The equilibrium and post-buckling behavior of an elastic curve governed by a non-convex energy

RICHARD D. JAMES

University of Minnesota, Department of Aerospace Engineering and Mechanics, 107 Aero, 110 Union St. S. E., Minneapolis, Minnesota 55455, USA

It has been argued elsewhere [1–5] that some of the features of anelastic deformation, change of phase, "necking", and yield are predicted by nonlinear elastic and thermoelastic theories. For the most part, the analyses have been confined to the simpler static nonlinear rod and bar theories, or to three-dimensional problems that can be treated by local analyses or that can be reduced to one dimension [6], since occasionally unfamiliar methods of nonlinear analysis are required. The results of these simple theories do, however, provide incentive for further investigation; in the simplest elastic bar theory designed to describe the cold drawing of polymers, even the positions and number of phase boundaries that form can be predicted.

Still, serious gaps remain in our understanding of the capability of nonlinear elasticity theory to describe such phenomena, and there remains some degree of inconsistency in the various approaches that have been followed. Two notable observations of certain importance have been ignored; first, the effect of surface energy of a phase boundary, and second, the observation that the symmetry of a body may change after the passage of a phase boundary, though the latter has been analyzed for second order phase transitions by Ericksen [7].[1]

Most of these investigations begin with the assumption that the free energy $W(\boldsymbol{F}, T)$ regarded as a function of the deformation gradient and temperature defined on a domain $S \times \mathbb{R}^1$, *fails to be elliptic in part of its domain*:

$$\frac{\partial^2 W}{\partial F^\alpha_i \partial F^\beta_j}(\boldsymbol{F}, T)a_i a_j b^\alpha b^\beta < 0 \qquad \forall \boldsymbol{F} \in S_1 \subset S$$

$$\forall \boldsymbol{a}, \boldsymbol{b} \in \mathbb{E}^3. \tag{0.1}$$

This kind of assumption permits stationary shock surfaces to exist in elastostatic solutions and slow moving large amplitude shocks to propagate in elastodynamic ones.

---

[1] In the present work I shall be concerned with transitions ordinarily designated as first order transitions.

In one dimensional theories the analogue of (0.1) is simply the failure of convexity of the stored energy function. Thus, in the theory of the elastica, in which the stored energy is a function only of the curvature of a plane curve, $W = W(\kappa)$, the requirement is that on some proper subdomain of the stored energy

$$\frac{d^2 W}{d\kappa^2}(\kappa) < 0. \tag{0.2}$$

A reasonable assumption would make (0.2) hold on each of a pair of intervals bounded away from, and separated by, $\kappa = 0$. On the remaining intervals (including an open set about $\kappa = 0$) the reversed inequality would hold. Fosdick and James [1] adopt this kind of assumption in the theory of the elastica and classify those configurations that can be produced by pure bending. When a certain value of the applied bending moment is reached, the configuration of minimum energy consists of arcs of circles of two different radii joined together with a continuously turning tangent. The jump in curvature across each discontinuity, as well as the critical bending moment, can be calculated from the constitutive function, whereas the portion of the curve having one or the other curvature is indeterminant. As a curious side issue, it was found that certain weak relative minimizers of the total energy (which, of course, must satisfy the Euler equation) do not satisfy the first integral, or "energy integral", of the Euler equation.

In the present work, I treat the problem of an elastica with a nonconvex energy function loaded by a dead load. I discuss in full detail only the traditional problem of an elastic curve with a fixed tangent at one end and a force parallel to this tangent at the other end. Most of the results of the analysis apply without change to the fixed-fixed, the fixed-pinned or the pinned-pinned elastica, since all of these problems lead to essentially the same expression for the total energy (equation (1.10)). The fixed-fixed elastica does, however, have certain configurations of equilibrium which I do not discuss in detail. All possible configurations of strong relative minimizers of the total energy must look qualitatively like parts of one of the graphs in Figure 3. To arrive at this picture, I must again confront possible failure of the energy integral. It is proved that an energy integral can be deduced for strong relative minimizers of the total energy. Each loop of the elastica associated with a strong relative minimizer must contain either two discontinuities or no discontinuities of the curvature. The appearance of this pair of discontinuities is governed by a surprisingly simple relation (equation (4.17)) involving the applied force, the angle of the elastic curve at the point where the force is applied, and a certain constitutive constant. The region along the curve between the discontinuities is a region of large curvature relative to the adjacent regions.

The traditional postbuckling problem is discussed in Section 6. The results are fundamentally different from those obtained in the classical theory ($W = c\kappa^2$, $c =$ const.) and, needless to say, from those obtained in the linearized version of the classical theory. Like the classical theory, I find that some columns buckle first at the Euler load, but then, as the load is increased, a discontinuity forms at the base of the column and moves up the column. The region between the discontinuity and the

base is a region of large curvature relative to the rest of the column. However, unlike the classical theory, I find that for some constitutive relations the column can buckle over abruptly at loads *below* the Euler buckling load. In these abruptly buckled configurations, the discontinuity can appear some distance up the column.

The region between the discontinuity and the base of the column is a region of large relative curvature and is associated with "plastic" or "permanent" deformation, or the formation of a new phase. The appearance of the discontinuity then represents true failure of the column, though this kind of "failure" is often desirable. I refer here to the isothermal behavior of certain shape-memory alloys in which just this permanent deformation is sought. A complete description of thin rods of shape memory alloy would require a thermodynamic analysis, though their isothermal behavior appears consistent with the description I shall present.

1. Theory of the elastica

In [1] the foundation of the theory of the elastica has been presented with a view toward deducing it from second grade elasticity theory. Building upon this foundation, I shall begin with the prescription of a **stored energy function**

$$W = W(\kappa), \qquad \kappa = \text{curvature}. \tag{1.1}$$

The **moment** corresponding to the curvature κ is then defined by[2]

$$M = \frac{dW}{d\kappa}(\kappa). \tag{1.2}$$

The variable $s \in [0, L]$ will always denote the arclength, and primes will indicate derivatives with respect to s. The curvature may be calculated in the following way. Given a continuously differentiable plane curve

$$\boldsymbol{x}(s) = x(s)\boldsymbol{i} + y(s)\boldsymbol{j}, \tag{1.3}$$

representing a **placement** of the elastica, we associate with it a function $\theta = \theta(\cdot)$ defined uniquely by the equations

$$\cos \theta(s) = x'(s), \qquad \sin \theta(s) = y'(s), \qquad 0 \leqq \theta(0) < 2\pi. \tag{1.4}$$

The value $\theta(s)$ represents the angle between $\boldsymbol{x}'(s)$ and $\boldsymbol{i}$, measured counterclockwise from $\boldsymbol{i}$. Then, the **curvature** at $\boldsymbol{x}(s)$ is defined by

$$\kappa = \theta'(s). \tag{1.5}$$

Conversely, the function $\theta(s)$ always determines a placement according to the

---

[2] Explicit smoothness assumptions will be deferred until the end of this section. Occasionally, we shall use the subscript κ to denote the derivative with respect to κ.

rules

$$x(s) = \int_0^s \cos \theta(r) \, dr + \text{const.}$$

$$y(s) = \int_0^s \sin \theta(r) \, dr + \text{const.}$$

(1.6)

We shall always assume that $x(0)$ or $x(L)$ is assigned, so that this determination is unique.

In the basic problem we shall consider, the elastica will be loaded at $x = x(L)$ by a **dead load**. A constant force $F = F_1 i + F_2 j$ is prescribed and the **total energy** is then defined by

$$E[\theta] = \int_0^L W(\theta'(s)) \, ds - F \cdot x(L).$$

(1.7)

We shall adjoin the side condition

$$\theta(0) = \theta_0 \qquad 0 \leq \theta_0 < 2\pi,$$

(1.8)

θ_0 being an assigned constant; for the classical postbuckling problem $\theta_0 = 0$ and $F_2 = 0$. In (1.7) the first term is interpreted as the energy stored in the elastica, and the second is the energy of the dead loading device. We interpret (1.7) and (1.8) as delivering the total energy of an elastic ribbon or thin rod, fixed in position and tangent at one end, and loaded at the other with a constant force.

Leaving momentarily aside precise conditions of smoothness and the exact definition of the competing class of functions, my aim is this: *to find the function $\theta(s)$ which minimizes $E[\cdot]$ subject to the side condition $\theta(0) = \theta_0$.* In particular, I shall be concerned with *relative minimizers* which are defined precisely at the beginning of Section 3. According to the usual interpretation of the energy criterion for stability, the functions we seek deliver stable or, in the case of relative minimizers, metastable configurations of the fixed-free elastica. I shall not attempt to connect this interpretation of stability with criteria for dynamic stability.

By using the definitions $(1.6)_{1,2}$ and assuming $x(0) = 0$, equations (1.7) and (1.8) may be written

$$E[\theta] = \int_0^L \{W(\theta'(s)) - F_1 \cos \theta(s) - F_2 \sin \theta(s)\} \, ds$$

$$\theta(0) = \theta_0.$$

(1.9)

It is always possible to rotate the elastica about $x = 0$, use the condition that the stored energy is Galilean invariant and thereby obtain a new problem which is equivalent to the old one in a sense presently to be made definite. Let ψ be a constant direction satisfying $F_1 \sin \psi = F_2 \cos \psi$, and define $R \equiv -(F_1 \cos \psi + F_2 \sin \psi)$, $\hat{E}[\theta] \equiv E[\theta + \psi]$, $\hat{\theta}_0 \equiv \theta_0 + \psi$. Then (1.9) becomes

$$\hat{E}[\theta] = \int_0^L \{W(\theta'(s)) + R \cos \theta(s)\} \, ds.$$
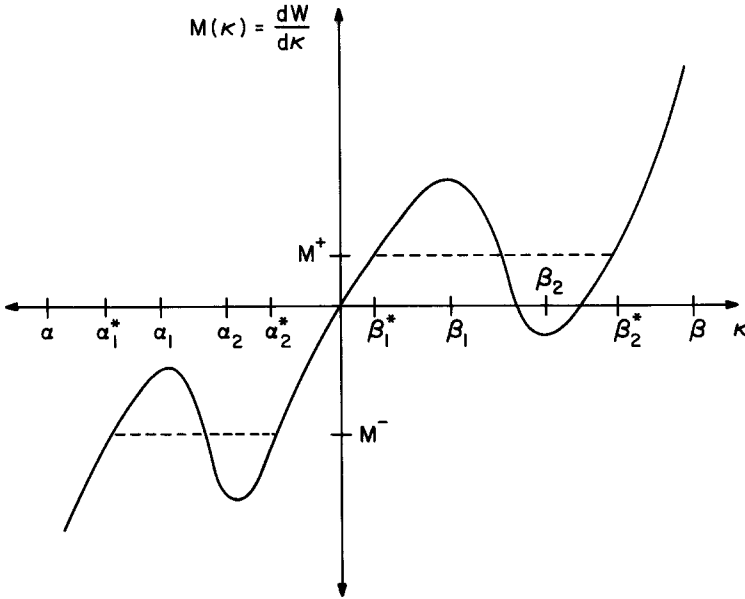
$$\theta(0) = \hat{\theta}_0$$

(1.10)

Figure 1. The moment-curvature relation.

Any minimizer (relative, weak or strong[3]) of (1.9) is also a minimizer (relative, weak or strong) of (1.10), and *vice versa*. Of course, ψ can always be chosen such that $0 \leqq \hat{\theta}_0 < 2\pi$.

