^n) \text{ in } W^{1,p}(\Omega).$$

We let $\| \cdot \|_{\Omega, 1, p}$ denote the $W^{1,p}$ norm on Ω. Thus, if $\mathbf{f} : \Omega \to \mathbb{R}^n$ is weakly differentiable

$$\| \mathbf{f} \|_{\Omega, 1, p} = \left\{ \int_\Omega [| \mathbf{f}(\mathbf{x}) |^p + | \nabla \mathbf{f}(\mathbf{x}) |^p \, d\mathbf{x} \right\}^{1/p}.$$

Note that the elements of $W^{1,p}(\Omega)$ or $W_0^{1,p}(\Omega)$ are equivalence classes of functions. Since in this paper we wish to distinguish a function from its equivalence class, we use the notation $\{ \mathbf{f} \} \in \mathscr{A}$ to mean that f belongs to an equivalence class that is contained in $\mathscr{A}$.

3. DEFORMATION AND STORED ENERGY

We consider a *homogeneous* body that, for convenience, we identify with the region $\bar{\Omega}$ that it occupies in a fixed homogeneous reference configuration. Let $1 \leqq p < n$. We call a function $\mathbf{f} : \bar{\Omega} \to \mathbb{R}^n$ a **deformation** of the body provided that

(i) $\mathbf{f}$ is one-to-one on $\bar{\Omega}$;

(ii) $\{ \mathbf{f} \} \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$.

We denote by $\mathrm{Def}(\Omega)$ the set of all such deformations.

Remark 3.1. – Additional restrictions are needed in order to insure that a function $\mathbf{f} \in \mathrm{Def}(\Omega)$ corresponds to a reasonable physical notion of a deformation. One such restriction is that for every $\mathscr{D} \subset \Omega$, $\mathbf{f}(\mathscr{D})$ has measure zero whenever $\mathscr{D}$ has measure zero. Without such a restriction one can construct (*cf*. Besicovitch [1950]) a function that is equal to the identity everywhere except on a flat surface, contained in Ω, which is mapped onto the unit cube. The fact that such a deformation is equal to the identity almost everywhere is one of the reasons we require our deformations to be functions rather than equivalence classes.

We assume that the body is hyperelastic with continuous **stored energy** function $W : \mathrm{Lin} \to \mathbb{R}^{\geqq} \cup \{+\infty\}$. W gives the energy stored per unit volume in Ω,

$$W(\nabla \mathbf{f}(\mathbf{x}))$$

at any point $\mathbf{x} \in \Omega$ when the body is deformed by a smooth deformation $\mathbf{f}$. We further assume that W is continuous and satisfies $W = +\infty$ on $\mathrm{Lin}^{\leqq}$.

In Section 5 we will restrict our attention to three dimensions and consider *isotropic materials*, that is, materials for which there is a symmetric function $\Phi : (\mathbb{R}^{>})^3 \to \mathbb{R}^{\geqq}$ with the property that for every $\mathbf{F} \in \mathrm{Lin}^{>}$

$$W(\mathbf{F}) = \Phi(\lambda_1(\mathbf{F}), \lambda_2(\mathbf{F}) . \lambda_3(\mathbf{F})), \tag{3.1}$$

where $\lambda_1(\mathbf{F})$, $\lambda_2(\mathbf{F})$, and $\lambda_3(\mathbf{F})$ are the *principal stretches*, *i.e.* the eigenvalues of $(\mathbf{F}\mathbf{F}^T)^{1/2}$.

4. THE $W^{1,p}$-QUASICONVEXITY OF MINIMIZERS

We assume that there is a potential $\beta \in C^1(\bar{\Omega} \times \mathbb{R}^n, \mathbb{R})$ such that

$$\mathbf{b}_{\mathbf{f}}(\mathbf{x}) := \frac{\partial}{\partial \mathbf{f}} \beta(\mathbf{x}, \mathbf{f}(\mathbf{x}))$$

gives the **body force** exerted by the environment on the material at the point $\mathbf{x}$ when the body is deformed by a smooth deformation $\mathbf{f}$. We let

$$E(\mathbf{f}, \Omega) := \int_{\Omega} [W(\nabla \mathbf{f}(\mathbf{x})) - \beta(\mathbf{x}, \mathbf{f}(\mathbf{x}))] \, d\mathbf{x} \tag{4.1}$$

denote the **total energy** when the body is deformed by $\mathbf{f}$.

Let $\mathbf{d} \in C^1(\bar{\Omega}, \mathbb{R}^n)$ be one-to-one. We are interested in deformations that are local minimizers of the total energy and that have the *same boundary-values and orientation* as $\mathbf{d}$. We therefore let

$$\mathrm{Kin}_{\mathbf{d}}(\Omega) := \{\mathbf{f} \in \mathrm{Def}(\Omega) : \{\mathbf{f} - \mathbf{d}\} \in W_0^{1,p}(\Omega),\ \mathbf{f}(\Omega) \subset \mathbf{d}(\Omega)\};$$

the set of *kinematically admissible* deformations.

Remark 4.1. — The constraint $\mathbf{f}(\Omega) \subset \mathbf{d}(\Omega)$ is used in the proof of Theorem 4.2, where a certain deformation is altered on a sphere in Ω. It seems to us a reasonable restriction on deformations that are allowed to compete for a minimum. This constraint may be a consequence of other conditions one might impose in order to insure that a function $\mathbf{f} \in \mathrm{Def}(\Omega)$ corresponds to a reasonable physical notion of a deformation that has finite energy and satisfies the boundary condition $\mathbf{f} = \mathbf{d}$ on $\partial\Omega$.

Let $\mathbf{f} \in \mathrm{Kin}_{\mathbf{d}}(\Omega)$. We say that $\mathbf{f}$ is a **strong local minimizer (in** $W^{1,p}(\Omega) \cap L^\infty(\Omega)$) of the energy E provided that $E(\mathbf{f}, \Omega) < +\infty$ and there is an $\varepsilon > 0$ such that

$$E(\mathbf{f}, \Omega) \leqq E(\mathbf{g}, \Omega)$$

for every $\mathbf{g} \in \mathrm{Kin}_{\mathbf{d}}(\Omega)$ that satisfies

$$\|\mathbf{f} - \mathbf{g}\|_{\Omega, \infty} + \|\mathbf{f} - \mathbf{g}\|_{\Omega, 1, p} < \varepsilon. \tag{4.2}$$

THEOREM 4.2. — *Let* $\mathbf{f} \in \mathrm{Kin}_{\mathbf{d}}(\Omega)$ *be a strong local minimizer of* $E(., \Omega)$. *Suppose that* $\mathbf{f}$ *is* C^1 *in a neighborhood of* $\mathbf{x}_0 \in \Omega$ *and let*

$$\mathbf{F}_0 := \nabla \mathbf{f}(\mathbf{x}_0); \qquad \mathbf{f}_0(\mathbf{x}) := \mathbf{F}_0(\mathbf{x} - \mathbf{x}_0) + \mathbf{f}(\mathbf{x}_0), \qquad \mathbf{x} \in \mathbb{R}^n.$$

Assume that $W(\mathbf{F}_0) < \infty$. *Then for every regular region* $\mathscr{D} \subset \mathbb{R}^n$

$$\int_{\mathscr{D}} W(\mathbf{F}_0)\, d\mathbf{x} \leqq \int_{\mathscr{D}} W(\mathbf{F}_0 + \nabla \mathbf{u}(\mathbf{x}))\, d\mathbf{x} \tag{4.3}$$

whenever $\mathbf{f}_0 + \mathbf{u} \in \mathrm{Kin}_{\mathbf{f}_0}(\mathscr{D})$.

