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Internal variables and fine-scale oscillations in micromagnetics

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We derive the field energy of micromagnetics by computing the limiting energy of a lattice of dipoles as a typical lattice parameter goes to zero. Different limiting energies are obtained, depending upon whether the dipoles do not oscillate (strong convergence), oscillate on a scale much larger than the scale of the lattice (weak-long) or oscillate on the scale of the lattice (weak-short). The analysis suggests a framework and some methods for the direct derivation of continuum theory from molecular theory.

1 Introduction

The theory of micromagnetics developed by W. F. Brown [1963] is based on the free energy

$$\int_\Omega \nabla \mathbf{m} \cdot \mathbf{A} \nabla \mathbf{m} + \varphi(\mathbf{m}) d\mathbf{x} + \frac{1}{2} \int_{\mathbb{R}^3} |\nabla \psi|^2 d\mathbf{x} \tag{1.1}$$

where ψ is determined by $\mathbf{m}$ from the equation

$$\operatorname{div} (-\nabla \psi + \mathbf{m}) = 0 \text{ on } \mathbb{R}^3. \tag{1.2}$$

Here, $\mathbf{m} : \mathbb{R}^3 \to \mathbb{R}^3$ is the **magnetization**, $\psi : \mathbb{R}^3 \to \mathbb{R}$ is the **magnetostatic potential**, $\mathbf{h} = -\nabla \psi$ is the **magnetic field** and $\mathbf{b} = -\nabla \psi + \mathbf{m}$ is the **magnetic induction**. The magnetiztion is assumed to vanish on $\mathbb{R}^3 \backslash \Omega$ and the expression (1.1) is subject to a saturation condition $|\mathbf{m}(\mathbf{x})| = \mathrm{f}(\theta)$, $\mathbf{x} \in \Omega$, where θ is the temperature and $\mathrm{f} : \mathbb{R} \to \mathbb{R}$ vanishes at the Curie temperature, $\mathrm{f}(\theta_c) = 0$. The term $\varphi(\mathbf{m})$ represents the **anisotropy energy** density and the term $\nabla \mathbf{m} \cdot \mathbf{A} \nabla \mathbf{m}$ represents the **exchange energy** density. The remaining term

$$\frac{1}{2} \int_{\mathbb{R}^3} |\nabla \psi|^2 d\mathbf{x} \tag{1.3}$$

is the **field energy**. This paper is concerned with the derivation of the field energy.

With improvements in high resolution microscopy, it is becoming evident that a variety of materials develop lattice-scale oscillations of field quantities at certain temperatures, stresses or fields. Examples of such oscillations are tweed structures (Van Tendeloo, Chandrasekaran, Lovey [1986], Tanner, Schryvers and Shapiro [1990], Schryvers [1991], Sethna, Kartha, Castán, Krumhansl [1992]), 9R/18R structures in CuAl (Adachi, Perkins, Wayman [1988], Lovey, Van Tendeloo, Van Landuyt, Delaey, Amelinckx [1984]) microtwinning (Schryvers [1992]) and various incommensurate phases (Van Landuyt, Van Tendeloo, Amelinckx [1988]) in materials that undergo martensitic or structural transformations and antiferromagnetism or ferrimagnetism (and related phases) in magnetic materials. The appearance of such fine scale structures calls into question the derivation of the standard continuum theories and, in particular, the theory of micromagnetics. In such cases the question arises "Is it a phase or is it a microstructure?" That is, the observed structure is evidently a microstructure if it is modeled by one of the standard theories but with an extremely small surface energy; in the context of micromagnetics this would mean that the contribution of the exchange energy is small relative to the other energies. In that case the appearance of fine-scale oscillations is attributed to the frustration of the energy (1.1), when the exchange energy is omitted (see *e.g.,* James and Kinderlehrer [1990]). On the other hand, it is a phase if there is in some sense a special potential-well of the free energy density assigned to the observed structure. Usually, it is only possible to assign such a potential well if new variables are introduced into the continuum theory to describe the geometry or nature of the observed structure. In nonclassical continuum theories these variables are called "internal" or "hidden" variables.

Our general goal is to revisit the underlying molecular basis for such continuum theories. We want to determine whether the standard continuum theory persists in the context of lattice-scale oscillations and, if not, to prescribe the necessary revision. We are particularly interested in what information from the molecular picture survives at continuum level and what information can be safely neglected at the continuum level. In this paper, we confine attention to the field energy term (1.3) in a classical framework.