We shall confine attention to a certain class of non-convex stored energy functions. Let constants $\alpha < \alpha_1 < \alpha_2 < 0 < \beta_1 < \beta_2 < \beta$ be prescribed, and assume $W(\kappa)$ satisfies the requirements,

$$W : [\alpha, \beta] \to \mathbb{R}^1,$$

$$W \in C^2[\alpha, \beta],$$

$$W_{\kappa\kappa} \begin{cases} > 0 & \kappa \in (\alpha, \alpha_1) \cup (\alpha_2, \beta_1) \cup (\beta_2, \beta), \\ < 0 & \kappa \in (\alpha_1, \alpha_2) \cup (\beta_1, \beta_2), \end{cases} \tag{1.11}$$

$$W_{\kappa}(\beta) \geqq W_{\kappa}(\beta_1), \qquad W_{\kappa}(\alpha) \leqq W_{\kappa}(\alpha_2),$$

$$W(0) = 0.$$

Figure 1 shows an example of $M(\kappa) = dW/d\kappa$ *versus* κ. The two dotted lines are called **Maxwell lines**; they cut off equal areas of the curve above and below. The points M^+ and M^- are the ordinates corresponding to the Maxwell lines. We shall also assume that the constitutive function is defined such that

$$M^+ > 0 > M^-. \tag{1.12}$$

The Maxwell lines have an important significance for the mathematical analysis

---

[3] Cf. equations (3.2) through (3.5) for precise definitions.

which shall follow. They delineate a certain domain

$$\mathcal{D} = [\alpha, \alpha_1^*] \cup [\alpha_2^*, \beta_1^*] \cup [\beta_2^*, \beta], \tag{1.13}$$

pictured in Figure 1. All points in $\mathcal{D}$ have the property that they are points of convexity of the function W. Therefore, we shall term $\mathcal{D}$ the **domain of convexity**. The proofs of these remarks are contained in [5].

When permitting nonconvex stored energy functions, it is to be expected[4] that minimizers (or relative minimizers) $\theta(s)$ of (1.10) will not have everywhere continuous derivatives. Hence, we shall assume that $\theta(s)$ is only absolutely continuous and that

$$\alpha \leqq \theta'(s) \leqq \beta \tag{1.14}$$

holds at almost every s in $[0, L]$. Thus, the basic function space of this problem will be

$$\mathcal{F}(\theta_0) = \{\theta : [0, L] \rightarrow \mathbb{R}^1 \mid \theta(0) = \theta_0, \; \theta \text{ is absolutely}$$
$$\text{continuous, and } \alpha \leqq \theta'(s) \leqq \beta \text{ on } [0, L]\}. \tag{1.15}$$

A placement of the elastic curve delivered by a function in $\mathcal{F}(\theta_0)$ always has a continuous tangent vector, though the curvature may be discontinuous. It is preferable to use the class of functions $\mathcal{F}(\theta_0)$ given by (1.15), rather than just the set of piecewise differentiable functions, since it is in $\mathcal{F}(\theta_0)$ that one should expect to be able to prove existence of minimizers of $\hat{E}[\theta]$ (see [10] for a prototype of this kind of proof).

We shall think of points s at which $\theta'(s)$ exists and belongs to the set $[\alpha, \alpha_1] \cup [\beta_2, \beta]$ as points in a different **phase** from points where $\theta'(s)$ belongs to $[\alpha_2, \beta_1]$. In this sense two phases are represented in the graph of Figure 1, one of large curvature and one of small curvature. Note that points of the two different phases may correspond to the same value of the moment.

2. Related problems

Though we shall treat in detail only the classical "fixed-free" postbuckling problem, two other problems can be framed in a similar way. If both ends of the elastica are pinned, the loading device contributes no energy, so the total energy is calculated according to the rule

$$E_P = \int_0^L W(\theta'(s)) \, ds, \tag{2.1}$$

to which is adjoined the constraint

$$|\boldsymbol{x}(L)| = d, \tag{2.2}$$

---

[4] See [1], [2], [3] or [5]. James [5] has given arguments in support of the choice $\mathcal{F}(\theta_0)$ as the underlying function space in a related problem.

d being the assigned distance between the pins. By appeal to Galilean invariance, we can replace (2.2) by

$$x(L) = d,$$
$$y(L) = 0,$$

(2.3)

or alternatively,

$$\int_0^L \cos \theta(s) \, ds = d,$$

$$\int_0^L \sin \theta(s) \, ds = 0.$$

(2.4)

It is customary to remove the constraints by using Lagrange multipliers. Let Lagrange multipliers λ_1, λ_2 be associated with the constraints $(2.4)_1$ and $(2.4)_2$, respectively. If we seek minimizers, or relative minimizers, of

$$\bar{E}_P = \int_0^L \{ W(\theta'(s)) - \lambda_1 \cos \theta - \lambda_2 \sin \theta \} \, ds,$$

(2.5)

from among all $\theta \in \mathscr{F}(\cdot)$, and we find one which satisfies the constraint, then it will also be a minimizer or relative minimizer of the energy E_P under the constraint (2.4). It is not at all clear, however, that every relative minimizer of the constrained problem can be obtained in this way, especially when the minimizers are not smooth. Nevertheless, the functional (2.5) is the same as the functional (1.9), so every property of minimizers of the fixed-free problem will also be shared by minimizers of the functional $\bar{E}_p$. As far as I know, the equivalence between the constrained problem and the associated problem with Lagrange multipliers is only formal.

There is, however, a rigorous connection between the "fixed-free" and the "pinned-pinned" problems. It can be shown directly that strong relative minimizers (see equations (3.2) and (3.3) for the precise definition) of the energy E_p under the constraint (2.3) satisfy the Euler equation (3.6) and the condition of convexity (3.7). We shall show in the next section that (3.6) and (3.7), in turn, are sufficient for the validity of the energy integral. In Section 4 we shall make a complete study of functions which satisfy the energy integral, unrestricted by side conditions, so the analysis will apply also to the constrained problem. It is from the energy integral that we develop a detailed picture of placements of the elastica.

In the "fixed-pinned" problem, we wish to minimize the functional E_p subject not only to the constraint (2.3) but also to the condition

$$\theta(0) = \theta_0.$$

(2.6)

The "fixed-fixed" problem adjoins to the problem just mentioned the additional condition

$$\theta(L) = \theta_L.$$

(2.7)

The remarks we have made about the "pinned-pinned" problem apply also to the

"fixed-pinned" problem. The "fixed-fixed" elastica, however, may equilibrate in certain placements, derived from non-inflectional solutions of the energy integral (Section 4), which I shall not discuss.

3. The energy integral

We seek a proof of the energy integral for strong relative minimizers of the total energy

$$E[\theta] = \int_0^L \{W(\theta'(s)) + R \cos \theta(s)\} \, ds. \tag{3.1}$$

A function $\tilde{\theta} \in \mathscr{F}(\theta_0)$ is a **strong relative minimizer** of $E[\cdot]$ if for some $\varepsilon > 0$

$$E[\tilde{\theta}] \leqq E[\theta], \tag{3.2}$$

whenever $\theta \in \mathscr{F}(\theta_0)$ and

$$\max_{s \in [0,L]} |\theta(s) - \tilde{\theta}(s)| < \varepsilon. \tag{3.3}$$

We shall also require that $\tilde{\theta}'$ lie in the interior of the domain of W; for some sufficiently small $\delta > 0$,

$$\alpha + \delta \leqq \tilde{\theta}'(s) \leqq \beta - \delta. \tag{3.4}$$

When (3.4) is not satisfied, certain difficulties arise which have been discussed in [5]; in particular, the familiar Gâteaux variation $\theta'(s) + \xi \eta'(s)$ is not necessarily contained in the domain of $W(\cdot)$ for ξ sufficiently small unless (3.4) is met. A function $\tilde{\theta}$ is a **weak relative minimizer** of $E[\cdot]$ if (3.2) holds for all $\theta \in \mathscr{F}(\theta_0)$ such that

$$\sup_{s \in [0,L]} |\theta'(s) - \tilde{\theta}'(s)| < \varepsilon, \tag{3.5}$$

for some preassigned $\varepsilon > 0$.

If $\tilde{\theta}$ is a strong relative minimizer satisfying (3.4), then two conditions must be fulfilled:

1. the *Euler equation* (or balance of moments)

$$\frac{dW}{d\kappa}(\tilde{\theta}'(s)) \doteq R \int_s^L \sin \tilde{\theta}(r) \, dr \quad \text{a.e.,} \tag{3.6}$$

2. *convexity at $\tilde{\theta}'(s)$* – for almost every $s \in [0, L]$,

$$W(\kappa) \text{ is convex at } \kappa = \tilde{\theta}'(s).^5 \tag{3.7}$$

For the validity of the Euler equation it is sufficient that $\tilde{\theta}$ be only a weak relative

---

[5] For the proofs of these statements see [5]. If $\tilde{\theta}$ is a weak relative minimizer, $W(\kappa)$ is only locally convex at $\kappa = \tilde{\theta}'(s)$.

minimizer; however, neither the convexity condition nor the energy integral (equation (3.27)) need follow for weak relative minimizers. The condition of convexity is equivalent to the familiar Weierstrass condition: for almost every fixed $s \in [0, L]$,

$$W(\tilde{\theta}'(s) + \mu) - W(\tilde{\theta}'(s)) - \mu \frac{dW}{d\kappa}(\tilde{\theta}'(s)) \geqq 0$$

(3.8)

$\forall \mu$ such that $\alpha \leqq \tilde{\theta}'(s) + \mu \leqq \beta$.

The condition (3.7) is also equivalent to the restriction $\tilde{\theta}'(s) \in \mathcal{D}$ a.e., $\mathcal{D}$ being defined by (1.13).

Our procedure will be the following: assume the existence of solutions of (3.6) and (3.7)[6] show that these solutions have only a finite number of discontinuities of the curvature, establish that they satisfy the energy integral, and prove that certain of them are strong relative minimizers. Along the way, we shall develop a detailed picture of the solutions and explicit formulae for the positions and number of discontinuities which occur.

To begin we assume $\tilde{\theta}(s) \in \mathcal{F}(\tilde{\theta}_0)$ satisfies the Euler equation (3.6) and the restriction (3.7).[7] $\tilde{\theta}'(s)$ exists only almost everywhere but we may adjust its value on a set of measure zero, and thereby not change the function $\tilde{\theta}(s)$, so that (3.6) holds everywhere. We do this in the following manner. Give $\tilde{\theta}'(s)$ any values in $[\alpha, \beta]$ on the set where it does not exist, so it is then defined everywhere. Now suppose (3.6) does not hold at a point $\bar{s}$. There is a sequence of points $\{s_m\}_{m=1}^{\infty}$ such that

(i) $s_m \to \bar{s}$ as $m \to \infty$, and
(ii) (3.6) holds at s_m, $m = 1, 2, 3, \ldots$

Since the range of $dW/d\kappa(\cdot)$ is closed, there is a value $\bar{\kappa}$ in $[\alpha, \beta]$ such that if $\tilde{\theta}' = \bar{\kappa}$, and if the right hand side of (3.6) is evaluated at $\bar{s}$, then (3.6) is satisfied. We simply redefine

$$\tilde{\theta}'(\bar{s}) \equiv \bar{\kappa}$$

(3.9)

Having carried out this procedure for each such point $\bar{s}$, *we may assume* (3.6) *holds everywhere.*

The Weierstrass condition also holds only almost everywhere. Suppose it does not hold at $\bar{\bar{s}}$. Then, $\bar{\bar{\kappa}} = \tilde{\theta}'(\bar{\bar{s}})$ does not belong to the domain of convexity of W, that is, $\alpha_1^* < \bar{\bar{\kappa}} < \alpha_2^*$ or $\beta_1^* < \bar{\bar{\kappa}} < \beta_2^*$. Therefore, at $\bar{\bar{\kappa}}$, $W_\kappa(\cdot)$ has a triple valued inverse; in fact, there is always a value $\hat{\kappa}$ such that

$$\frac{dW}{d\kappa}(\bar{\bar{\kappa}}) = \frac{dW}{d\kappa}(\hat{\kappa}),$$

(3.10)

and $\hat{\kappa}$ lies in the domain of convexity of W. We redefine

$$\tilde{\theta}'(\bar{\bar{s}}) \equiv \hat{\kappa}.$$

(3.11)

---

[6] In Section 4 we shall prove the existence of such solutions, and in Section 5 we shall explicitly construct those for which $\tilde{\theta}_0 = 0$.
[7] See footnote p. 246.