In other words a necessary condition for $\mathbf{f}$ to be a strong local minimizer is that, at each point $\mathbf{x}_0$ of smoothness of $\mathbf{f}$, the affine deformation $\mathbf{f}_0(\mathbf{x}) = \nabla \mathbf{f}(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0) + \mathbf{f}(\mathbf{x}_0)$ is a global minimizer of the total energy of any body that is composed of the same material but is not subjected to body forces.

Remark 4.3. — Theorem 4.2 remains valid if the constraint $\mathbf{f}(\Omega) \subset \mathbf{d}(\Omega)$ and the hypothesis of invertibility are dropped. In the terminology of Ball and Murat [1984] our proof of Theorem 4.2 shows that if $\mathbf{f}$ is a strong local minimizer then the stored energy function W must be $W^{1,p}$-quasiconvex at $\mathbf{F}_0 = \nabla \mathbf{f}(\mathbf{x}_0)$ for every point $\mathbf{x}_0$ at which $\mathbf{f}$ is smooth. If the variations $\mathbf{u}$ are required to be C^∞ then eq. (4.3) is Morrey's [1952] quasiconvexity condition and Theorem 4.2 is therefore related to a result of Meyers [1965, pp. 128-131] who shows that quasiconvexity is a necessary condition for a local minimum.

Remark 4.4. — We do not know if Theorem 4.2 is true for general integrands $W(\mathbf{x}, \mathbf{f}, \nabla\mathbf{f})$. The technique used in our proof requires that W be independent of its first two arguments, although we have included body forces.

Remark 4.5. — Theorem 4.2 remains valid if the definition of a strong local minimizer omits the term $\|\mathbf{f}-\mathbf{g}\|_{\Omega,\infty}$ provided a growth condition is put on the body force potential β.

Remark 4.6. — In James and Spector [1991] we introduce constitutive hypotheses on the stored energy W that promote the formation of voids by forcing (4.3) to be violated at certain $\mathbf{F}_0$. Thus these constitutive hypotheses, as well the hypotheses of all the other authors who consider deformations that create holes, imply that W is not $W^{1,p}$-quasiconvex. A result of Ball and Murat [1984, Corollary 3.2] then implies that, for such energy functions, $E(.,\Omega)$ is not sequentially weakly lower semicontinuous. Therefore it is not clear that a minimizer of $E(.,\Omega)$ need exist (*cf.* Ball and Murat [1984, Theorem 5.1]). Even if a minimizer does indeed exist it may be difficult to find since it may be approached weakly by only very special minimizing sequences, a situation that arises in the study of phase transitions (*cf.* Ball and James [1987]).

Proof of Theorem 4.2. — Let $\mathbf{f}\in\mathrm{Kin}_{\mathbf{d}}(\Omega)$ be C^1 in a neighborhood of $\mathbf{x}_0$ with $W(\mathbf{F}_0)$ finite. Then our basic assumptions on W imply that there is an $\varepsilon\in(0, 1)$ such that $\mathbf{f}$ is C^1 with $\det\nabla\mathbf{f}>0$ and $W(\nabla\mathbf{f}(.))$ finite on $\mathscr{B}_{3\varepsilon}$. (Here, and in what follows, $\mathscr{B}_\rho=\mathscr{B}_\rho(\mathbf{x}_0)$ the open ball of radius $\rho>0$ centered at $\mathbf{x}_0$.)

To prove this theorem, we show that to each competitor $\mathbf{g}:=\mathbf{f}_0+\mathbf{u}\in\mathrm{Kin}_{\mathbf{f}_0}(\mathscr{D})$ we can associate a sequence $\mathbf{f}_\alpha$ that converges to $\mathbf{f}$ in $W^{1,p}(\Omega)\cap L^\infty(\Omega)$ as $\alpha\to 0^+$. The hypothesis $E(\mathbf{f},\Omega)\leqq E(\mathbf{f}_\alpha,\Omega)$, which holds for α sufficiently small, then implies (4.3). The constants ε and δ (introduced below) will be fixed throughout the proof.

Let $\theta\in C^\infty(\mathbb{R},\mathbb{R})$ be a monotone decreasing bridging function satisfying

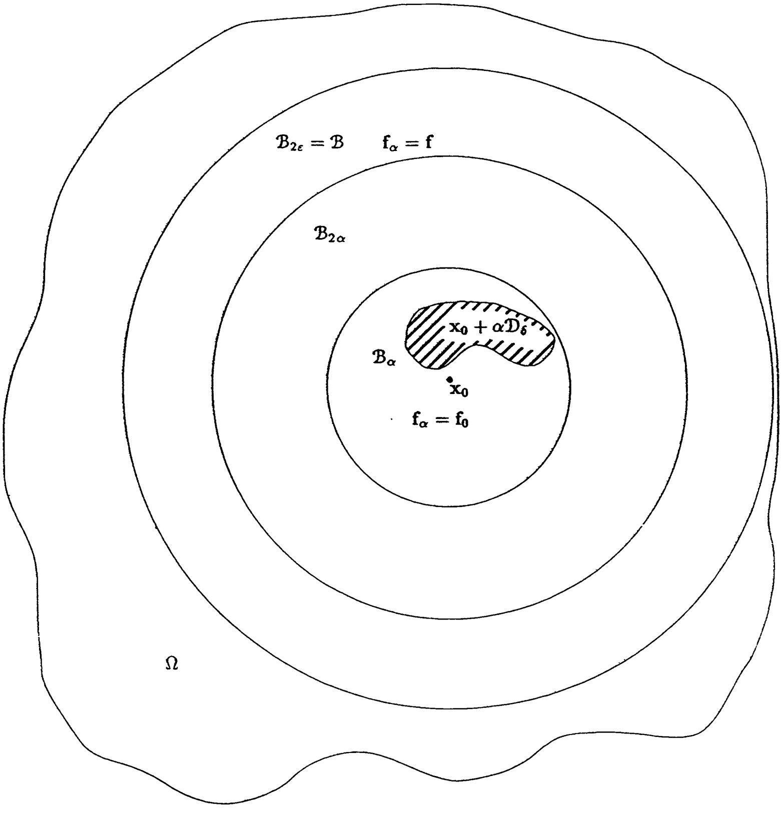
$$\theta(t)=\left\{\begin{matrix} 1, & t\leqq 1 \\ 0, & t\geqq 4 \end{matrix}\right\}. \tag{4.4}$$