As explained by Brown [1962, 1963, 1966], or in the corresponding dielectric context by Toupin [1956], the derivation of (1.3) is related to Lorentz's [1909] calculation of the field of a lattice of dipoles. We adopt this framework except that we consider a family of Bravais lattices $L_\lambda, \lambda \in (0, 1]$, in which λ is a typical lattice parameter, $L_\lambda = \lambda L_1$. We place a dipole of moment $\mathbf{d}_\lambda(\mathbf{x})$ on each lattice point $\mathbf{x} \in L_\lambda$ and we identify continuum quantities as limits of corresponding lattice quantities as $\lambda \to 0$. Two issues arise in the calculation of these limits. First, the field of a dipole has non-integrable singularities at each lattice point and at ∞. Second, the calculation of the limiting energy involves products of oscillating sequences, so weak-convergence techniques are useful, with the added feature that weak limits are affected by the relation between the wavelength of a typical oscillation and the scale of the lattice.

In Section 2 we explain the notation and the method of treating the nonintegrable dipole field. In Section 3 we compare different methods of imposing convergence conditions on the dipole field $\mathbf{d}_\lambda$. For the purpose of this introduction we may think of extending $\mathbf{d}_\lambda(\mathbf{x})$ to all of $\mathbb{R}^3$, using a piecewise constant extension (constant on each unit cell), and then multiplying by λ^3 vol. (U) where U is a unit cell of the lattice L_1. Convergence conditions can then be imposed on the extended and scaled field, $\tilde{\mathbf{d}}_\lambda(\mathbf{z}) = \lambda^3 \text{ vol. } U\mathbf{d}_\lambda(\mathbf{x})$, $\mathbf{z} \in \mathbf{x} + \lambda U$. In our terminology strong convergence of $\tilde{\mathbf{d}}_\lambda$ in L^2 corresponds to "no oscillations," while fine-scale lattice oscillations are modeled by weak convergence $\tilde{\mathbf{d}}_\lambda \rightharpoonup \mathbf{m}$ in L^2.

We discuss the limiting fields in Section 5. The limiting magnetic induction $\mathbf{b}$ arises in our calculation by cutting out a region rR around each dipole and by replacing the dipole field by a nonsingular divergence-free field. For the calculation of macroscopic fields it turns out not to matter whether r is much larger, much smaller, or of the same order as the lattice scale λ. Also, the limiting fields $\mathbf{b}, \mathbf{h}, \mathbf{m}$ and the relations $\mathbf{b} = \mathbf{h} + \mathbf{m}$, $\operatorname{div}\mathbf{b} = 0$ are insensitive as to whether the dipoles oscillate on the lattice scale or on longer scales.

The insensitivity of field quantities to the nature of the oscillations is not shared by the energy. For the case of strong convergence $\tilde{\mathbf{d}}_\lambda \to \mathbf{m}$ in L^2, we obtain the classical formula for the field energy (Section 6). The case of weak convergence divides naturally into two subcases that we label weak-long and weak-short; these terms are defined by looking at correlations of $\tilde{\mathbf{d}}_\lambda(\cdot)$ with its translates $\tilde{\mathbf{d}}_\lambda(\cdot + \mathbf{z})$ where $|\mathbf{z}| \leq \lambda$ (Section 7, $(7.2)_{ff}$). The weak-long case corresponds to oscillations with wavelength much larger than the lattice spacing but small compared to the macroscopic dimensions. We show that the weak-long limiting energy is still given by the standard formula (essentially (1.3) with (1.2)), but that this formula is *not* evaluated at the magnetization but rather on the sequence $\tilde{\mathbf{d}}_\lambda$ (Proposition 7.1). We give some specific examples of the weak-long energy in Examples 7.4, 7.5 and 7.6.