Because of (3.10) *this redefinition will not alter* (3.6), *which will still hold everywhere.* In this whole process of redefinition we have only changed $\tilde{\theta}'(s)$ on a set of measure zero, so $\tilde{\theta}(s)$ remains unaltered. In a sense, we have removed all the bogus singularities.

From now on we assume that (3.6) and (3.7) hold at every $s \in [0, L]$. The restriction (3.7) now holds at every s so that $\tilde{\theta}'(s)$ is contained in the domain of convexity of W for every $s \in [0, L]$. Let us denote

$$\tilde{M}(s) \equiv \frac{dW}{d\kappa}(\tilde{\theta}'(s)). \tag{3.12}$$

The Euler equation (3.6) implies that $\tilde{M}(s)$ is continuously differentiable. Suppose $\tilde{M}(s_0) \neq M^+$ or M^-. Then there is a neighborhood of values of s near s_0 where $\tilde{M}(s) \neq M^+$ or M^-. When s lies in this neighborhood, $W_\kappa(\cdot)$ is uniquely invertible on the domain of convexity of W:

$$\tilde{\theta}'(s) = W_\kappa^{-1}\left(R \int_s^L \sin \tilde{\theta}(r) \, dr \right). \tag{3.13}$$

Therefore we have

LEMMA 1. *If* $W_\kappa(\tilde{\theta}'(s_0)) \neq M^+$ *or* M^-, $\tilde{\theta}(s)$ *is twice continuously differentiable in a neighborhood of* s_0.

Now we wish to show that $\tilde{\theta}'(s)$ can only have a finite number of discontinuities in the bounded interval $[0, L]$. Suppose not; Then $\tilde{\theta}'(s)$ has an infinite number of discontinuities at points $s_1, s_2, s_3, \ldots$, all contained in $[0, L]$. From Lemma 1 it follows that

$$\tilde{M}(s_n) = M_+ \text{ or } M_-, \qquad n = 1, 2, 3, \ldots \tag{3.14}$$

Since $[0, L]$ is compact, there is a subsequence $\{s_{n_l}\}_{l=1}^\infty$ which converges: $s_{n_l} \to s_0$ as $l \to \infty$. We may assume by starting sufficiently far out in the subsequence that

$$\begin{aligned} \tilde{M}(s_{n_l}) &= M^+, \quad \text{or} \\ \tilde{M}(s_{n_l}) &= M^-, \end{aligned} \qquad l = N, N+1, N+2, \ldots \tag{3.15}$$

In the case $(3.15)_1$, since $M^+ > 0$ and $M(\cdot) \in C^1_{[0,L]}$, $\tilde{\theta}'(s)$ must be bounded away from zero:

$$\tilde{\theta}'(s) > K > 0, \qquad K = \text{const.} > 0, \quad s \text{ near } s_0. \tag{3.16}$$

On the other hand, by Rolle's theorem applied to $(3.15)_1$,

$$\tilde{M}'(s_{n_l}^*) = 0 \text{ for some } s_{n_l}^* \in [s_{n_l}, s_{n_{l+1}}], \tag{3.17}$$

which, upon differentiation of (3.6), implies that

$$R \sin \tilde{\theta}(s_{n_l}^*) = 0. \tag{3.18}$$

If $R = 0$, (3.6) has the unique solution $\tilde{\theta}(s) = \text{const.} = \tilde{\theta}_0$. If $R \neq 0$, we have for l

sufficiently large,

$$\tilde{\theta}(s_{n_l}^*) = \frac{\pi}{2} + m\pi, \qquad l = N, N+1, N+2, \ldots, \tag{3.19}$$

m being some fixed integer. But (3.19) and (3.16) contradict each other. If $(3.15)_2$ holds we are led to the same conclusion with $\tilde{\theta}'(s) < -K$. Thus, summarizing this argument and related facts, we have

LEMMA 2. *$\tilde{\theta}'(s)$ has only a finite number of discontinuities in $[0, L]$. If $\tilde{\theta}'(s)$ has a discontinuity at s_0, the limiting values $\tilde{\theta}'(s_0 + 0)$ and $\tilde{\theta}'(s_0 - 0)$ exist and,*

$$\frac{dW}{d\kappa}(\tilde{\theta}'(s_0 + 0)) = \frac{dW}{d\kappa}(\tilde{\theta}'(s_0 - 0)) = M^+ \text{ or } M^-. \tag{3.20}$$

If $W_\kappa(\tilde{\theta}'(s_0 + 0)) = M^+$, then

$$\begin{aligned}
\tilde{\theta}'(s_0 + 0) &= \beta_2^* (\text{or } \beta_1^*) \\
\theta'(s_0 - 0) &= \beta_1^* (\text{or } \beta_2^*, \text{ respectively}).
\end{aligned} \tag{3.21}$$

Otherwise $W_\kappa(\tilde{\theta}'(s_0 + 0)) = M^-$ and

$$\begin{aligned}
\tilde{\theta}'(s_0 + 0) &= \alpha_2^* \ (\text{or } \alpha_1^*) \\
\tilde{\theta}'(s_0 - 0) &= \alpha_1^* \ (\text{or } \alpha_2^*, \text{ respectively}).
\end{aligned} \tag{3.22}$$

Between each consecutive pair of points of discontinuity of $\tilde{\theta}'(s)$, the energy integral is satisfied by a classical argument. Hence, if s_0 and s_1 are two such consecutive points,

$$W(\tilde{\theta}'(s)) - \tilde{\theta}'(s)W_\kappa(\tilde{\theta}'(s)) - R\cos\tilde{\theta}(s) = c_1 = \text{const.}, \qquad s \in (s_0, s_1). \tag{3.23}$$

On the next interval of continuous differentiability of $\tilde{\theta}$, say (s_1, s_2), it follows by the same argument that

$$W(\tilde{\theta}'(s)) - \tilde{\theta}'(s)W_\kappa(\tilde{\theta}'(s)) - R\cos\tilde{\theta}(s) = c_2 = \text{const.}, \qquad s \in (s_1, s_2) \tag{3.24}$$

If we evaluate (3.23) at $s_1 - 0$ and (3.24) at $s_1 + 0$, use the continuity of $\tilde{\theta}(\cdot)$ and $W_\kappa(\tilde{\theta}'(\cdot))$, and subtract (3.23) from (3.24), we obtain

$$W(\tilde{\theta}'(s_1 + 0)) - W(\tilde{\theta}'(s_1 - 0)) - (\tilde{\theta}'(s_1 + 0) - \tilde{\theta}'(s_1 - 0))W_\kappa(\tilde{\theta}'(s_1)) = c_2 - c_1. \tag{3.25}$$

Assume $W_\kappa(\tilde{\theta}'(s_1)) = M^+$ and $\tilde{\theta}'(s_1 + 0) = \beta_2^*$. Then, (3.25) becomes

$$W(\beta_2^*) - W(\beta_1^*) - (\beta_2^* - \beta_1^*)M^+ = c_2 - c_1. \tag{3.26}$$

However, the signed area under the moment curvature relation in Figure 1 between β_1^* and β_2^* is zero; thus, the left hand side of (3.26) is zero, so $c_1 = c_2$. In other cases not covered by the assumption given just before (3.26) the same conclusion is reached. The argument is partially reversible; if $\tilde{\theta}(\cdot)$ is continuous and has piecewise continuous first and second derivatives, if $W_\kappa(\tilde{\theta}'(\cdot))$ is continuous and vanishes at $s = L$, and if $\tilde{\theta}'(\cdot)$ vanishes only on a set of measure zero, then $\tilde{\theta}$ satisfies the Euler

equation. The proof of this statement follows simply by differentiation of the energy integral. Thus, we have

THEOREM 1. *Suppose* $\tilde{\theta}(s)$ *satisfies the Euler equation* (3.6) *and the condition of convexity* (3.7) *at every* $s \in [0, L]$. *Then, the energy integral holds*:

$$W(\tilde{\theta}'(s)) - \tilde{\theta}'(s) \frac{dW}{d\kappa}(\tilde{\theta}'(s)) + R \cos \tilde{\theta}(s) = c = \text{const.}, \qquad s \in [0, L]. \tag{3.27}$$

Therefore, if $\tilde{\theta}$ *is a strong relative minimizer of* $E[\cdot]$ *in the class* $\mathcal{F}(\theta_0)$, *then* (3.27) *is satisfied.*

Conversely, suppose the energy integral (3.27) *is satisfied by a continuous, once and twice piecewise continuously differentiable function* $\theta(\cdot)$ *such that*

(i) $W_\kappa(\theta'(\cdot))$ *is continuous,*
(ii) $W_\kappa(\theta'(L)) = 0$,
(iii) $\theta' = 0$ *on at most a set of measure zero.* $\tag{3.28}$

Then, the Euler equation (3.6) *holds for* $\theta(\cdot)$.

The advantage of the energy integral is that it transforms the Euler equation to a standard initial value problem. In the next section we shall find and characterize solutions $\tilde{\theta}$ of the energy integral such that $\tilde{\theta}'(s)$ lies in the domain of convexity of W. Thus, we shall obtain a characterization of strong relative minimizers. As discussed in [1], weak relative minimizers need not satisfy (3.27), even though they satisfy the Euler equation. This kind of "failure" of weak relative minimizers is brought about by the failure of convexity of the stored energy function.

Incidentally, I see no way of proving the energy integral (3.27) directly from (3.6) and (3.7) without first showing that there are only a finite number of discontinuities of the curvature.

4. Structure of solutions of the energy integral

We now shall find and characterize solutions of the Euler equation (3.6) which also fulfill the condition of convexity (3.7). If there is such a solution θ, then it must satisfy

$$\theta'(L) = 0 \tag{4.1}$$

according to (3.6). The condition (4.1) distinguishes the several problems mentioned in Section 2. Both the dead loaded elastica and the one pinned at $s = L$ will yield (4.1), but the elastica with tangents fixed at both ends will generally not. The condition (4.1) distinguishes the *inflectional elastica*, which always contains a zero of θ', from the *noninflectional elastica* which contains no zero of θ'. We shall study in detail only the inflectional elastica; it is easy to construct the non-inflectional solution from the methods below.

Each function which is a minimizer of the total energy will satisfy an assigned initial condition $\theta(0) = \theta_0$. To classify all possible solutions of (3.6) and (3.7) we find it easier to disregard this initial condition and, instead, to classify solutions according to their terminal directions $\theta(L)$.