Let $\mathscr{B}:=\mathscr{B}_{2\varepsilon}$ and define $\mathbf{h}_\alpha\in C^1(\bar{\mathscr{B}},\mathbb{R}^n)$, for $0<\alpha<\varepsilon$, by

$$\mathbf{h}_\alpha(\mathbf{x}):=\theta(\alpha^{-2}|\mathbf{x}-\mathbf{x}_0|^2)\,\mathbf{f}_0(\mathbf{x})+[1-\theta(\alpha^{-2}|\mathbf{x}-\mathbf{x}_0|^2)]\,\mathbf{f}(\mathbf{x}) \tag{4.5}$$

and note that

$$\nabla\mathbf{h}_\alpha=\nabla\mathbf{f}+\theta[\mathbf{F}_0-\nabla\mathbf{f}]+2\alpha^{-2}\dot\theta[\mathbf{f}-\mathbf{f}_0]\otimes[\mathbf{x}-\mathbf{x}_0]. \tag{4.6}$$



Construction used in the proof of Theorem 4.2.

We claim that for α sufficiently small $\mathbf{h}_\alpha\in \mathrm{Kin}_{\mathbf{f}}(\mathscr{B})$. To show this we first combine (4.4) and (4.6) to conclude that

$$\max_{\bar{\mathscr{B}}}|\nabla \mathbf{h}_\alpha-\nabla \mathbf{f}|\leqq\max_{\bar{\mathscr{B}}_{2\alpha}}|\mathbf{F}_0-\nabla \mathbf{f}|+4\alpha^{-1}(\max|\dot{\theta}|)\max_{\bar{\mathscr{B}}_{2\alpha}}|\mathbf{f}-\mathbf{f}_0|. \quad (4.7)$$

Since $\mathbf{f}\in C^1(\bar{\mathscr{B}}, \mathbb{R}^n)$, $\mathbf{f}(\mathbf{x}_0)=\mathbf{f}_0(\mathbf{x}_0)$, and $\nabla \mathbf{f}_0(\mathbf{x})=\nabla \mathbf{f}(\mathbf{x}_0)$, we find that

$$\alpha^{-1}\max_{\bar{\mathscr{B}}_{2\alpha}}|\mathbf{f}-\mathbf{f}_0|\to 0 \quad \text{as } \alpha\to 0^+$$

and thus, with the aid of (4.7), we find that $\nabla \mathbf{h}_\alpha\to\nabla \mathbf{f}$ uniformly on $\bar{\mathscr{B}}$. We note that $\mathbf{f}$ is one-to-one with $\det\nabla \mathbf{f}>0$ on $\bar{\mathscr{B}}$ and $\mathbf{h}_\alpha=\mathbf{f}$ on $\partial\mathscr{B}$. Thus a standard theorem (*cf.*, e.g., Ciarlet [1988, Theorem 5.5-1]) yields an $\alpha_0\in(0, \varepsilon)$ such that $\mathbf{h}_\alpha$ is one-to-one on $\bar{\mathscr{B}}$ with $\mathbf{h}_\alpha(\mathscr{B})=\mathbf{f}(\mathscr{B})$ for $0<\alpha<\alpha_0$. Thus, $\mathbf{h}_\alpha\in\mathrm{Kin}_{\mathbf{f}}(\mathscr{B})$.

We now suppose that $\mathscr{D}\subset\mathbb{R}^n$ is a regular region and $\mathbf{g}\in\mathrm{Kin}_{\mathbf{f}_0}(\mathscr{D})$ is a kinematically admissible deformation of $\mathscr{D}$. Define

$$\gamma:=\int_{\mathscr{D}}[\mathrm{W}(\nabla\mathbf{g}(\mathbf{x}))-\mathrm{W}(\mathbf{F}_0)]\,d\mathbf{x} \tag{4.8}$$

and note that the desired result [equation (4.3)] is that $\gamma\geqq 0$. If $\gamma=+\infty$ we are done. Otherwise, choose $\delta>0$ so that (see *Fig.*)

$$\mathbf{x}_0+\mathscr{D}_\delta\subset\mathscr{B},\qquad \mathscr{D}_\delta:=\delta\mathscr{D}.$$

For $0<\alpha<\alpha_0$ and $\mathbf{x}\in\mathbf{x}_0+\alpha\mathscr{D}_\delta$, let

$$\mathbf{e}_\alpha(\mathbf{x}):=\alpha\delta\,[\mathbf{g}((\mathbf{x}-\mathbf{x}_0)/\alpha\delta)-\mathbf{f}_0((\mathbf{x}-\mathbf{x}_0)/\alpha\delta)]+\mathbf{f}_0(\mathbf{x}). \tag{4.9}$$

Observe that since the sum of the last two terms of (4.9) does not depend on $\mathbf{x}$, $\mathbf{e}_\alpha(\mathbf{x})$ is $1-1$ on $\mathbf{x}_0+\alpha\mathscr{D}_\delta$.

Define $\mathbf{f}_\alpha:\bar{\Omega}\to\mathbb{R}^n$ by

$$\mathbf{f}_\alpha(\mathbf{x}):=\left\{\begin{array}{ll}\mathbf{e}_\alpha(\mathbf{x}), & \mathbf{x}\in\mathbf{x}_0+\alpha\mathscr{D}_\delta\\ \mathbf{h}_\alpha(\mathbf{x}), & \mathbf{x}\in\bar{\mathscr{B}}-(\mathbf{x}_0+\alpha\mathscr{D}_\delta)\\ \mathbf{f}(\mathbf{x}), & \mathbf{x}\in\bar{\Omega}-\bar{\mathscr{B}}\end{array}\right\}. \tag{4.10}$$

Since $\mathbf{f}\in\mathrm{Kin}_{\mathbf{d}}(\Omega)$, $\mathbf{h}_\alpha\in\mathrm{Kin}_{\mathbf{f}}(\mathscr{B})$ for $0<\alpha<\alpha_0$, and $\mathbf{g}\in\mathrm{Kin}_{\mathbf{f}_0}(\mathscr{D})$, we conclude that $\mathbf{f}_\alpha\in\mathrm{Kin}_{\mathbf{d}}(\Omega)$ for $0<\alpha<\alpha_0$.