New expressions for the micromagnetic field energy are found in the weak-short case, which models oscillations that are truly on the scale of the lattice (Section 8). In the periodic case (Section 9) the excess field energy is found to depend on a finite set of vector fields $\hat{\mathbf{d}}_1, \ldots, \hat{\mathbf{d}}_n, \hat{\mathbf{d}}_i : \Omega \to \mathbb{R}^3$. The $\hat{\mathbf{d}}_1, \ldots \hat{\mathbf{d}}_n$ are not the actual lattice-scale dipole moments but are related to them by (9.12)-(9.14). The form of the excess field energy in the weak-short case is

$$\frac{1}{2}\int_\Omega \hat{\mathbf{d}}^{(i)}(\mathbf{x}) \cdot \mathbf{S}^{ij}\hat{\mathbf{d}}^{(j)}(\mathbf{x})\mathrm{d}\mathbf{x}, \tag{1.4}$$

where $\mathbf{S}^{ij}$ are certain trace-free symmetric matrices given explicitly by certain remnant lattice sums (they are independent of the dipole field $\tilde{\mathbf{d}}_\lambda$). Further discussion is given in Section 9.

The three cases – strong, weak-long and weak-short – can be summarized briefly as follows. In this summary, $\mathbf{m}_\lambda \to \mathbf{m}$ in the appropriate sense and $e_\lambda(\mathbf{p})$ denotes the energy on L_λ of the dipole field $\mathbf{p}$.

strong: $\quad e_\lambda(\mathbf{m}_\lambda) \to e_0(\mathbf{m})$ as $\lambda \to 0$,

$$\text{weak-long:} \quad e_\lambda(\mathbf{m}_\lambda) \to \lim_{\lambda\to 0} e_0(\mathbf{m}_\lambda) \neq e_0(\mathbf{m}) \ ,$$

$$\text{weak-short:} \quad e_\lambda(\mathbf{m}_\lambda) \to \lim_{\lambda\to 0} e_1(\mathbf{m}_\lambda) \begin{cases} \neq e_0(\mathbf{m}_1), \\ \neq \lim\limits_{\lambda\to 0} e_0(\mathbf{m}_\lambda) \ . \end{cases}$$

Here, $\neq$ is read "not in general equal to" (see Firoozye [1993] for a more precise statement). The operator e_0 depends only on the lattice L_1, while e_1 depends on both the lattice and the sequence $\mathbf{m}_\lambda$.

Our general impression after doing these calculations is that there is a huge variety of ways that a material can store energy in weak-long and weak-short oscillations, this energy not being reflected by the magnetization of the material. Also, it appears to us that the weak-long/weak-short paradigm will emerge in the molecular derivation of other continuum theories. In this context the reader is referred to recent work of Gérard [1990-91] and Lions and Paul [1992].

Dipole calculations of the kind given here also appear in the theory of bubbly liquids and in the theory of dielectrics (Rayleigh [1892], Sangani and Acrivos [1983], Toupin [1956]). We do not pursue the application to bubbly liquids. Throughout the paper, we use the terminology of micromagnetics rather than the corresponding electric terminology. Many of the mathematical results we use can be gathered from books on singular integrals; for the most part we avoid making extensive reference to these books and present complete arguments. Several of the arguments given here have been recently generalized by Firoozye [1993].

2 Preliminaries

The field of a dipole is

$$\mathbf{K}(\mathbf{x}) := -\frac{1}{4\pi|\mathbf{x}|^3}\left\{\mathbf{1} - 3\frac{\mathbf{x}}{|\mathbf{x}|} \otimes \frac{\mathbf{x}}{|\mathbf{x}|}\right\}, \quad \mathbf{x} \in \mathbb{R}^3. \tag{2.1}$$

Except at the origin, $\mathbf{K}$ is smooth and satisfies

$$\mathbf{K}(\mathbf{x}) = -\nabla\left(\frac{\mathbf{x}}{4\pi|\mathbf{x}|^3}\right) = \nabla\nabla\left(\frac{1}{4\pi|\mathbf{x}|}\right). \tag{2.2}$$

We also use the notation $\mathbf{u}(\mathbf{x}) = \frac{\mathbf{x}}{4\pi|\mathbf{x}|^3}$ to denote the monopole field.

Because the field of a dipole exhibits a r^{-3} singularity, it is not a locally integrable function. For this reason, straightforward volume averages of arrays of dipoles are meaningless. A natural way to interpret both derivatives and averages in this context is by using the theory of distributions. In this section we collect several results on distributions that we use later.