Thus, let $\theta(L) = \theta_L$, and define

$$f(\kappa) \equiv \kappa \frac{dW}{d\kappa} - W. \tag{4.2}$$

The energy integral (3.27) can be written

$$f(\theta'(s)) = R(\cos \theta(s) - \cos \theta_L), \qquad s \in [0, L]. \tag{4.3}$$

To obtain (4.3) we have evaluated the constant c in (3.27) by using (4.1), so the form (4.3) is only valid for the inflectional elastica.

From (4.2) and $(1.11)_4$, we deduce that $f(0) = 0$; from (3.8) it follows that if κ belongs to the domain of convexity of W,

$$f(\kappa) = W(0) - W(\kappa) - (0 - \kappa)W_\kappa(\kappa) \geqq 0. \tag{4.4}$$

Furthermore,

$$f_\kappa = \kappa W_{\kappa\kappa} \tag{4.5}$$

and

$$\begin{aligned}
f(\beta_2^*) - f(\beta_1^*) &= W(\beta_1^*) - W(\beta_2^*) - (\beta_1^* - \beta_2^*)M^+ = 0, \\
f(\alpha_2^*) - f(\alpha_1^*) &= W(\alpha_1^*) - W(\alpha_2^*) - (\alpha_1^* - \alpha_2^*)M^- = 0.
\end{aligned} \tag{4.6}$$

From (1.11), (4.4), (4.5), (4.6) and the requirement $f(0) = 0$ which follows from (4.2) and (1.11) we build up the graph of $f(\kappa)$ shown in Figure 2. It will be convenient to
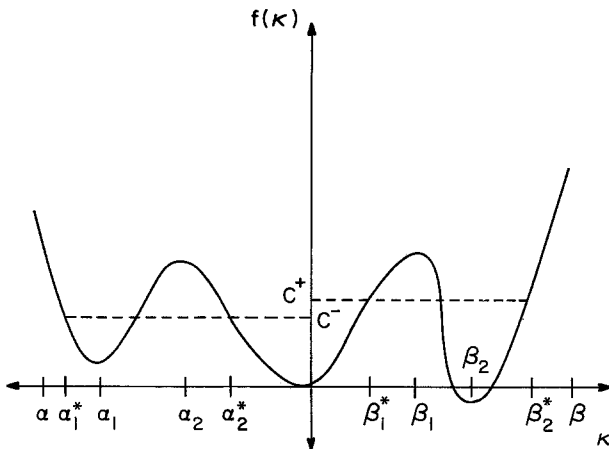

Figure 2. $f(\kappa) = \kappa W_\kappa - W$.

call

$$C^+ \equiv f(\beta_1^*) \qquad C^- \equiv f(\alpha_1^*).$$

We shall obtain a picture of the placement $x(s)$ associated with $\theta(s)$ by solving (4.3) backward from $s = L$. If $R = 0$, we have the solution $\theta = 0$, which is the only solution of (4.3) if we restrict θ' to lie on the domain of convexity of W. Now assume $R \neq 0$.

Near $s = L$, $\tilde{\theta} \in C^2$ by Lemma 1 and (4.1). Thus we may differentiate (3.6) to determine the behavior of $\tilde{\theta}$ near $s = L$:

$$W_{\kappa\kappa}(\theta'(s))\theta''(s) = -R \sin \theta(s)$$

$$\theta'(L) = 0 \tag{4.8}$$

$$\theta(L) = \theta_L.$$

The only solution of (3.6) possible when $\theta_L = n\pi$, $n = $ integer, is the constant solution $\theta(s) = \theta_L$. That is, any solution $\theta(s)$ of (3.6) must satisfy (4.8) on some neighborhood $\mathcal{N}$ of L. But $\theta(s) = n\pi$ is the unique solution of $(4.8)_{1,2}$ in $\mathcal{N}$ under the condition $\theta(L) = n\pi$. Suppose $\theta(s) = n\pi$ continues to hold on the *maximal* closed interval $s \in [a, L]$; the solution again being C^2 in a neighborhood of $s = a$ we may apply again the same argument at $s = a$ and conclude that $\theta(s) = n\pi$ in a neighborhood of a. But this statement contradicts the assumption that $[a, L]$ is maximal. Thus, $\theta(s) = n\pi$ is the unique solution of (3.6) on $[0, L]$ which satisfies $\theta_L = n\pi$.

Having dispensed with this case, we assume from now on that

$$\theta_L \neq n\pi, \qquad n = \text{any integer}. \tag{4.9}$$

We shall also assume, without loss of generality,[8] that

$$0 < \theta_L < \pi. \tag{4.10}$$

Then, it follows from (4.8) that in some neighborhood $[a, L]$ of L

1. $R > 0 \Rightarrow \theta'$ strictly monotone decreasing, $\tag{4.11}$

2. $R < 0 \Rightarrow \theta'$ strictly monotone increasing, $\tag{4.12}$

and within this neighborhood the solution of (4.8) is unique. We treat separately the two possible cases.

Case A. $R > 0$

Beginning at the point a, we continue to construct $\theta(s)$ by solving the energy integral (4.3). Since by (4.11) and $(4.8)_3$ $\theta'(a) > 0$, the energy integral can be regarded as a standard initial value problem with initial value $\theta(a)$, which we begin to solve backward from $s = a$. By a simple application of the comparison theorem for ordinary differential equations to (4.3) [see 11, Corollary 4.2], we see that as s decreases from a, $\theta'(s)$ increases and $\theta(s)$ decreases. We continue solving for $\theta(s)$ in

---

[8] The assertion of "no loss of generality" is not plain at this point. It will become clear that solutions with other end positions are either covered by the change $R \to -R$ or contradict the assumption $0 \leq \theta_0 < 2\pi$.

this fashion until one of the following occurs:

(i) $\theta(s)$ reaches zero, or

(ii) $R(\cos\theta - \cos\theta_L)$ reaches C^+.

Suppose case (ii) occurs before case (i) at a point $\bar{s}$, i.e.

$$R(\cos\theta(\bar{s}) - \cos\theta_L) = C^+, \qquad \theta(\bar{s}) > 0.$$

We cannot continue solving the energy integral in the ordinary sense for $s < \bar{s}$ because $\theta'(s)$ would then no longer belong to the domain of convexity of W, and so our solution would violate (3.7). However, if we pose the initial value problem (4.3), with $\theta(\bar{s})$ obtained from the solution already constructed but with $\theta'(s) \in [\beta_2^*, \beta]$, we may continue[9] solving (4.3) until $\theta = 0$ is reached; assume this occurs at the point s_0:

$$\theta(s_0) = 0. \tag{4.14}$$

Equation (4.14) now coincides with condition (i) above, so by continuing from (4.14) we shall cover the other case. For $s < s_0$, define

$$\theta(s) \equiv -\theta(2s_0 - s). \tag{4.15}$$

Extended in this fashion, $\theta(s)$ will continue to satisfy (4.3) until θ reaches $-\theta_L$ at $s = 2s_0 - L$.

The function $\theta(s)$, $s \in [2s_0 - L, L]$, constructed in this manner is the unique solution of (3.6) and (3.7) in the class of absolutely continuous functions which satisfy $\theta(L) = \theta_L$, $0 < \theta_L < \pi$. That is, there is only one solution to the initial value problem (4.8) in this class when s is near L. Having shown $\theta'(a) > 0$, we see that there is a unique solution of the energy integral until (4.13) or (4.14) occurs. If (4.13) occurs, we use the continuity of θ to pose a new initial value problem, which again has a unique solution if θ' is restricted to belong to the domain of convexity of W. The extension of θ via (4.15) is also unique; it solves a well posed initial value problem until (4.13) occurs again. At this point, by accounting for (3.7), we begin a new initial value problem which has a unique solution.

This completes the construction of $\theta(s)$ for one "loop" of the elastica when $R > 0$. I shall omit the similar analysis for succeeding loops. Several examples of placements of the elastica are pictured in Figure 3. In the second and third of them I have pictured the situation in which $R(\cos\theta - \cos\theta_L)$ reaches C^+ before θ reaches zero; thus each loop pictured below the x-axis contains exactly two discontinuities of the curvature. Each loop above(below) the x-axis is a translate of every other one below(above) the x-axis, but the loops below the axis are not necessarily mirror images of the ones above the axis unless, of course, the stored energy function is even. In fact, it is possible to construct solutions for certain non-even energy functions in which the loops above the axis contain two discontinuities and the loops below contain none. As R is increased, θ_L being held fixed, the arclength between

---

[9] I tacitly assume here that β is large enough, i.e. the domain of the stored energy is large enough, so that this can be done. Hence, I ignore questions of non-existence which can arise when the solution hits the boundary of the domain of the constitutive function.

A. R > 0, θ_L small.
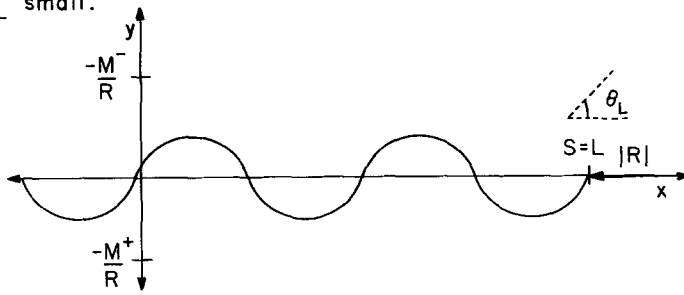
B. R > 0,
   $R(1 - \cos \theta_L) >$
   C^+ and C^-.
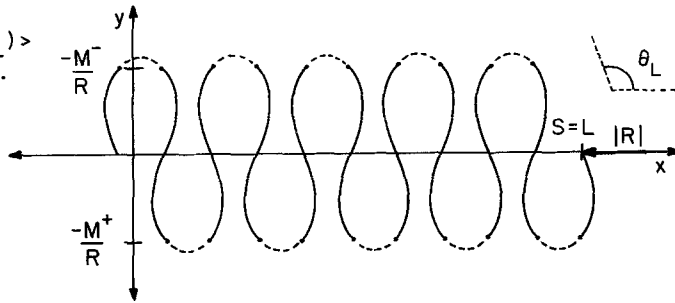
C. R < 0,
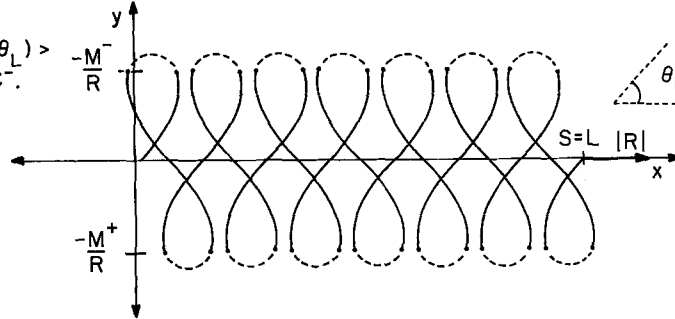 $R(1 - \cos \theta_L) >$
   C^+ and C^-.