We next show that $\mathbf{f}_\alpha\to\mathbf{f}$ in $\mathrm{W}^{1,p}(\Omega)\cap\mathrm{L}^\infty(\Omega)$ as $\alpha\to 0^+$. First, by (4.9) we note that for $\mathbf{x}\in\mathbf{x}_0+\alpha\mathscr{D}_\delta$,

$$|\mathbf{e}_\alpha(\mathbf{x})-\mathbf{f}(\mathbf{x})|\leqq\alpha\delta\,[\,|\mathbf{g}((\mathbf{x}-\mathbf{x}_0)/\alpha\delta)|+|\mathbf{f}_0((\mathbf{x}-\mathbf{x}_0)/\alpha\delta)|\,]+|\mathbf{f}_0(\mathbf{x})-\mathbf{f}(\mathbf{x})|,$$

and so we find, with the aid of (4.4), (4.5), and (4.10), that

$$\|\mathbf{f}_\alpha-\mathbf{f}\|_{\Omega,\infty}\leqq\alpha\delta\,\{\|\mathbf{g}\|_{\mathscr{D},\infty}+\|\mathbf{f}_0\|_{\mathscr{D},\infty}\}+\|\mathbf{f}_0-\mathbf{f}\|_{\mathscr{B}_{2\alpha},\infty}.$$

Since $\mathbf{f}$ is continuous on $\mathscr{B}$ we conclude that $\mathbf{f}_\alpha\to\mathbf{f}$ in $\mathrm{L}^\infty(\Omega)$ as $\alpha\to 0^+$. By (4.9), the triangle inequality, Hölders inequality and the change of variables $\mathbf{y}=(\mathbf{x}-\mathbf{x}_0)/\alpha$ we find that

$$\|\nabla\mathbf{e}_\alpha-\nabla\mathbf{f}\|_{\mathbf{x}_0+\alpha\mathscr{D}_\delta,\,p}\leqq\alpha^{n/p}\left\{\begin{array}{l}\left(\displaystyle\int_{\mathscr{D}_\delta}|\nabla\mathbf{g}-\mathbf{F}_0|^p\,d\mathbf{y}\right)^{1/p}\\ +\mathrm{Vol}(\mathscr{D}_\delta)^{1/p}\|\mathbf{F}_0-\nabla\mathbf{f}\|_{\mathscr{B},\infty}\end{array}\right\}. \tag{4.11}$$

Therefore, since $\mathbf{h}_\alpha\to\mathbf{f}$ and $\nabla\mathbf{h}_\alpha\to\nabla\mathbf{f}$ uniformly on $\bar{\mathscr{B}}$ (4.11) shows that $\mathbf{f}_\alpha\to\mathbf{f}$ in $\mathrm{W}^{1,p}(\Omega)$.

Finally, we compute the total energy of the deformations $\mathbf{f}_\alpha$, $0<\alpha<\alpha_0$. The definition (4.10) of $\mathbf{f}_\alpha$ gives

$$\int_\Omega [W(\nabla \mathbf{f}_\alpha)-W(\nabla \mathbf{f})]\,d\mathbf{x}=\left\{\begin{array}{l}\displaystyle\int_{\mathbf{x}_0+\alpha\mathscr{D}_\delta}[W(\nabla \mathbf{e}_\alpha)-W(\mathbf{F}_0)]\,d\mathbf{x}\\ \displaystyle+\int_{\mathscr{B}_\alpha}[W(\mathbf{F}_0)-W(\nabla \mathbf{f})]\,d\mathbf{x}\\ \displaystyle+\int_{\mathscr{B}_{2\alpha}-\mathscr{B}_\alpha}[W(\nabla \mathbf{h}_\alpha)-W(\nabla \mathbf{f})]\,d\mathbf{x}\end{array}\right\}. \qquad (4.12)$$

If we define

$$\sigma_\alpha:=\|W(\mathbf{F}_0)-W(\nabla \mathbf{f})\|_{\mathscr{B}_\alpha,\,\infty}+\|W(\nabla \mathbf{h}_\alpha)-W(\nabla \mathbf{f})\|_{\mathscr{B},\,\infty},$$

and make the change of variables $\mathbf{y}=(\mathbf{x}-\mathbf{x}_0)/\alpha\delta$ in the first integral on the right hand side of (4.12) we find, with the aid of (4.8) and (4.9), that

$$\int_\Omega [W(\nabla \mathbf{f}_\alpha)-W(\nabla \mathbf{f})]\,d\mathbf{x}\leqq\alpha^n[\gamma\delta^n+2^n\,\mathrm{vol}(\mathscr{B}_1)\,\sigma_\alpha]. \qquad (4.13)$$

Similarly, (4.10) implies that

$$\int_\Omega [\beta(\mathbf{x},\mathbf{f}_\alpha(\mathbf{x}))-\beta(\mathbf{x},\mathbf{f}(\mathbf{x}))]\,d\mathbf{x}$$

$$=\left\{\begin{array}{l}\displaystyle\int_{\mathbf{x}_0+\alpha\mathscr{D}_\delta}[\beta(\mathbf{x},\mathbf{e}_\alpha(\mathbf{x}))-\beta(\mathbf{x},\mathbf{f}_0(\mathbf{x}))]\,d\mathbf{x}\\ \displaystyle+\int_{\mathscr{B}_\alpha}[\beta(\mathbf{x},\mathbf{f}_0(\mathbf{x}))-\beta(\mathbf{x},\mathbf{f}(\mathbf{x}))]\,d\mathbf{x}\\ \displaystyle+\int_{\mathscr{B}_{2\alpha}-\mathscr{B}_\alpha}[\beta(\mathbf{x},\mathbf{h}_\alpha(\mathbf{x}))-\beta(\mathbf{x},\mathbf{f}(\mathbf{x}))]\,d\mathbf{x}\end{array}\right\}. \qquad (4.14)$$

Let

$$\tau_\alpha:=\|\beta(.,\mathbf{f}_0(.))-\beta(.,\mathbf{f}(.))\|_{\mathscr{B}_\alpha,\,\infty}+\|\beta(.,\mathbf{h}_\alpha(.))-\beta(.,\mathbf{f}(.))\|_{\mathscr{B},\,\infty},$$

and

$$\omega_\alpha:=\int_{\mathscr{D}}[\beta(\mathbf{x}_0+\alpha\,\delta\mathbf{y},\,\alpha\delta[\mathbf{g}(\mathbf{y})-\mathbf{f}_0(\mathbf{y})]+\mathbf{f}_0(\mathbf{x}_0+\alpha\,\delta\mathbf{y}))$$
$$-\beta(\mathbf{x}_0+\alpha\,\delta\mathbf{y},\,\mathbf{f}_0(\mathbf{x}_0+\alpha\,\delta\mathbf{y})]\,d\mathbf{y}$$

We make the change of variables $\mathbf{y}=(\mathbf{x}-\mathbf{x}_0)/\alpha\delta$ in the first integral on the right hand side of (4.14). Then we combine (4.13) with (4.14) to get

$$E(\mathbf{f}_\alpha,\Omega)-E(\mathbf{f},\Omega)\leqq\alpha^n[(\gamma+|\omega_\alpha|)\,\delta^n+2^n\,\mathrm{vol}(\mathscr{B}_1)(\sigma_\alpha+\tau_\alpha)]. \qquad (4.15)$$