The set $C_o^\infty(\mathbb{R}^n)$ denotes the set of infinitely differentiable functions with compact support on $\mathbb{R}^n$. A *distribution* is a continuous linear functional on $C_o^\infty(\mathbb{R}^n)$, and the set of distributions is denoted by $D'(\mathbb{R}^n)$. Here, the continuity of a distribution T refers to the statement $T(\psi_i) \to T(\psi)$ whenever the support

of each ψ_i is contained in a fixed compact set K and all derivatives of $\psi_i - \psi$ converge uniformly to zero (see Rudin [1991, Theorem 6.4]). We say that $T_i \to T$ in D' if and only if $T_i(\psi) \to T(\psi)$ for every $\psi \in C_0^\infty(\mathbb{R}^n)$. We use the standard notation $\delta_{\mathbf{a}}(\psi) = \psi(\mathbf{a})$ for the Dirac distribution.

Our expression for the field of an array of dipoles will have the form of a convolution of a distribution with a macroscopic dipole moment per unit volume. Generally, the convolution of a distribution will be defined as follows. Let $\psi \in C_0^\infty(\mathbb{R}^n)$ and let $\tau_{\mathbf{x}}(\mathbf{y}) := \psi(\mathbf{x} - \mathbf{y})$, $\mathbf{y} \in \mathbb{R}^n$. The convolution of a distribution with $\psi \in C_0^\infty(\mathbb{R}^n)$ is the function $(T * \psi) : \mathbb{R}^n \to \mathbb{R}$ defined by

$$(T * \psi)(\mathbf{x}) = T(\tau_{\mathbf{x}}). \tag{2.3}$$

We shall also need several variants of the convolution. As a general purpose notation, we will write for example $T(\psi(\mathbf{x} - \cdot))$ for the function $T(\tau_{\mathbf{x}})$ where $\tau_{\mathbf{x}}$ is defined above. Using this notation,

$$(T * \psi)(\mathbf{x}) = T(\psi(\mathbf{x} - \cdot)). \tag{2.4}$$

Another kind of convolution that will appear often is

$$(T \diamond \psi)(\mathbf{x}) := T(\psi(\mathbf{x} + \cdot)). \tag{2.5}$$

Let T be a distribution. The derivative of T is the distribution DT defined by

$$DT(\psi) = -T(\nabla\psi). \tag{2.6}$$

Higher derivatives are defined in a similar way. Our distributions will usually be vector or matrix-valued which just means that each component is a distribution. Thus, if a^i, $i = 1, \ldots, n$ is a vector-valued distribution, div $\mathbf{a}$ and curl $\mathbf{a}$ are the distributions defined by

$$(\text{curl } \mathbf{a}(\psi))_i := -\varepsilon_{ijk} a^j(\psi_{,k}), \quad \text{div } \mathbf{a}(\psi) := -a^i(\psi_{,i}). \tag{2.7}$$

In (2.7) the summation convention is used and ε_{ijk} is the permutation symbol. Similarly, the divergence of an $n \times n$ matrix- valued distribution $\mathbf{M}$ is the vector-valued distribution div $\mathbf{M}$ defined by

$$\text{div}\mathbf{M}(\psi)^j := -M^{ij}(\psi_{,i}). \tag{2.8}$$

Note that we adopt the convention that *div operates on the first index of a matrix-valued distribution.* For any locally integrable function $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ we say that "div $\mathbf{f} = 0$ in the sense of distributions" if the distribution

$$F(\psi) := -\int_{\mathbb{R}^n} \mathbf{f} \cdot \nabla\psi d\mathbf{x}, \quad \psi \in C_0^\infty(\mathbb{R}^n) \tag{2.9}$$

vanishes; this is consistent with $(2.7)_2$ if we associate a function and its distribution in the usual way.

Throughout this paper we interpret the field of a dipole as the matrix-valued distribution

$$\mathbf{K}(\psi) := \int_{\mathbb{R}^3} \mathbf{u}(\mathbf{x}) \otimes \nabla\psi(\mathbf{x}) d\mathbf{x}, \quad \psi \in C_o^\infty(\mathbb{R}^3), \tag{2.10}$$

where $\mathbf{u}(\mathbf{x}) = \frac{\mathbf{x}}{4\pi|\mathbf{x}|^3}$ is the monopole field. $\mathbf{K}$ is the well-defined because the monopole field is locally integrable. It will be clear from the context whether $\mathbf{K}$ stands for the distribution (2.10) or the function (2.1).