Figure 3. Examples of possible placements of the inflectional elastica. Dots indicate jump discontinuities of the curvature and the solid and the dashed lines distinguish the two phases.

the discontinuities likewise increases. The discontinuities on the set of loops lying below(above) the x-axis all fall on the horizontal line

$$y = \frac{-M^+}{R} \left(= \frac{-M^-}{R} \right), \qquad (4.16)$$

a relation which follows immediately from (3.6) evaluated at the point of discontinuity, equation (1.6), and the condition $y(L) = 0$. Equation (4.16) appears to provide the easiest way to determine the important constants M^+ and M^- from experiment. In the sense indicated by (4.16), *the discontinuity always forms so that* M^+ *(or* M^-*), which depends on the constitutive equation alone, is the moment of the force R acting on the point of discontinuity.*

We emphasize again that all of the results of this section, since they are consequences of the energy integral and the condition of convexity, need not hold for weak relative minimizers of the total energy. Hence, weak relative minimizers need not have only two discontinuities on each loop and need not lead to (4.16). In fact, examples of this kind are not difficult to construct.

Case B. $R < 0$

Analytically, the situation $R < 0$ is handled the same way as $R > 0$. As s decreases from L, θ will increase until it reaches π. In doing so, if $R(\cos \theta - \cos \theta_L)$ becomes equal to C^- before θ reaches π, the loop being constructed will contain exactly two discontinuities. Again, simple relations like (4.16) emerge, and if θ_L is held fixed the discontinuities move apart as R increases. In Figure 3 I have pictured placements derived from an even stored energy function so that the placements for $R > 0$ can be obtained from those for $R < 0$ by the transformation $s \to L - s$. This conclusion follows directly from the energy integral and the observation that if $W(\cdot)$ is even the domain of convexity of W is symmetric about $\kappa = 0$.

A simple relation governs the appearance of discontinuities. The analysis of Case A., when $R > 0$, shows that a pair of discontinuities will first begin to appear at the apex of each loop when (4.13) and (4.14) are simultaneously fulfilled, that is, when

$$R(1 - \cos \theta_L) = \begin{cases} C^+ & \text{if } \theta'(\cdot) > 0 \text{ on that loop,} \\ C^- & \text{if } \theta'(\cdot) < 0 \text{ on that loop.} \end{cases} \tag{4.17}$$

Two discontinuities will always be present on a loop if $=$ in (4.17) is replaced by $>$. If $R < 0$ (Case B) a pair of discontinuities will begin to form in each loop when

$$-R(1 + \cos \theta_L) = \begin{cases} C^+ & \text{if } \theta'(\cdot) > 0 \text{ on that loop,} \\ C^- & \text{if } \theta'(\cdot) < 0 \text{ on that loop.} \end{cases} \tag{4.18}$$

Again, if $>$ replaces $=$ in (4.18) then there will always be two discontinuities present on the appropriate loops.

Equations (4.17) and (4.18) have an easy consequence: *if* $|R| < \frac{1}{2}C^+(|R| < \frac{1}{2}C^-)$ *no discontinuities will appear on a loop of positive(negative) curvature.*

The relations (4.17) (or (4.18)) and (4.16) are independent consequences of the theory – neither by itself can be derived from the other. Since both of these relations involve only overall forces, distances and constitutive parameters, they are easily made subject to experiment and provide an important test for the theory. In Section 7 I shall discuss the implications of these formulae in full detail.

5. Postbuckling

Having given a picture of all possible solutions of the energy integral (4.3) in the preceding section, we shall here seek those particular solutions which satisfy

$$\theta(0) = \theta_0 = 0. \tag{5.1}$$

Among these are the null solutions

$$\theta(s) \equiv 0 \tag{5.2}$$

possible for any R. We shall look for solutions containing only one half of a loop of the elastica, so-called "first mode" solutions. Owing to the results of the preceding section, these either satisfy (5.2) or have the property that $\theta'(s) \neq 0$ on $[0, L]$.[10] Thus, excluding the null solutions (5.2) θ is an invertible function of s on $[0, L]$. If there exists one of these non-null first mode solutions $\tilde{\theta}(s)$ satisfying $\tilde{\theta}(0) = 0$, then it must satisfy the energy integral

$$f(\tilde{\theta}') = R(\cos \tilde{\theta} - \cos \tilde{\theta}_L), \tag{5.3}$$

which, because $\tilde{\theta}'$ is assumed to have one sign, may be rewritten

$$\tilde{\theta}' = k(R(\cos \tilde{\theta} - \cos \tilde{\theta}_L)), \tag{5.4}$$

k being the inverse function of f restricted to either $[\alpha, \alpha_1^*] \cup [\alpha_2^*, 0]$ or $[0, \beta_1^*] \cup [\beta_2^*, \beta]$, depending upon the sign of the curvature on the particular loop under consideration. Referring to Figure 2 we see that k has a jump discontinuity at its argument C^+ if $k \geqq 0$, or at C^- if $k \leqq 0$. To fix the discussion, we shall assume that $\theta' > 0$, so that k is the positive valued inverse of f. Since we have chosen κ to have values only on the domain of convexity of W, (3.7) will be satisfied by $\tilde{\theta}$. From (5.4),

$$\frac{\tilde{\theta}'(s)}{k(R(\cos \tilde{\theta} - \cos \tilde{\theta}_L))} = 1, \qquad s \in [0, L], \tag{5.5}$$

which yields

$$\int_0^s \frac{\tilde{\theta}'(\xi) \, d\xi}{k(R(\cos \tilde{\theta}(\xi) - \tilde{\theta}_L))} = s. \tag{5.6}$$

If we interpret (5.6) as a Stieltjes integral, we may change variables $\xi \leftrightarrow \tilde{\theta}$ in (5.6) to obtain the identity

$$\int_0^{\tilde{\theta}} \frac{d\theta}{k(R(\cos \theta - \cos \theta_L))} = s(\tilde{\theta}). \tag{5.7}$$

Equation (5.7) delivers an explicit representation for the inverse of the function $\tilde{\theta}(s)$. Conversely, if $s(\tilde{\theta})$ is *defined* by (5.7) and if the parameter θ_L can be chosen so that

$$\int_0^{\theta_L} \frac{d\theta}{k(R(\cos \theta - \cos \theta_L))} = L, \tag{5.8}$$

then the inverse of $s(\tilde{\theta})$ is a solution of the energy integral (5.3) and the initial condition $\tilde{\theta}(0) = 0$. Furthermore, this inverse is strictly monotone and its curvature lies on the domain of convexity of W. By our method of construction of this inverse, the conditions (3.28) will also be fulfilled. Thus, our method of construction will deliver a solution of the Euler equation (3.6) and condition of convexity (3.7).

---

[10] Recall that we have sought solutions in which $\theta'(s)$ is defined everywhere; $\theta'(s)$ may be discontinuous at a single point for the "first mode" solutions.

Our study of "first mode" (monotone) solutions of the Euler equation, the condition of convexity and the initial condition (5.1) has been reduced to the study of (5.8).

In (5.8) we view θ_L as a parameter so adjusted that (5.8) holds for an assigned value L. It is quite possible that no such value θ_L can be found for some values of L; to clarify the situation we change variables;

$$\psi_L \equiv \tfrac{1}{2}\theta_L, \qquad \psi \equiv \tfrac{1}{2}\theta \tag{5.9}$$

and rewrite (5.8) as

$$\int_0^{\psi_L} \frac{d\psi}{k(R(\sin^2 \psi_L - \sin^2 \psi)} = \frac{L}{2}. \tag{5.10}$$

Let ϕ be defined by

$$\sin \phi = \frac{\sin \psi}{\sin \psi_L}, \tag{5.11}$$

assuming, of course, that $0 < \psi_L < \pi/2$. The change of variables induced by (5.11) puts (5.10) in a form reminiscent of the Jacobian elliptic integrals:

$$\int_0^{\pi/2} \frac{\sin \psi_L \cos \phi \, d\phi}{\sqrt{1 - \sin^2 \psi_L \sin^2 \phi} \; k(2R(\sin^2 \psi_L \cos^2 \phi))} = \frac{L}{2}. \tag{5.12}$$

The connection with elliptic functions is best illuminated if we define[11]

$$h(y) \equiv \frac{\sqrt{y}}{k(y)}. \tag{5.13}$$

After the introduction of (5.3), the basic condition (5.12) becomes

$$H(\psi_L) \equiv \frac{1}{\sqrt{2R}} \int_0^{\pi/2} \frac{h(2R \sin^2 \psi_L \cos^2 \phi) \, d\phi}{\sqrt{1 - \sin^2 \psi_L \sin^2 \phi}} = \frac{L}{2} \tag{5.14}$$

in which I have denoted the left hand side by $H(\psi_L)$.

Now we observe that

$$h(0) = \lim_{y \to 0} \frac{\sqrt{y}}{k(y)} = \lim_{\kappa \to 0} \frac{\sqrt{f(\kappa)}}{\kappa} = \lim_{\kappa \to 0} \frac{\sqrt{w_2 \dfrac{\kappa^2}{2} + o(\kappa^2)}}{\kappa}$$

$$= (\tfrac{1}{2} w_2)^{1/2}, \tag{5.15}$$

the constant w_2 being defined by

$$w_2 = W_{\kappa\kappa}(0) > 0. \tag{5.16}$$

If we evaluate (5.14) in the limit as ψ_L tends to zero, and use the elementary theory

---

[11] In the classical theory, $W = c\kappa^2$, $c = \text{const.}$, the function h is simply $c^{1/2}$.

of elliptic integrals [12], we obtain

$$\lim_{\psi_L \to 0} H(\psi_L) = \frac{\pi}{2}\left(\frac{w_2}{4R}\right)^{1/2}, \tag{5.17}$$

according to the same theory

$$\lim_{\psi_L \to \pi/2} H(\psi_L) = \infty. \tag{5.18}$$

If $H(\cdot)$ happens to be monotone increasing, as it is in the clasical theory, then its smallest value is the right hand side of (5.17), and to each length L greater than

$$\pi\left(\frac{w_2}{4R}\right)^{1/2} \tag{5.19}$$

there corresponds one and only one nontrivial solution ψ_L of (5.14). However, this will *not* always be the case, and in the generality embraced here we can only state that each length greater than $\pi(w_2/4R)^{1/2}$ will produce *at least one* nontrivial solution. Equivalently, regarding the length as assigned, each force R greater than

$$\frac{w_2 \pi^2}{4L^2}, \tag{5.20}$$

will produce a nontrivial solution. The constant $w_2\pi^2/4L^2$ is the familiar Euler buckling load associated with the "fixed-free" problem.

Thus, the possibility is left open that there be nontrivial solutions corresponding to lengths less than (5.19), or equivalently, to loads less than the Euler buckling load. This will be possible if H decreases near $\psi_L = 0$. Hence, we are led to examine the derivative $H'(\psi_L)$ when ψ_L is small. This derivative will certainly exist when ψ_L is sufficiently small, because h experiences a discontinuity only if its argument equals $C^+ > 0$. By looking at this derivative in detail, we establish

THEOREM 2. *Let* $W(\kappa)$ *be represented by*

$$W(\kappa) = \kappa \Sigma(\kappa). \tag{5.21}$$

Then

$$h'(y) = \frac{\Sigma_{\kappa\kappa}}{2h\kappa W_{\kappa\kappa}}, \tag{5.22}$$

κ *being evaluated at* $k(y)$. *If* $W \in C^3$ *near* $\kappa = 0$, *then*

$$W_{\kappa\kappa\kappa}(0) \begin{cases} < 0 \\ > 0 \\ = 0 \end{cases} \Rightarrow H'(0) \begin{cases} < 0 \\ > 0 \\ = 0 \end{cases} \tag{5.23}$$

respectively. If $\Sigma_{\kappa\kappa}(\kappa) \geqq 0$ *for* $\kappa \in (0, \beta_1^*) \cup (\beta_2^*, \beta)$, *then*

$$H'(\psi_L) > 0, \qquad 0 < \psi_L < \pi/2; \tag{5.24}$$

that is, the convexity of Σ is a sufficient condition that H be strictly monotonically increasing.