Now, $\nabla \mathbf{h}_\alpha \to \nabla \mathbf{f}$ uniformly on $\bar{\mathscr{B}}$ and $\nabla \mathbf{f}$ is continuous on $\mathscr{B}$ with $\nabla \mathbf{f}(\mathbf{x}_0)=\mathbf{F}_0$. Thus since W is continuous we find that

$$\sigma_\alpha \to 0 \quad \text{as } \alpha \to 0^+. \tag{4.16}$$

Also $\mathbf{h}_\alpha \to \mathbf{f}$ uniformly on $\bar{\mathscr{B}}$ and $\mathbf{f}_0$ and $\mathbf{f}$ are continuous on $\mathscr{B}$ with $\mathbf{f}(\mathbf{x}_0)=\mathbf{f}_0(\mathbf{x}_0)$. Thus since β is continuous we conclude that

$$\tau_\alpha \to 0 \quad \text{as } \alpha \to 0^+. \tag{4.17}$$

Finally, $\{\mathbf{g}\} \in L^\infty(\mathscr{D})$ and β and $\mathbf{f}_0$ are continuous. Therefore, by the bounded convergence theorem we find that

$$\omega_\alpha \to 0 \quad \text{as } \alpha \to 0^+. \tag{4.18}$$

To finish the proof we note that $\mathbf{f}$ is a strong local minimizer of E and that $\mathbf{f}_\alpha \to \mathbf{f}$ in $W^{1,p}(\Omega) \cap L^\infty(\Omega)$. Thus, for α sufficiently small the left hand side of (4.15) is nonnegative. Therefore, if we divide (4.15) by α^n and let $\alpha \to 0^+$ we conclude from (4.16)-(4.18) that $\gamma \geqq 0$, completing the proof. □

Remark 4.7. – The standard proof (*see* Ball [1977], Theorem 3.1) that $W^{1,\infty}$-quasiconvexity is a necessary condition for a local minimum uses the comparison functions

$$\mathbf{f}_\alpha(\mathbf{x}) = \begin{cases} \mathbf{f}(\mathbf{x}) + \alpha\,\mathbf{u}((\mathbf{x}-\mathbf{x}_0)/\alpha), & (\mathbf{x}-\mathbf{x}_0)/\alpha \in D \\ \mathbf{f}(\mathbf{x}) & \text{otherwise} \end{cases}$$

rather than those given by (4.10). In the absence of body forces the assumption that $\mathbf{f}$ is a strong local minimizer yields, with the aid of the change of variables $\mathbf{y}=(\mathbf{x}-\mathbf{x}_0)/\alpha$,

$$\int_D W(\nabla \mathbf{f}(\mathbf{x}_0+\alpha\mathbf{y}) + \nabla \mathbf{u}(\mathbf{y}))\,d\mathbf{y} \geqq \int_D W(\nabla \mathbf{f}(\mathbf{x}_0+\alpha\mathbf{y}))\,d\mathbf{y}. \tag{4.19}$$

The final step is to let $\alpha \to 0^+$. If $\{\mathbf{u}\} \in W^{1,\infty}(D)$ then the bounded convergence theorem lets one interchange the limit with the integration to deduce that W is $W^{1,\infty}$-quasiconvex at $\nabla \mathbf{f}(\mathbf{x}_0)$. This last step is not valid in general for $\{\mathbf{u}\} \in W^{1,p}(\mathscr{D})$ due to the fact that $W(\mathbf{F}) = +\infty$ if $\det \mathbf{F} \leqq 0$. Our proof avoids this issue by first replacing $\mathbf{f}$ by a linear function near $\mathbf{x}_0$ and then inserting the test function.

Alternatively, it is possible to give a simple proof that $W^{1,p}$-quasiconvexity is a necessary condition for a strong local minimum under mild growth conditions on W which, however, contradict the hypothesis that $W(\mathbf{F}) = +\infty$ for $\det \mathbf{F} \leqq 0$ (We are grateful to the reviewer for the following remark). For example, if W satisfies the condition

$$|W(\mathbf{H}+\mathbf{G})| \leqq c(\eta)(1+W(\mathbf{H})) \tag{4.20}$$

for all $\mathbf{G}$ with $|\mathbf{G}|<\eta$, then it is easy to pass to the limit $\alpha \to 0^+$ in the inequality (4.19) for $\mathbf{u} \in W^{1,p}(\Omega)$. This is accomplished by putting

$\mathbf{H}=\nabla \mathbf{f}(\mathbf{x}_0)+\nabla \mathbf{u}(\mathbf{y})$, $\mathbf{G}=\nabla \mathbf{f}(\mathbf{x}_0+\alpha \mathbf{y})-\nabla \mathbf{f}(\mathbf{x}_0)$ and then using the fact that $\mathbf{f}$ is C^1 in a neighborhood of $\mathbf{x}_0$ and W is continuous. The inequality (4.20) allows polynomial or exponential growth of W but implies that W is everywhere finite.

5. INTERPENETRATION OF MATTER AND FUNCTION SPACES FOR ELASTICITY

In our definition of a deformation we explicitly included the hypothesis that deformations are 1-1, a reasonable requirement for elasticity. In this section we let $\Omega \subset \mathbb{R}^3$ and show that if this requirement is dropped and one just minimizes the total energy in the space

$$\{\{\mathbf{f}\}\in W^{1,p}(\Omega)\cap L^\infty(\Omega): \det \nabla \mathbf{f}>0 \text{ a.e.}\},$$

$1 \leqq p<3$, then one obtains physically unreasonable results for very simple boundary value problems under very mild constitutive hypotheses.

The idea behind these results is that the local invertibility constraint $\det \nabla \mathbf{f}>0$ a.e. is not sufficiently strong to prevent interpenetration of matter in the space $W^{1,p}(\Omega)\cap L^\infty(\Omega)$, $1 \leqq p<3$, and under severe compressive boundary conditions, the material can relax severe compressive strains and therefore reduce the energy by interpenetrating.

We do not know what is the appropriate function space for elasticity. However, we think that such a space should allow for the formation of spherical and filamentary voids, should satisfy the constraint $\mathbf{f}(\Omega)\subset \mathbf{d}(\Omega)$, should restrict minimizers to be 1-1, and should permit $h_\lambda(\mathbf{x}):=\lambda \mathbf{x}$, $\lambda \ll 1$, to be a minimizer for the displacement problem for a large class of reasonable stored energy functions.

Remark 5.1. — Ball [1981], Ciarlet and Nečas [1987] and Šverák [1988] have obtained results in which the global invertibility of a function is a consequence of the local constraint $\det \nabla \mathbf{f}>0$. However the underlying function spaces that they use do not allow deformations that create holes.

For simplicity we now assume that the body in its reference configuration occupies the unit ball

$$\mathscr{B}:=\{\mathbf{x}\in \mathbb{R}^3: |\mathbf{x}|<1\}.$$

For $1 \leqq p \leqq \infty$, we let

$$A_\lambda^p:=\{\mathbf{f}:\mathscr{B}\to \mathbb{R}^3:\{\mathbf{f}\}\in W^{1,p}(\mathscr{B})\cap L^\infty(\mathscr{B}),\\ \det \nabla \mathbf{f}>0 \text{ a.e. and } \mathbf{f}(\mathbf{x})=\lambda \mathbf{x} \text{ on } \partial \mathscr{B}\}.$$

A function $\mathbf{f}:\mathscr{B}\to \mathbb{R}^3$ is said to be *radial* if it has the form

$$\mathbf{f}(\mathbf{x})=\frac{\rho(\mathrm{P})}{\mathrm{P}}\mathbf{x}, \qquad \mathrm{P}=|\mathbf{x}|\neq 0, \tag{5.1}$$

for some $\rho:[0, 1]\to\mathbb{R}$.