We shall often have occasion to extend the domain of a convolution $T * \psi$ from C_o^∞ to a larger complete normed linear space X, typically L^2. The extension is immediate if we can first prove the bound

$$||T * \psi||_X \leq c||\psi||_X \quad \text{for all} \quad \psi \in C_o^\infty(\mathbb{R}^n). \tag{2.11}$$

That is, if $\psi^k \in C_o^\infty(\mathbb{R}^n)$ is a Cauchy sequence in X, (2.11) shows that

$$||T * \psi^k - T * \psi^m||_X \leq c||\psi^k - \psi^m||_X, \tag{2.12}$$

so that $T * \psi^k$ is a Cauchy sequence in X. Hence we define for $\psi \in X$,

$$T(\psi) := \lim_{k\to\infty} T(\psi^{(k)}) \tag{2.13}$$

where $\psi^k \to \psi$ in X and $\psi^k \in C_o^\infty(\mathbb{R}^n)$. By (2.11) it follows that $T(\psi)$ is independent of the choice of approximating sequence.

Unless explicity stated, all norms $\|\cdot\|$ in this paper represent L^2 norms and $< \cdot, \cdot >$ denotes the standard inner product on L^2.

3 Precursor fields for a single dipole

In addition to the basic field $\mathbf{K}$ of a single dipole, there are other fields derived from $\mathbf{K}$ that arise in a natural way in lattice calculations. These are the magnetic induction, the Maxwell self-field and the Lorentz local field. In this section we define these fields and their associated distributions for a single dipole.

To motivate our expression for the magnetic induction, consider the function

$$\mathbf{B}_o^{(r)}(\mathbf{x}) := \begin{cases} \mathbf{K}(\mathbf{x}), & |\mathbf{x}| > r, \\ \frac{1}{2\pi r^3}\mathbf{1}, & |\mathbf{x}| \leq r. \end{cases} \tag{3.1}$$

Since the jump condition

$$[\![\mathbf{B}_o^{(r)}(\mathbf{x})]\!]\,\mathbf{n} = 0, \quad \mathbf{n} := \frac{\mathbf{x}}{|\mathbf{x}|}, \tag{3.2}$$

holds at $|\mathbf{x}| = r$, it follows that $\mathbf{B}_o^{(r)}$ is divergence-free in the sense of distributions. Also, if we write

$$\mathbf{B}_o^{(r)}(\mathbf{x}) = \mathbf{K}(\mathbf{x}) + \mathbf{M}_o^{(r)}(\mathbf{x}), \quad \mathbf{x} \neq 0\,, \tag{3.3}$$

it follows that $\mathbf{M}_0^{(r)}$ has compact support on the ball of radius r. The formula (3.3) is the precursor of the decomposition of the magnetic induction into the sum of the magnetic field and the magnetization.

More generally, it is enough to adopt the scaling represented in (3.1). Let $R \subset \mathbb{R}^3$ be bounded and open with a smooth boundary and with $0 \in R$. Let $\mathbf{K}_* \in L^1(R, M^{3\times 3})$ be chosen such that the function

$$\mathbf{B}^{(r)}(\mathbf{x}) := \begin{cases} \mathbf{K}(\mathbf{x}), & \mathbf{x} \in \mathbb{R}^3 - rR, \\ \dfrac{1}{r^3}\mathbf{K}_*\left(\dfrac{\mathbf{x}}{r}\right), & \mathbf{x} \in rR, \end{cases} \tag{3.4}$$

satisfies

$$\operatorname{div} \mathbf{B}^{(1)}(\mathbf{x}) = 0 \quad \text{in} \quad D'. \tag{3.5}$$

This can always be accomplished by first solving

$$\begin{aligned} &\triangle \mathbf{p} = \mathbf{0} \quad \text{on} \quad R, \\ &\frac{\partial \mathbf{p}}{\partial \mathbf{n}} = \mathbf{K}(\mathbf{y})\mathbf{n} \quad \text{for} \quad \mathbf{y} \in \partial R, \end{aligned} \tag{3.6}$$

and then by setting

$$\mathbf{K}_* := (\nabla \mathbf{p})^T. \tag{3.7}$$

(recall that divergence operates on the first index of a matrix). It is easily checked that the Neumann problem (3.6) is indeed solvable. Note that $\operatorname{div} \mathbf{B}^{(1)}(\mathbf{x}) = 0$ implies that $\operatorname{div}\mathbf{B}^{(r)}(\mathbf{x}) = 0$. The definitions (3.4) and (3.5) define the precursor of the *magnetic induction* of a dipole.