Finally, suppose that $W \in C^4$ near $\kappa = 0$ and that $W_{\kappa\kappa\kappa}(0) = 0$. Assume that

$$W_{\kappa\kappa\kappa\kappa}(0) < \frac{-w_2^2}{R}. \tag{5.25}$$

Then $H(\psi_L)$ is a strictly decreasing function of ψ_L for ψ_L sufficiently small.

A proof of this theorem is contained in the appendix.

COROLLARY. *If $W(\kappa)$ is even, then $H'(0) = 0$.*

The proof of the corollary follows from $(5.23)_3$; if W is even, then $W_{\kappa\kappa\kappa}(0) = 0$ so that $H'(0) = 0$.

We have mentioned above the possibility that abrupt buckling occurs before the Euler buckling load is reached. The theorem shows that this situation is indeed possible, even if the energy function is symmetric about $\kappa = 0$. The test for this possibility is an easy one; we only need to check the simple condition (5.25).

If the energy function were not even, we might expect this abrupt kind of buckling to occur, but according to (5.25) it can also occur when W is even. *In fact, (5.25) implies that if $W_{\kappa\kappa\kappa\kappa}(0) < 0$, there is always a value of R such that $H(\cdot)$ will decrease from zero.* This value of R may have to be large, so that the length L associated with the bifurcation, that is, the length calculated from (5.19), is small.

Away from $\psi_L = 0$ the behavior of H has not been determined, except for the qualitative result (5.18). It also can be easily shown that H is continuous and has a single discontinuity in its derivative when $(4.17)_1$ is satisfied:

$$R(1 - \cos 2\psi_L) = C^+. \tag{5.26}$$

This condition, as before, delivers the value of ψ_L at which a discontinuity of the curvature first forms at the base of the column. This discontinuity moves up the column as ψ_L is increased, and the region between the discontinuity and the base will be a region of large curvature ($\kappa \in [\beta_2^*, \beta]$). If we had done the same analysis for $\kappa \leq 0$, we would have found that H is smooth at $\kappa = 0$. From these facts I have constructed the schematic bifurcation diagrams of Figure 4. The significance of the dotted lines shown has not yet been explained, but will become apparent in the next section.

It can be easily shown by an argument based upon the appropriate form of (4.16) that the discontinuity of $H'(\cdot)$ can occur either on a monotone decreasing branch of $H(\cdot)$, or not. Of course, it is easiest in simple cases to numerically integrate (5.14) to determine this point of discontinuity. Therefore, it is possible to produce the solutions which buckle over abruptly at loads below the Euler load, and for which the buckled solution contains a region of the phase of large curvature.

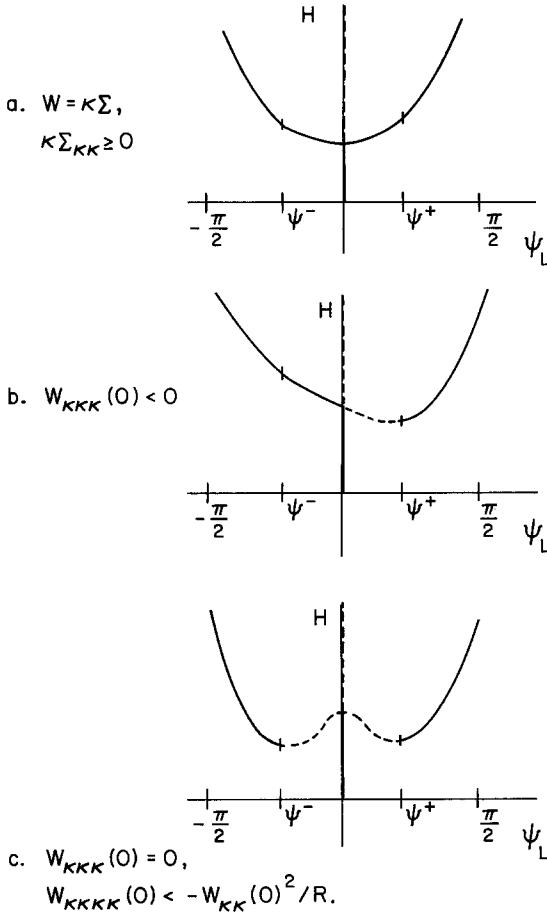Theorem 2 has an obvious counterpart for negative curvatures.

a. $W = \kappa\Sigma,$
   $\kappa\Sigma_{\kappa\kappa} \geq 0$

b. $W_{\kappa\kappa\kappa}(0) < 0$

c. $W_{\kappa\kappa\kappa}(0) = 0,$
   $W_{\kappa\kappa\kappa\kappa}(0) < -W_{\kappa\kappa}(0)^2/R.$

Figure 4. Schematic bifurcation diagrams. A monotone placement corresponds to each solution ψ_L of $H(\psi_L) = L/2$. H has a discontinuous derivative at ψ^- and ψ^+, these being the solutions of $R(1-\cos 2\psi^+) = C^+$, $0 < \psi^+ < \pi/2$, and of $R(1-\cos 2\psi^-) = C^-$, $-\pi/2 < \psi^- < 0$. $\psi_L \equiv \frac{1}{2}\theta(L)$.

6. Stability

Although the functions $\tilde{\theta}$ found in the preceding section are solutions of the Euler equation (3.6) which also fulfill the condition of convexity (3.7), they do not all turn out to be minimizers of the total energy (1.10). I propose in this section to lay down sufficient conditions that certain of them be strong relative minimizers of the total energy (Theorem 3).

Classical methods for the study of relative minimizers based upon the work of Weierstrass, Jacobi, and Hilbert generally fail when the minimizers are not smooth, or the energy is not convex. The importance and subtleness of nonconvex problems

were recognized later by Carathéodory [13] and Graves [14], but the thrust of their work was directed toward the parametric problem with fixed end points. Therefore, I have adopted another procedure, not wholly divorced from "field theory" of the calculus of variations, but which bypasses the construction of a field of extremals in favor of a direct approach based on the energy integral.

I shall focus attention upon the monotone solutions $\tilde{\theta}(\cdot)$, $\tilde{\theta}(0) = 0$, obtained in the preceding section. Two results of a local nature can be deduced from the classical theory, which I state below without proof:[12]

(i) the straight solution $\tilde{\theta} \equiv 0$ is a strong relative minimizer of $E[\cdot]$ in the class $\mathscr{F}(0)$ for loads R less than the Euler buckling load $w_2 \pi^2 / 4L^2$, and

(ii) the straight solution ceases to be even a weak relative minimizer for loads R greater than the Euler buckling load; there is always a function $\psi \in \mathscr{F}(0)$, arbitrarily close to $\tilde{\theta} \equiv 0$ in the sense that $|\psi|$ and $|\psi'|$ are less than any given positive number, for which $E[\psi] < E[\tilde{\theta}]$.

The monotone solutions $\tilde{\theta}$ of the energy integral whose curvature lies on the domain of convexity of W satisfy (cf. equation (5.4))

$$\tilde{\theta}'(s) = k(R(\cos \tilde{\theta} - \cos \tilde{\theta}_L)), \tag{6.1}$$

$k(y)$ being the inverse of the function $f(\kappa) = \kappa W_\kappa - W$ (cf. equation (4.2)), with κ restricted to $[0, \beta_1^*] \cup [\beta_2^*, \beta]$ for the solutions which buckle upward and with κ restricted to $[\alpha, \alpha_1^*] \cup [\alpha_2^*, 0]$ for the ones which buckle downward. To fix the discussion, we shall assume that $k(\cdot) \geqq 0$, and that R is greater than the Euler buckling load so that (6.1) has a nontrivial solution $\tilde{\theta}(\cdot)$. We shall silence the dependence of the right hand side of (6.1) upon R and write

$$K(\theta, \theta_L) \equiv k(R(\cos \theta - \cos \theta_L)); \tag{6.2}$$

note that K is only defined for $R(\cos \theta - \cos \theta_L) > 0$.

Let $\theta(\cdot) \in \mathscr{F}(0)$ be a competitor for the minimum of $E[\cdot]$. Suppose that $\theta(s)$ satisfies

$$\pi > \theta_L = \theta(L) \geqq |\theta(s)|, \qquad s \in [0, L]. \tag{6.3}$$

Since (6.3) implies that $K(\theta(s), \theta_L)$ is well defined, we may write

$$\begin{aligned}
E[\theta] &= \int_0^L \{W(\theta'(s)) + R \cos \theta(s)\} \, ds \\
&= \int_0^L \{W(\theta'(s)) - W(K(\theta(s), \theta_L)) - [\theta'(s) - K(\theta(s), \theta_L)] W_\kappa(K(\theta(s), \theta_L)) \\
&\quad + [\theta'(s) - K(\theta(s), \theta_L)] W_\kappa(K(\theta(s), \theta_L)) \\
&\quad + R \cos \theta(s) + W(K(\theta(s), \theta_L))\} \, ds \\
&= \int_0^L \mathscr{E}(\theta'(s), K(\theta(s), \theta_L)) \, ds + A[\theta], \tag{6.4}
\end{aligned}$$

---

[12] In the usual proofs of these results, it must also be assumed that $W \in C^3_{[\alpha, \beta]}$. This remark does not apply to Theorem 3.

in which the Weierstrass excess function $\mathscr{E}(\cdot, \cdot)$ is defined by

$$\mathscr{E}(b, a) \equiv W(b) - W(a) - (b-a)W_\kappa(a), \tag{6.5}$$

and $A[\cdot]$ is the functional

$$\begin{aligned}
A[\phi] \equiv \int_0^L & \{[\phi'(s) - K(\phi(s), \phi_L)]W_\kappa(K(\phi(s), \phi_L)) \\
& + R \cos \phi(s) + W(K(\phi(s), \phi_L))\} \, ds.
\end{aligned} \tag{6.6}$$

$A[\theta]$ is well defined for functions $\theta(s)$ which satisfy (6.3). By virtue of (3.8) and the fact that K has values only on the domain of convexity of W we have

$$\mathscr{E}(\theta'(s), K(\theta(s), \theta_L)) \geqq 0. \tag{6.7}$$

Also, since $\tilde{\theta}$ is monotone and satisfies (6.1),

$$A[\tilde{\theta}] = E[\tilde{\theta}]. \tag{6.8}$$

We shall show that $A[\psi]$, while it appears to be a functional of ψ, actually depends solely as a function upon ψ_L. That is, since k is the inverse of f, then by (6.2) and (4.2),

$$K(\psi, \psi_L)W_\kappa(K(\psi, \psi_L)) - W(K(\psi, \psi_L)) = R(\cos \psi - \cos \psi_L). \tag{6.9}$$

Thus, (6.6) can be written

$$A[\psi] = \int_0^L \{\psi'(s)W_\kappa(K(\psi(s), \psi_L)) + R \cos \psi_L\} \, ds, \tag{6.10}$$

or equivalently,

$$A[\psi] = \int_0^{\psi_L} W_\kappa(K(\mu, \psi_L)) \, d\mu + RL \cos \psi_L. \tag{6.11}$$