LEMMA 5.2 (Ball [1982, p. 566]). — *Let* $\mathbf{f}:\mathscr{B}\to\mathbb{R}^3$ *satisfy* (5.1). *Then* $\{\mathbf{f}\}\in W^{1,q}(\mathscr{B})$ *if and only if* ρ *is absolutely continuous on every closed subinterval of* (0, 1) *and*

$$\int_0^1\left[\,|\dot\rho(P)|^q+\left|\frac{\rho(P)}{P}\right|^q\right]P^2\,dP<+\infty. \tag{5.2}$$

In this case the weak derivatives of $\mathbf{f}$ *are given by*

$$\nabla\mathbf{f}(\mathbf{x})=\frac{\rho(P)}{P}\mathbf{I}+\left(\dot\rho(P)-\frac{\rho(P)}{P}\right)\frac{\mathbf{x}}{P}\otimes\frac{\mathbf{x}}{P}\quad \text{a.e. } \mathbf{x}\in\mathscr{B}$$

and hence the principal stretches are given by

$$|\dot\rho(P)|,\left|\frac{\rho(P)}{P}\right|,\left|\frac{\rho(P)}{P}\right|,\quad \text{a.e. } \mathbf{x}\in\mathscr{B}. \tag{5.3}$$

Recall the definition of an isotropic material given in (3.1).

PROPOSITION 5.3. — *Suppose that the stored energy of an isotropic material satisfies the following constitutive hypotheses* ([2]):

(I1) $\lim_{\lambda\to 0^+}\Phi(\lambda,\lambda,\lambda)=+\infty$,

(I2) $t\mapsto\Phi(t^{1-q}, t, t)\,t^{q-1}\in L^1((0, 1))$,

(I3) $t\mapsto\Phi(t^{1-r}, t, t)\,t^{-4}\in L^1((1, \infty))$,

for some $q\in(1,\infty)$ *and* $r\in(0,\infty)$. *Then there is a* $\lambda_0>0$ *such that for every* $\lambda\in(0,\lambda_0)$

$$\inf_{\mathbf{f}\in A_\lambda^p} E(\mathbf{f},\mathscr{B})<E(\lambda\mathbf{x},\mathscr{B})$$

whenever $p<\min\{3, q/(q-1)\}$.

Remark 5.4. — The competitor $\mathbf{f}_\lambda$ used to prove Proposition 5.3 belongs to A_λ^p and therefore satisfies $\det\nabla\mathbf{f}_\lambda>0$ a.e. but is not 1-1. However, $\mathbf{f}_\lambda$ does satisfy the constraint $\mathbf{f}_\lambda(\mathscr{B})\subset\mathbf{d}_\lambda(\mathscr{B})$ where $\mathbf{d}_\lambda(\mathbf{x})=\lambda\mathbf{x}$, $\mathbf{x}\in\overline{\mathscr{B}}$ (*see* Remark 4.1). Note that the constitutive hypotheses (I1)-(I3) are extremely mild.

For an elastic fluid we obtain a more precise result.

PROPOSITION 5.5. — *Let the stored energy be given by*

$$W(\mathbf{F})=\delta(\det\mathbf{F}), \tag{5.4}$$

([2]) Actually we need not make I1 as a separate hypotheses since it is a consequence of the continuity of W and the fact that $W(\mathbf{F})=+\infty$ for $\det\mathbf{F}\leq 0$.

where δ *has a strict global minimum at* $\Delta>0$. *Then, for* $p\in[1, 3/2)$ *and* $\lambda^3\neq\Delta$,

$$\inf_{\mathbf{f}\in\mathcal{A}_\lambda^p} \mathrm{E}(\mathbf{f}, \mathcal{B})<\mathrm{E}(\lambda\mathbf{x}, \mathcal{B}).$$

Moreover, a global minimizer of the total energy $\mathrm{E}(.,\mathcal{B})$ *in* $\mathcal{A}_\lambda^p$ *is given by*

$$\mathbf{f}_\lambda(\mathbf{x}):=\frac{\rho(\mathrm{P})}{\mathrm{P}}\mathbf{x}, \qquad \rho(\mathrm{P}):=(\Delta\mathrm{P}^3+\lambda^3-\Delta)^{1/3}. \tag{5.5}$$

which exhibits interpenetration of matter for $\lambda^3<\Delta$ *and cavitation for* $\lambda^3>\Delta$. *If* $\lambda^3>\Delta$, $\mathbf{f}_\lambda$ *as defined by* (5.5) *is a global minimizer of the total energy* $\mathrm{E}(.,\mathcal{B})$ *in* $\mathcal{A}_\lambda^p$ *for* $1\leqq p<3$.

Remark 5.6. — In the terminology of Ball and Murat [1984] Proposition 5.5 implies that, for $1\leqq p<3/2$, the stored energy given by (5.4) is not $\mathrm{W}^{1,p}$-quasiconvex at $\lambda\mathbf{I}$ unless λ^3 is the global minimizer of δ. A related result of Ball and Murat [1984, Theorem 4.1 (iii)] is that, for $1\leqq p<3$, W as given by (5.4) is $\mathrm{W}^{1,p}$-quasiconvex at *every* $\mathbf{A}\in\mathrm{Lin}$ if and only if δ is constant.

Proof of Proposition 5.3. — Let $\lambda\in(0, 1/3)$, consider the function

$$\mathbf{f}_\lambda(\mathbf{x}):=\frac{\rho(\mathrm{P})}{\mathrm{P}}\mathbf{x}, \qquad \rho(\mathrm{P}):=\left\{\begin{array}{ll} -(\lambda^r-\mathrm{P}^r)^{r^*}, & 0\leqq\mathrm{P}<\lambda 2^{-r^*} \\ -(\mathrm{A}^q-\mathrm{P}^q)^{q^*}, & \lambda 2^{-r^*}\leqq\mathrm{P}<\mathrm{A} \\ (\mathrm{P}^q-\mathrm{A}^q)^{q^*}, & \mathrm{A}\leqq\mathrm{P}<\hat{\lambda}\mathrm{A} \\ \lambda\mathrm{P}, & \hat{\lambda}\mathrm{A}\leqq\mathrm{P}<1 \end{array}\right\}, \tag{5.6}$$

where $\mathrm{A}:=\lambda/2^{(r^*-q^*)}$, $\hat{\lambda}:=(1-\lambda^q)^{-q^*}$, $q^*:=1/q$, and $r^*:=1/r$.