A different kind of decomposition of $\mathbf{K}$ arises naturally in lattice calculations of energy and corresponds to the principal value decomposition. For this purpose we define

$$\mathbf{P}_{MS}^{(r)}(\mathbf{x}) = \begin{cases} \mathbf{K}(\mathbf{x}), & |\mathbf{x}| > r, \\ \mathbf{0}, & |\mathbf{x}| \le r. \end{cases} \tag{3.8}$$

The subscript *MS* here refers to the *Maxwell self-field* of a dipole and identifies $\mathbf{P}_{MS}^{(r)}$ as the precursor of this field. The remaining part of the field,

$$\mathbf{P}_{LL}^{(r)} := \mathbf{K} - \mathbf{P}_{MS}^{(r)}, \tag{3.9}$$

is the precursor of the *Lorentz local field* of a dipole. To make the resulting formulae correspond with Lorentz's, we have used a spherical cut-out in (3.8) rather than the more general cut-out as in (3.4).

We first show that the dipole field $\mathbf{K}$ is approximated (in the sense of distributions) by either $\mathbf{B}^{(r)}$ or $\mathbf{P}_{MS}^{(r)}$ and certain Dirac masses at the origin.

Proposition 3.1. *Let* $\mathbf{G}_* \in L^1(R,\ M^{3x3})$ *and define for* $r > 0$,

$$\mathbf{G}^{(r)}(\mathbf{x}) := \begin{cases} \mathbf{K}(\mathbf{x}), & \mathbf{x} \in \mathbb{R}^3 - rR, \\ \dfrac{1}{r^3}\mathbf{G}_*\left(\dfrac{\mathbf{x}}{r}\right), & \mathbf{x} \in rR. \end{cases} \tag{3.10}$$

Then,

$$\mathbf{G}^{(r)} \to \mathbf{K} + \mathbf{M}\delta_{\mathbf{0}} \text{ in } D' \text{ as } r \to 0, \tag{3.11}$$

where $\delta_{\mathbf{0}}$ *is a Dirac mass at the origin and*

$$\mathbf{M} := \int_{\partial R} \frac{1}{4\pi}\frac{\mathbf{x}}{|\mathbf{x}|^3} \otimes \mathbf{n}\, \mathrm{dS} + \int_R \mathbf{G}_* d\mathbf{x}. \tag{3.12}$$

Furthermore, if $\operatorname{div} \mathbf{G}^{(1)} = \mathbf{0}$ *in the sense of distributions, then* $\mathbf{M} = \mathbf{1}$ *and, conversely, if* $\mathbf{M}=\mathbf{1}$ *then* $\operatorname{div} \mathbf{G}^{(r)} \to \mathbf{0}$ *in* D'.

It follows that the precursors of the magnetic induction and Maxwell self-field satisfy

$$\begin{aligned} \mathbf{B}^{(r)} &\to \mathbf{K} + \mathbf{1}\delta_{\mathbf{0}}, \\ \mathbf{P}^{(r)}_{MS} &\to \mathbf{K} + \frac{1}{3}\mathbf{1}\delta_{\mathbf{0}}, \end{aligned} \tag{3.13}$$

in D' *as* $r \to 0$.

Proof. From the definition of $\mathbf{G}^{(r)}$ we have for any $\psi \in C_0^\infty(\mathbb{R}^3)$,