Equation (6.11) shows that $A[\psi]$ is equal to a certain function of ψ_L, which we shall denote by $a(\psi_L)$:

$$A[\psi] = a(\psi_L) \equiv \int_0^{\psi_L} W_\kappa(K(\mu, \psi_L)) \, d\mu + RL \cos \psi_L. \tag{6.12}$$

The function of $a(\cdot)$ is continuous because K is continuous except when its value equals β_1^* or β_2^*, but $W_\kappa(\beta_1^*) = W_\kappa(\beta_2^*)$. It is also differentiable except possibly at the single point where

$$\psi_L = \arccos\left(1 - \frac{C^+}{R}\right). \tag{6.13}$$

We wish to show that $a(\cdot)$ has a minimum value at $\psi_L = \tilde{\theta}_L$. By differentiating (6.9) with respect to ψ_L, we derive that

$$W_{\kappa\kappa}(K(\psi, \psi_L)) \frac{\partial K}{\partial \psi_L} K(\psi, \psi_L) = R \sin \psi_L, \tag{6.14}$$

which permits us to write the derivative of (6.12) in the following way:

$$
\frac{da}{d\psi_L} = W_\kappa(K(\psi_L, \psi_L)) + R \sin \psi_L\left\{\int_0^{\psi_L} \frac{d\mu}{K(\mu, \psi_L)} - L\right\}
$$
$$
= R \sin \psi_L\left\{\int_0^{\psi_L} \frac{d\mu}{K(\mu, \psi_L)} - L\right\}. \tag{6.15}
$$

But $(6.15)_2$ is simply

$$
\frac{da}{d\psi_L} = 2(R \sin \psi_L)\left(H(\tfrac{1}{2}\psi_L) - \frac{L}{2}\right), \tag{6.16}
$$

and from the results of Section 5 (summarized by the bifurcation diagrams of Figure 4), we deduce that

$$
\frac{da}{d\psi_L} = 0 \quad \text{when} \quad \psi_L = \tilde{\theta}_L. \tag{6.17}
$$

If H is monotonically increasing on a neighborhood of $\tilde{\theta}_L$, then a has a minimum at $\psi_L = \tilde{\theta}_L$. Thus, if H is monotonically increasing on a neighborhood of $\tilde{\theta}$,

$$
A[\theta] \geqq A[\tilde{\theta}] = E[\tilde{\theta}], \quad \theta_L \text{ near } \tilde{\theta}_L, \tag{6.18}
$$

so (6.4) becomes

$$
E[\theta] - E[\tilde{\theta}] \geqq \int_0^L \mathscr{E}(\theta'(s), K(\theta(s), \theta_L)) \, ds \geqq 0, \tag{6.19}
$$

whence

LEMMA 3. *Let* $\tilde{\theta}(s)$ *be a monotone solution of the energy integral* (6.1) *with* k *restricted to have values on the domain of convexity of* W. *Suppose the function* $H(\psi_L)$ *defined by* (5.14) *is monotone increasing (decreasing) in a neighborhood* $\mathscr{N} \equiv [\tfrac{1}{2}\tilde{\theta}_L - a, \tfrac{1}{2}\tilde{\theta}_L + b]$, $a, b > 0$, *of* $\tfrac{1}{2}\tilde{\theta}_L > 0$ (<0). *Let* $\theta(s) \in \mathscr{F}(0)$ *satisfy the condition*

$$
\pi > \theta_L \geqq |\theta(s)| \quad (-\pi < \theta_L \leqq |\theta(s)|), \quad s \in [0, L], \tag{6.20}
$$

and assume that

$$
\theta_L \in \mathscr{N}. \tag{6.21}
$$

Then,

$$
E[\tilde{\theta}] \leqq E[\theta]. \tag{6.22}
$$

Lemma 3 states simply that the "first mode" solutions are minimizers with respect to functions $\theta(\cdot)$ whose terminal values satisfy (6.20) and (6.21). The assumption (6.21) is necessary for the proof, but (6.20) is actually not essential. To see this, let $\theta \in \mathscr{F}(0)$ be a competitor and assume

$$
\pi > \theta_m = \max_{[0,L]} |\theta(s)| = \theta(\bar{s}) > 0. \tag{6.23}
$$

Of course $\theta(s)$ may not satisfy (6.20), but the function

$$\bar{\theta} = \begin{cases} \theta(s) & s \in [0, \bar{s}] \\ \theta_m & s \in (\bar{s}, L], \end{cases} \tag{6.24}$$

does satisfy (6.20). Since $\cos \theta(s) \geqq \cos \bar{\theta}(s)$, $s \in [0, L]$, and $W(0) = 0$, then for $R > 0$,

$$E[\theta] - E[\bar{\theta}] = \int_{\bar{s}}^{L} \{W(\theta'(s)) + R(\cos \theta(s) - \cos \bar{\theta}(s))\} \, ds.$$

$$\geqq 0. \tag{6.25}$$

Thus we can evidently weaken (6.20), and we summarize the improved result below.

THEOREM 3. *Let $\tilde{\theta}(s)$ be a monotone solution of the energy integral with k restricted to have values on the domain of convexity of W. Suppose the function $H(\psi_L)$ defined by (5.14) is monotone increasing (decreasing) in a neighborhood $\mathcal{N} \equiv (\frac{1}{2}\tilde{\theta}_L - a, \frac{1}{2}\tilde{\theta}_L + b)$, $a, b > 0$, of $\frac{1}{2}\tilde{\theta}_L > 0$ (<0). Let $\theta(s) \in \mathcal{F}(0)$ satisfy the condition*

$$\max_{[0,L]} \theta(s) \in \mathcal{N} \qquad (\min_{[0,L]} \theta(s) \in \mathcal{N}). \tag{6.26}$$

Then,

$$E[\tilde{\theta}] \leqq E[\theta]. \tag{6.27}$$

The theorem implies that the monotone solutions $\tilde{\theta} \not\equiv 0$ of the energy integral are strong relative minimizers of the total energy if H is monotone increasing, when $\tilde{\theta}' \geqq 0$, or monotone decreasing, when $\tilde{\theta}' \leqq 0$, but the theorem has global implications as well. In particular, *if $W = \kappa\Sigma$ is an even function and $\kappa\Sigma_{\kappa\kappa} \geqq 0$ whenever κ belongs to the domain of convexity of W, then each of the two possible monotone solutions is a global minimizer.* That is, if $\kappa\Sigma_{\kappa\kappa} \geqq 0$ on the domain of convexity of W, Theorem 2 shows that H is monotone increasing when its argument is positive. Theorem 3 then shows that $\tilde{\theta}(\cdot)$ is a minimizer relative to functions $\theta(\cdot)$ for which $\theta(L) > 0$; but since W is even $\tilde{\theta}(\cdot)$ has the same total energy as $-\tilde{\theta}(\cdot)$, which by the same argument has less total energy than all $\theta(\cdot)$ with $\theta(L) < 0$. Thus $\tilde{\theta}(\cdot)$ and $-\tilde{\theta}(\cdot)$ are global minimizers, proving the assertion.

7. Discussion

While Theorem 3 clarified the status of the first mode solutions whose half terminal angles lie on a monotone increasing branch of H (cf. (5.14) and Figure 4), other possibilities are left open. If $H(\psi_L)$ decreases for $\psi_L > 0$ and ψ_L small, then there will be nonstraight solutions with lengths less than the length corresponding to the Euler buckling load. The straight solution is a strong relative minimizer for lengths less than the Euler length, but the classical theorem which delivers that result gives no

information about the size of the neighborhood in which the straight solution is a minimizer. Therefore, without deeper analysis we cannot deduce anything about solutions of the energy integral whose half terminal angles $\psi_L = \frac{1}{2}\theta_L$ lie on monotone decreasing branches of $H(\cdot)$. I conjecture that they are not strong relative minimizers of the total energy.

I wish to emphasize again that the phenomenon of abrupt buckling at loads below the Euler buckling load is not necessarily connected with the failure of convexity of the stored energy, although it can be accompanied by the presence of a new phase if the energy does lose convexity. To have a positive Euler buckling load, we must assume $W_{\kappa\kappa}(0) > 0$. If $W_{\kappa\kappa}(0) = 0$, the question of abrupt buckling below the Euler load is meaningless, since only the straight solution is possible for $R < 0$. The inequality $W_{\kappa\kappa\kappa}(0) \neq 0$ or the inequality $W_{\kappa\kappa\kappa\kappa}(0) < -W_{\kappa\kappa}(0)^2/R$ (cf. Theorem 2) is sufficient that there shall be a nontrivial solution of the Euler equation which also fulfills the condition of convexity for loads below the Euler buckling load. These inequalities place no restriction upon the domain of convexity W.

It is common to obtain the moment-curvature relation by comparison with the solution in linear elasticity for pure bending, and to assert that the resulting relation $(M(\kappa) = EI\kappa$; E = extensional modulus; I = the appropriate moment of inertia) represents well the behavior of a thin rod for small values of the curvature. This constitutive equation does *not* permit abrupt buckling at loads below the Euler buckling load. That is, if $M(\kappa) = EI\kappa$ then $W(\kappa) = \frac{1}{2}EI\kappa^2$ so that $W_{\kappa\kappa\kappa}(0) = W_{\kappa\kappa\kappa\kappa}(0) = 0$; in fact, based on this constitutive equation, $H(\psi_L)$ is monotone increasing for $\psi_L > 0$ and is even. Since the analysis of Theorem 2 was carried out under the restriction of arbitrarily small curvature, the observation just made provides another example of a phenomenon for which the deformation is arbitrarily small, but the linear theory is misleading.

For the purpose of the design of structures to prevent abrupt buckling at or below the Euler load, one should avoid materials for which $W_{\kappa\kappa\kappa}(0) \neq 0$ or $W_{\kappa\kappa\kappa\kappa}(0) < -W_{\kappa\kappa}(0)/R^2$. It is best to insure that $H(\cdot)$ has a strict minimum at $\psi_L = 0$; this will be true if $\kappa\Sigma_{\kappa\kappa} \geqq 0$, whenever κ belongs to the domain of convexity of W (cf. Theorem 2). If "failure" attributed to the presence of new phases is to be avoided, it is best to choose materials for which M^+ and $-M^-$ are as large as possible. Suppose for simplicity that $M = M^+ = -M^-$. Since the moment of the applied force acting on the point of discontinuity at least for strong relative minimizers is M, the deflection of the elastica (that is, $|y(L) - y(0)|$) which must be reached to just produce a new phase will increase linearly with M.