We will show that $\{\mathbf{f}_\lambda\}\in\mathrm{W}^{1,p}(\mathcal{B})$ and that, for λ sufficiently small, $\mathbf{f}_\lambda$ has less total energy than the homogeneous deformation $\mathbf{f}(\mathbf{x})=\lambda\mathbf{x}$.

We first compute the total energy of $\mathbf{f}_\lambda$. By Lemma 5.2 we find, with the aid of (3.1), (5.3) and (5.6), that

$$\mathrm{E}(\mathbf{f}_\lambda, \mathcal{B})=4\pi\lambda^3[\mathrm{E}_1+\mathrm{E}_2+\mathrm{E}_3+\mathrm{E}_4],$$

$$\mathrm{E}_1:=\lambda^{-3}\int_0^{\lambda/2^{r^*}}\Phi\left(\dot{\rho}, -\frac{\rho}{\mathrm{P}}, -\frac{\rho}{\mathrm{P}}\right)\mathrm{P}^2\,d\mathrm{P},$$

$$\mathrm{E}_2:=\lambda^{-3}\int_{\lambda/2^{r^*}}^{\mathrm{A}}\Phi\left(\dot{\rho}, -\frac{\rho}{\mathrm{P}}, -\frac{\rho}{\mathrm{P}}\right)\mathrm{P}^2\,d\mathrm{P},$$

$$\mathrm{E}_3:=\lambda^{-3}\int_{\mathrm{A}}^{\hat{\lambda}\mathrm{A}}\Phi\left(\dot{\rho}, \frac{\rho}{\mathrm{P}}, \frac{\rho}{\mathrm{P}}\right)\mathrm{P}^2\,d\mathrm{P}, \qquad \mathrm{E}_4:=\lambda^{-3}\int_{\hat{\lambda}\mathrm{A}}^{1}\Phi\left(\dot{\rho}, \frac{\rho}{\mathrm{P}}, \frac{\rho}{\mathrm{P}}\right)\mathrm{P}^2\,d\mathrm{P}.$$

In order to compute E_1 we make the change of variables $w=-\rho/\mathrm{P}$ and note that, by (5.6)

$$\dot{\rho}=w^{1-r}, \qquad \mathrm{P}=\lambda/(1+w^r)^{r^*}$$

and hence

$$E_1 = \int_1^\infty \Phi(w^{1-r}, w, w)\, g_r(w)\, dw, \qquad g_r(w) := \frac{w^{r-1}}{(1+w^r)^{(r+3)/r}}.$$

Similarly, the change of variables $w = -\rho/P$ in E_2 and the change of variables $v = \rho/P$ in E_3 yield

$$E_2 = K \int_0^1 \Phi(w^{1-q}, w, w)\, g_q(w)\, dw, \qquad E_3 = K \int_0^\lambda \Phi(v^{1-q}, v, v)\, g_q(v)\, dv,$$

where $K = K(q, r) = 8^{q^* - r^*}$. It is now clear that I2 and I3 are necessary and sufficient for E_1, E_2, and E_3 to be finite.

We next compute

$$E(\lambda \mathbf{x}, \mathscr{B}) = 4\pi\lambda^3 \left[\lambda^{-3} \int_0^{\hat{\lambda} A} \Phi(\lambda, \lambda, \lambda)\, P^2\, dP + E_4 \right]$$
$$= 4\pi\lambda^3 \left[K\, \Phi(\lambda, \lambda, \lambda)/3\, (1-\lambda^q)^{3/q} + E_4 \right].$$

Thus

$$E(\mathbf{f}_\lambda, \mathscr{B}) - E(\lambda \mathbf{x}, \mathscr{B}) \leqq 4\pi\lambda^3 \left[E_1 + E_2 + E_3 - k\, \Phi(\lambda, \lambda, \lambda) \right]$$

for $\lambda \in (0, 1/3)$, where $k > 0$ is independent of λ.

We note that I2 and I3 imply that E_1, E_2, and E_3 are bounded independently of λ and hence that I1 implies that

$$E(\mathbf{f}_\lambda, \mathscr{B}) < E(\lambda \mathbf{x}, \mathscr{B})$$

for sufficiently small λ.

In order to prove that $\mathbf{f}_\lambda \in A_\lambda^p$ we first note that it is clear from (5.6) that $\{\mathbf{f}_\lambda\} \in L^\infty(\mathscr{B})$ and that $\mathbf{f}_\lambda(\mathbf{x}) = \lambda \mathbf{x}$ for $\mathbf{x} \in \partial\mathscr{B}$. In addition the definition (5.6) shows that ρ is continuous and piecewise differentiable. Thus, by Lemma 5.2, all we need to show is that ρ, as given by (5.6), satisfies

$$\hat{E} := \int_0^1 \left[|\dot{\rho}(P)|^p + 2 \left| \frac{\rho(P)}{P} \right|^p \right] P^2\, dP < +\infty. \tag{5.7}$$

Consider the function

$$\hat{W}(\mathbf{F}) = \hat{\Phi}(\lambda_1(\mathbf{F}), \lambda_2(\mathbf{F}), \lambda_3(\mathbf{F})) := \lambda_1^p + \lambda_2^p + \lambda_3^p.$$

Then, by (5.6), (5.7), and the changes of variables used in the first part of this proof we find that

$$\int_{\mathscr{B}} \hat{W}(\nabla \mathbf{f}_\lambda)\, d\mathbf{x} = 4\pi \hat{E} = 4\pi\lambda^3 [\hat{E}_1 + \hat{E}_2 + \hat{E}_3 + \hat{E}_4],$$

$$\hat{E}_1 := \int_1^\infty (w^{p(1-r)} + 2w^p)\, g_r(w)\, dw,$$

$$\hat{E}_2 := K \int_0^1 (w^{p(1-q)} + 2w^p)\, g_q(w)\, dw,$$

$$\hat{E}_3 := K \int_0^\lambda (v^{p(1-q)} + 2v^p)\, g_q(v)\, dv, \qquad \hat{E}_4 := \lambda^{-3} \int_{\hat{\lambda}A}^1 3\lambda^p P^2\, dP.$$

Thus $\hat{E}_1 < +\infty$ provided $p<3$ while $\hat{E}_2 < +\infty$ and $\hat{E}_3 < +\infty$ follow from $p<q/(q-1)$. □

Proof of Proposition 5.5. – If we cube (5.5), and then differentiate with respect to P we find that

$$\dot{\rho}(P)\frac{\rho(P)^2}{P^2} = \Delta, \qquad 0<P<1.$$

Thus by Lemma 5.2, (5.4), and the fact that δ has a strict global minimum at Δ we conclude that $\mathbf{f}_\lambda$, as given by (5.5), is a global minimizer of the total energy. It is clear that $\mathbf{f}_\lambda$ exhibits interpenetration of matter for $\lambda^3 < \Delta$ and cavitation for $\lambda^3 > \Delta$.