$$\begin{aligned} \int_{\mathbb{R}^3} \mathbf{G}^{(r)}\psi \mathrm{d}\mathbf{x} &= \int_{\mathbb{R}^3 - rR} \mathbf{K}\psi \mathrm{d}\mathbf{x} + \int_{rR} \frac{1}{r^3}\mathbf{G}_*\left(\frac{\mathbf{x}}{r}\right)\psi(\mathbf{x})\mathrm{d}\mathbf{x} \\ &= \int_{\mathbb{R}^3 - rR} \nabla\left(\frac{-\mathbf{x}}{4\pi|\mathbf{x}|^3}\right)\psi \mathrm{d}\mathbf{x} + \int_R \mathbf{G}_*(\mathbf{y})\psi(r\mathbf{y})\mathrm{d}\mathbf{y} \\ &= \int_{\partial(rR)} \frac{\psi(\mathbf{x})\mathbf{x}\otimes\mathbf{n}}{4\pi|\mathbf{x}|^3}\mathrm{dS} + \int_{\mathbb{R}^3 - rR} \frac{\mathbf{x}\otimes\nabla\psi}{4\pi|\mathbf{x}|^3}\mathrm{d}\mathbf{x} + \int_R \mathbf{G}_*(\mathbf{y})\psi(r\mathbf{y})\mathrm{d}\mathbf{y} \\ &= \int_{\partial R} \frac{\psi(r\mathbf{y})\mathbf{y}\otimes\mathbf{n}}{4\pi|\mathbf{y}|^3}\mathrm{dS} + \int_{\mathbb{R}^3 - rR} \frac{\mathbf{x}\otimes\nabla\psi}{4\pi|\mathbf{x}|^3}\mathrm{d}\mathbf{x} + \int_R \mathbf{G}_*(\mathbf{y})\psi(r\mathbf{y})\mathrm{d}\mathbf{y}. \end{aligned} \tag{3.14}$$

Here, $\mathbf{n}$ is the outer normal of rR. Using the facts that $\psi \in C_0^\infty(\mathbb{R}^3)$ and $\mathbf{G}_* \in \mathrm{L}^1(R, \mathrm{M}^{3\times3})$ and the definition of the distribution $\mathbf{K}$, we take the limit of $(3.14)_4$ as $r \to 0$ and get (3.11) and (3.12).

Now suppose that $\operatorname{div} \mathbf{G}^{(1)} = 0$ so $\operatorname{div} \mathbf{G}^{(r)} = 0$ in D' for all $r > 0$. Passing to the limit in (3.11) and recalling the definition (2.8) of the divergence, we get

$$\int_{\mathbb{R}^3} \frac{(\nabla^2\psi)\mathbf{x}}{4\pi|\mathbf{x}|^3}\mathrm{d}\mathbf{x} + \mathbf{M}^T\nabla\psi(0) = 0. \tag{3.15}$$

Since the monopole field $\mathbf{u}(\mathbf{x}) = \frac{\mathbf{x}}{4\pi|\mathbf{x}|^3}$ is in L^1, (3.15) implies that

$$\int_{\mathbb{R}^3 - B(\rho)} \frac{(\nabla^2\psi)\mathbf{x}}{4\pi|\mathbf{x}|^3} \, d\mathbf{x} + \mathbf{M}^T \nabla\psi(0) = o(1), \tag{3.16}$$

where $B(\rho)$ is the ball of radius ρ and $o(1) \to 0$ as $\rho \to 0$. Integrating the first term of (3.16) by parts, we get

$$-\int_{\partial B(\rho)} \frac{\nabla\psi}{4\pi\rho^3}\mathbf{x} \cdot \mathbf{n} dS - \int_{\mathbb{R}^3 - B(\rho)} (\mathrm{div}\mathbf{u})\nabla\psi d\mathbf{x} + \mathbf{M}^T\nabla\psi(0) = o(1). \tag{3.17}$$

Since $\mathrm{div}\mathbf{u} = 0$ on $\mathbb{R}^3 - B(\rho)$ and $\mathbf{n} = \frac{\mathbf{x}}{\rho}$ for $\mathrm{x} \in \partial B(\rho)$ we can take the limit of (3.17) as $\rho \to 0$ and get that $\mathbf{M} = \mathbf{1}$. Conversely, if $\mathbf{M} = \mathbf{1}$ then (3.17) holds so that (3.16) and (3.15) also hold; hence, $\mathrm{div}\mathbf{G}^{(r)} \to 0$ in D'. The first of (3.13) now follows immediately from (3.5), while the second of (3.13) follows from the identity

$$\frac{1}{4\pi}\int_{\partial B(1)} \mathbf{n} \otimes \mathbf{n} dS = \frac{1}{3}\mathbf{1}. \tag{3.18}$$ □

4 Long wavelength oscillations vs. oscillations on the scale of the lattice

We consider an array of dipoles lying on the points of a Bravais lattice. To each of these lattice points we assign a vector representing the dipole moment. As we refine the lattice by letting a typical lattice parameter go to zero, we will have to decrease the strength of this dipole field accordingly, so that the macroscopic magnetization (the dipole moment per unit volume) is bounded. The forms of macroscopic fields will be relatively insensitive to the type of convergence assumed for the dipole field, whereas the form of the energy will be sensitive to the type of convergence assumed. In this section we define and compare several types of convergence used later.