If one tries to obtain the stored energy function from experiment, one quickly encounters a delicate problem underlying the connection of theory and experiment. In practice the experiment is not difficult to carry out; we simply apply moments to the ends of a rod and measure the radius of the circular configuration, or the two different radii of the smoothly joined circular arcs, which the rod assumes. Solutions of this kind have been analyzed in detail in [1]. The moment-curvation relation obtained in this way will be defined on three disconnected sets, since, referring to Figure 1, the relation restricted to the intervals $[\alpha_1, \alpha_2]$ and $[\beta_1, \beta_2]$ can never be

obtained by a static experiment. This is, if

$$E[\theta] = \int_0^L W(\theta'(s)) \, ds + \mathscr{L} \tag{7.1}$$

$\mathscr{L}$ being the expression for the energy of any loading device which depends on the values of θ in an arbitrarily small neighborhood of $s = 0$ and $s = L$, then a necessary condition that $\tilde{\theta}(\cdot)$ be a weak relative minimizer of (7.1) is $W_{\kappa\kappa}(\tilde{\theta}'(s)) \geqq 0$ a.e. Interpreted in the usual way, this result indicates the instability of placements with curvatures on a falling portion of the moment-curvature relation. I do not assert that the moment-curvature relation cannot be determined in this "unstable region" in any way whatsoever. Constitutive relations can come from molecular theory, and statically unstable solutions can be taken on by dynamic solutions at certain isolated values of the time. But, typically, the moment-curvature relation will come from a static experiment, and since it will be defined on three disconnected intervals, three constants of integration will be needed to obtain the stored energy function on the same three intervals. One of these constants is unimportant since all results of the static theory are invariant under the transformation $W \to W + \text{const.}$; I have avoided this inconvenience by putting $W(0) = 0$ in (1.11). The other two constants can be evaluated by the relations

$$W(\beta_2^*) = W(\beta_1^*) + M^+(\beta_2^* - \beta_1^*)$$
$$W(\alpha_2^*) = W(\alpha_1^*) + M^-(\alpha_2^* - \alpha_1^*), \tag{7.2}$$

if the values M^+ and M^- are known. Of course, naively, these constants can be obtained from any strong relative minimizer by measurement of the applied force on the point of discontinuity (cf. (4.16)). But herein lies the difficulty. If what I have seen of such experiments is not the solitary exception, it is difficult to decide whether a solution is indeed "stable" (a strong relative minimizer) or just "metastable" (a weak relative minimizer). Sometimes a deformed configuration which appears stable will change its shape when excited by ultrasonic vibrations. The effect seems to depend not on the frequency but on the amplitude of the vibrations, a fact which indicates that the apparently stable configuration corresponds to a weak relative minimizer.

I have focussed attention upon the strong relative minimizers in this paper because they are easy to classify and because they are more stable than the weak relative minimizers, granted the usual interpretation of the theory. There is a huge class of weak relative minimizers; typically, if there is a single weak relative minimizer in the postbuckling problem, there are infinitely many others. They need not have a finite number of discontinuities of the curvature, need not satisfy the energy integral; the moment acting on a point of discontinuity need not be M^+ or M^-. On the other hand, weak relative minimizers may have several co-existent phases at zero applied force and moment, even if $M^- < 0 < M^+$. If configurations corresponding to weak relative minimizers are the ones actually observed, theory would indicate a decided lack of reproducibility of experiments.

Acknowledgement

REFERENCES

[1] Fosdick, Roger L. and Richard D. James, The elastica and the problem of pure bending for a nonconvex stored energy function. *Journal of Elasticity* 11 (1981) 165.
[2] Ericksen, J. L. Equilibrium of bars, *Journal of Elasticity* 5 (1976) 191.
[3] Knowles, J. K. and Eli Sternberg, On the failure of ellipticity and the emergence of discontinuous deformation gradients in plane finite elastostatics. *Journal of Elasticity* 8 (1978), 329.
[4] Knowles, J. K. and Eli Sternberg, On the failure of ellipticity of the equations for finite elastostatic plane strain. *Archive for Rational Mechanics and Analysis* 63 (1977) 321.
[5] James, Richard D. Co-existent phases in the one dimensinal static theory of elastic bars. *Archive for Rational Mechanics and Analysis* 72 (1979) 99–140.
[6] Abeyaratne, Rohan C. *Discontinuous deformation gradients in the finite twisting of an incompressible elastic tube.* Tech. Report #42 (ONR), Calif. Inst. of Eng. and Appl. Sci., June 1973.
[7] Ericksen, J. L. Some phases transitions in crystals, *Archive for Rational Mechanics and Analysis*, 73 (1980) 99–124.
[8] Antman, Stuart S. and Ernest R. Carbone, Shear and necking instabilities in nonlinear elasticity. *Journal of Elasticity* 7 (1977) 125.
[9] Antman, Stuart S., Nonuniqueness of equilibrium states for bars in tension. *Journal of Mathematical Analysis and Applications* 44 (1973) 333.
[10] Ekeland, Ivar and Roger Teman, *Convex Analysis and Variational Problems.* North-Holland Publishing Company, Amsterdam-Oxford, 1976.
[11] Hartman, Philip, *Ordinary Differential Equations.* John Wiley and Sons, New York 1964.
[12] Love, A. E. H., *A Treatise on the Mathematical Theory of Elasticity*, 4th ed. Cambridge University Press, Cambridge 1927.
[13] Carathéodory, C., *Über die diskontinuierlichen Lösungen in der Variationsrechung.* Dissertation, Göttingen (1904).
[14] Graves, Lawrence M., Discontinuous solutions in space problems of the calculus of variations. *American Journal of Mathematics* 52 (1930), 1.

Appendix

Proof of Theorem 2. Let $W(\kappa) = \kappa\Sigma(\kappa)$. By straight forward calculation

$$
\begin{aligned}
h'(y) &= \left(\frac{y^{1/2}}{k(y)}\right)' = \frac{1}{y^{1/2}k(y)^2}\left(\frac{k(y)}{2} - yk'(y)\right) \\
&= \frac{1}{y^{1/2}k(y)^2}\left(\frac{k(y)}{2} - \frac{f(k(y))}{f_\kappa(k(y))}\right).
\end{aligned}
\tag{A.1}
$$

However,

$$
f(\kappa) = \kappa W_\kappa - W = \kappa^2\Sigma_\kappa,
\tag{A.2}
$$

so that

$$\frac{\kappa}{2} f_\kappa = \frac{\kappa^3}{2} \Sigma_{\kappa\kappa} + f. \tag{A.3}$$

Alternatively,

$$f_\kappa = \kappa W_{\kappa\kappa}. \tag{A.4}$$

By substituting first (A.3), and then (A.4), into (A.1) we obtain

$$h'(y) = \frac{\Sigma_{\kappa\kappa}(k(y))}{2h(y)k(y) W_{\kappa\kappa}(k(y))}, \tag{A.5}$$

which proves (5.22). Now,

$$H'(\mu) = \frac{1}{(2R)^{1/2}} \int_0^{\pi/2} \left\{ \frac{h'(2R \sin^2 \mu \cos^2 \psi) 4R \sin \mu \cos \mu \cos^2 \psi}{(1 - \sin^2 \mu \sin^2 \psi)^{1/2}} \right.$$
$$\left. + \frac{h(2R \sin^2 \mu \cos^2 \psi) \sin \mu \cos \mu \sin^2 \psi}{(1 - \sin^2 \mu \sin^2 \psi)^{3/2}} \right\} d\psi \tag{A.6}$$

The second term under the integral sign is positive for $0 < \mu < \pi/2$, and according to (A.5) the first term is positive if $\Sigma_{\kappa\kappa} > 0$, $\kappa \in (0, \alpha_1^*) \cup (\alpha_2^*, \alpha)$. This proves (5.24).

By differentiation of (5.21) under the assumption that $W \in C^3$ near $\kappa = 0$, we see that

$$W_{\kappa\kappa\kappa} = \kappa \Sigma_{\kappa\kappa\kappa} + 3\Sigma_{\kappa\kappa} \tag{A.7}$$

so that

$$W_{\kappa\kappa\kappa}(0) = 3\Sigma_{\kappa\kappa}(0). \tag{A.8}$$

As μ tends to zero, the second term of (A.6) also tends to zero; by use of (A.5) the first term becomes

$$\frac{1}{(2R)^{1/2}} \int_0^{\pi/2} \frac{\Sigma_{\kappa\kappa}(k(y)) 2R \sin \mu \cos \mu \cos^2 \psi}{h(y)k(y) W_{\kappa\kappa}(k(y))(1 - \sin^2 \mu \sin^2 \psi)^{1/2}} \bigg|_{y = 2R \sin^2 \mu \cos^2 \psi} d\psi. \tag{A.9}$$

As μ tends to zero, the argument y in (A.9) tends uniformly to zero and

$$W_{\kappa\kappa} \to W_{\kappa\kappa}(0) = w_2 > 0,$$
$$h(y) \to (\tfrac{1}{2} w_2)^{1/2}, \quad \text{as} \quad \mu \to 0. \tag{A.10}$$
$$(1 - \sin^2 \mu \sin^2 \psi)^{1/2} \to 1.$$

Thus, the existence and sign of the limit of (A.9) as $\mu \to 0$ depends on the existence and sign of the limit of

$$\frac{\Sigma_{\kappa\kappa}(k(y)) \sin \mu}{k(y)} \bigg|_{y = 2R \sin^2 \mu \cos^2 \psi} \tag{A.11}$$

as $\mu \to 0$. However, $k(y)$ has the expansion

$$k(y) = \left(\frac{y}{\frac{1}{2}w^2 + o(1)}\right)^{1/2}, \qquad o(1) \to 0 \quad \text{as} \quad y \to 0, \tag{A.12}$$

which is simply the inverse of the expansion for $f(\kappa)$. Thus, (A.11) becomes

$$\frac{\Sigma_{\kappa\kappa}(k(y))(\frac{1}{2}w_2 + o(1))^{1/2}}{(2R)^{1/2} \cos \psi}\bigg|_{y=2R\sin^2\mu\cos^2\psi}, \tag{A.13}$$

and for μ small its sign is the same as the sign of $\Sigma_{\kappa\kappa}(0)$. If we put (A.13) back into (A.9), the conditions (5.23) emerge.

It remains to prove (5.25). We suppose $W_{\kappa\kappa\kappa}(0) = 0$ from which it follows that $H'(0) = 0$, and we examine closely (A.6) when μ is small, but positive. Since the second term of (A.6) is positive, we can bound H' in the following way:

$$H'(\mu) \leq \frac{1}{(2R)^{1/2}} \int_0^{\pi/2} \left\{ (1 - \sin^2 \mu \sin^2 \psi) \left[h'(y)4R \cos^2 \psi \right. \right.$$
$$\left. \left. + \frac{h(y)}{1-\varepsilon} \sin^2 \psi \right] \cos \mu \sin \mu \bigg|_{y=2R\sin^2\mu\cos^2\psi} \right\} d\psi, \tag{A.14}$$

in which $\varepsilon = \sin^2 \mu$. Because $W_{\kappa\kappa\kappa}(0) = 0$ and $W \in C^4$ near $\kappa = 0$, we may write

$$\Sigma_{\kappa\kappa}(\kappa) = \sigma\kappa + o(\kappa); \tag{A.15}$$

here

$$\sigma \equiv \Sigma_{\kappa\kappa\kappa}(0) = \frac{1}{4} W_{\kappa\kappa\kappa\kappa}(0). \tag{A.16}$$

The square bracketed term in (A.14) determines the sign of $H'(\mu)$ when μ is small. By use of (A.5) and (A.16) we may write it as

$$\left[\frac{\left(\sigma + \frac{o(\kappa)}{\kappa}\right)2R \cos^2 \psi}{hW_{\kappa\kappa}} + \frac{h}{1-\varepsilon} \sin^2 \psi \right]. \tag{A.17}$$

All of the terms of (A.17) have uniform limits as $\mu \to 0$ except $\cos^2 \psi$ and $\sin^2 \psi$, but these yield equal values when integrated from zero to $\pi/2$. Hence, $H'(\mu)$ will be negative when μ is small if

$$\sigma < \frac{-h(0)^2 W_{\kappa\kappa}(0)}{2R}, \tag{A.18}$$

i.e., if

$$W_{\kappa\kappa\kappa\kappa}(0) < \frac{-w_2^2}{R}. \tag{A.19}$$