In order to complete the proof we must determine the values of p for which $\mathbf{f}_\lambda \in A_\lambda^p$. If $\lambda^3 > \Delta$ then $\dot{\rho}$ is bounded on (0,1) and hence it is clear from (5.2) and (5.5) that $1 \leqq p < 3$. If $\lambda^3 < \Delta$ then $\dot{\rho}$ is singular at $P^3 = (\Delta - \lambda^3)/\Delta$. In this case the changes of variables used in the proof of Proposition 5.3 can be used to show that $\mathbf{f}_\lambda \in A_\lambda^p$ for $1 \leqq p < 3/2$. □

ACKNOWLEDGEMENTS

R. D. J. acknowledges the support of the National Science Foundation and the Air Force Office of Scientific Research through NSF/DMS-8718881. S. J. S. acknowledges the support the Institute for Mathematics and its Applications, the National Science Foundation through NSF/DMS-8810653, and the Air Force Office of Scientific Research through AFOSR-880200.

REFERENCES

[1950] A. S. BESICOVITCH, Parametric Surfaces, *Bull. Amer. Math. Soc.*, Vol. **56**, pp. 228-296.

[1952] C. B. MORREY, Quasi-Convexity and the Lower Semicontinuity of Multiple Integrals, *Pacific J. Math.*, Vol. **2**, pp. 25-53.

[1958] A. N. GENT and P. B. LINDLEY, Internal Rupture of Bonded Rubber Cylinders in Tension, *Proc. Roy. Soc. London*, Vol. **A 249**, pp. 195-205.

[1965] N. G. MEYERS, Quasi-Convexity and the Lower Semi-Continuity of Multiple Variational Integrals of Any Order, *Trans. Amer. Math. Soc.*, Vol. **119**, pp. 125-149.

[1975] R. ADAMS, *Sobolev Spaces*, Academic Press.

[1977] J. M. BALL, Convexity Conditions and Existence Theorems in Nonlinear Elasticity, *Arch. Rat. Mech. Anal.*, Vol. **63**, pp. 337-403.

[1981] J. M. BALL, Global Invertibility of Sobolev Functions and the Interpenetration of Matter, *Proc. Royal Soc. Edinburgh*, Vol. **88 A**, pp. 315-328.

[1982] J. M. BALL, Discontinuous Equilibrium Solutions and Cavitation in Nonlinear Elasticity, *Phil. Trans. Roy. Soc. London*, Vol. **A 306**, pp. 557-611.

[1984] J. M. BALL and F. MURAT, $W^{1,p}$-Quasiconvexity and Variational Problems for Multiple Integrals, *J. Funct. Anal.*, Vol. **58**, pp. 225-253.

[1985] C. A. STUART, Radially Symmetric Cavitation for Hyperelastic Materials, *Ann. Inst. Henri Poincaré: Anal. non linéaire*, Vol. **2**, pp. 33-66.

[1986] C. O. HORGAN and R. ABEYARATNE, A Bifurcation Problem for a Compressible Nonlinearly Elastic Medium: Growth of a Microvoid, *J. Elasticity*, Vol. **16**, pp. 189-200.

P. PODIO-GUIDUGLI, G. VERGARA CAFFARELLI and E. G. VIRGA, Discontinuous Energy Minimizers in Nonlinear Elastostatics: an Example of J. Ball Revisited, *J. Elasticity*, Vol. **16**, pp. 75-96.

[1986 *a*] J. SIVALOGANATHAN, Uniqueness of Regular and Singular Equilibria for Spherically Symmetric Problems of Nonlinear Elasticity, *Arch. Rational Mech. Anal.*, Vol. **96**, pp. 97-136.

[1986 *b*] J. SIVALOGANATHAN, A Field Theory Approach to Stability of Radial Equilibria in Nonlinear Elasticity, *Math. Proc. Camb. Phil. Soc.*, Vol. **99**, pp. 589-604.

[1987] S. S. ANTMAN and P. V. NEGRON-MARRERO, The Remarkable Nature of Radially Symmetric Equilibrium States of Aeolotropic Nonlinearly Elastic Bodies, *J. Elasticity*, Vol. **18**, pp. 131-164.

J. M. BALL and R. D. JAMES, Fine Phase Mixtures as Minimizers of Energy, *Arch. Rational Mech. Anal*, Vol. **100**, pp. 13-52.

P. G. CIARLET and J. NEČAS, Injectivity and Self-Contact in Nonlinear Elasticity, *Arch. Rat. Mech. Anal.*, Vol. **97**, pp. 171-188.

[1988] P. G. CIARLET, *Mathematical Elasticity*, Vol. **I**: Three- Dimensional Elasticity, Amsterdam: North-Holland.

K. A. PERICAK-SPECTOR and S. J. SPECTOR, Nonuniqueness for a Hyperbolic System: Cavitation in Nonlinear Elastodynamics, *Arch. Rational Mech. Anal.*, Vol. **101**, pp. 293-317.

V. ŠVERÁK, Regularity Properties of Deformations with Finite Energy, *Arch. Rational Mech. Anal.*, Vol. **100**, pp. 105-127.

S. MÜLLER, Weak Continuity of Determinants and Nonlinear Elasticity, *C. R. Acad. Sci. Paris*, 307, Séries I, pp. 501-506.

[1989] M. GIAQUINTA, G. MODICA and J. SOUČEK, Cartesian Currents, weak Diffeomorphisms and Existence Theorems in Nonlinear Elasticity, *Arch. Ration. Mech. Anal.*, Vol. **106**, pp. 97-159.

P. MARCELLINI, The Stored Energy of Some Discontinuous Deformations in Nonlinear Elasticity, in Partial Differential Equations and the Calculus of Variations Volume **II**, Essays in Honor of Ennio De Giorgi, F. COLOMBINI *et al.* Ed., Birkhaüser, pp. 767-786.

[1989 *a*] M. S. CHOU-WANG and C. O. HORGAN, Cavitation in Nonlinear Elastodynamics for Neo-Hookean Materials, *Int. J. Engng. Sci.*, Vol. **27**, pp. 967-973.

[1989 *b*] M. S. CHOU-WANG and C. O. HORGAN, Void Nucleation and Growth for a Class of Incompressible Nonlinearly Elastic Materials, *Int. J. Solids Structures*, Vol. **25**, pp. 1239-1254.

[1989 *a*] C. O. HORGAN and T. PENCE, Void Nucleation in Tensile Dead-Loading of a Composite Incompressible Nonlinearly Elastic Sphere, *J. Elasticity*, Vol. **21**, pp. 61-82.

[1989 *b*] C. O. HORGAN and T. PENCE, Cavity Formation at the Center of a Composite Incompressible Nonlinearly Elastic Sphere, *J. Appl. Mech.*, Vol. **111**, pp. 302-308.

[1991] R. D. JAMES and S. J. SPECTOR, The Formation of Filamentary Voids in Solids, *J. Mech. Phys. Solids*, Vol. **39**, pp. 783-813.

(*Manuscript received February* 26, 1990;
revised May 13, 1991.)