A *Bravais lattice* is the set of points given by

$$L(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) = \left\{\mathbf{x} \in \mathbb{R}^3 : \mathbf{x} = \nu^i\mathbf{e}_i \quad \text{where} \quad \nu^1, \nu^2, \nu^3 \in Z\right\}. \tag{4.1}$$

Here $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \in \mathbb{R}^3$ are linearly independent *lattice vectors* and Z denotes the integers. We fix $\{\mathbf{e}_i\}$ and write

$$L_\lambda := L(\lambda\mathbf{e}_1, \lambda\mathbf{e}_2, \lambda\mathbf{e}_3), \quad \lambda > 0. \tag{4.2}$$

For convenience we assume that the lattice vectors have been chosen such that the *unit cell*

$$U := \left\{\mathbf{x} \in \mathbb{R}^3 : \mathbf{x} = \alpha^i\mathbf{e}_i, 0 \leq \alpha_1, \alpha_2, \alpha_3 < 1\right\} \tag{4.3}$$

has unit volume. We also use the terminology *unit cell* for any translate of U by $\mathbf{x} \in L_1$.

Consider a discrete dipole field $\mathbf{d}_\lambda : L_\lambda \to \mathbb{R}^3$ defined for each $\lambda \in (0, 1)$. We wish to consider convergence of the dipole field $\mathbf{d}_\lambda$ as $\lambda \to 0$. As the $\mathbf{d}_\lambda$ are defined on different lattices for different values of λ, it is useful to consider extensions of $\mathbf{d}_\lambda$ to all of $\mathbb{R}^3$. Then we can consider the usual kinds of convergence for the extended fields. The *standard extension* of $\mathbf{d}_\lambda$ is the scaled piecewise constant extension $\mathbf{d}_\lambda^\sim$ given by

$$\mathbf{d}_\lambda^\sim(\mathbf{y}) = \lambda^{-3}\mathbf{d}_\lambda(\mathbf{x}) \quad \text{for} \quad \mathbf{x} \in L_\lambda \quad \text{and} \quad \mathbf{y} \in \mathbf{x} + \lambda U. \tag{4.4}$$

The field $\mathbf{d}_\lambda^\sim : \mathbb{R}^3 \to \mathbb{R}^3$ is constant on each unit cell of L_λ. We say that $\mathbf{d}_\lambda$ *converges strongly* as $\lambda \to 0$ to $\mathbf{m} \in L^2(\mathbb{R}^3, \mathbb{R}^3)$ if

$$\mathbf{d}_\lambda^\sim \to \mathbf{m} \quad \text{in} \quad L^2(\mathbb{R}^3, \mathbb{R}^3). \tag{4.5}$$

Similarly, we say that $\mathbf{d}_\lambda$ *converges weakly* to $\mathbf{m} \in L^2(\mathbb{R}^3, \mathbb{R}^3)$ if

$$\mathbf{d}_\lambda^\sim \rightharpoonup \mathbf{m} \quad \text{in} \quad L^2(\mathbb{R}^3, \mathbb{R}^3). \tag{4.6}$$

In the latter case we will also assume that

$$\lim_{\rho \to \infty} \sup_{\lambda \in (0,1]} \int_{\mathbb{R}^3 - B(\rho)} |\mathbf{d}_\lambda^\sim|^2 \, d\mathbf{x} = 0, \tag{4.7}$$

which insures that the dipole moment per unit volume does not "escape to infinity" as $\lambda \to 0$. In particular (4.7) holds if all the $\mathbf{d}_\lambda$ vanish outside a fixed bounded set. Note that because of the scaling λ^{-3} in (4.4), $\mathbf{m}$ represents a dipole moment per unit volume. In our terminology strong convergence corresponds to no oscillations. Fine-scale oscillations are modeled by weak convergence of $\mathbf{d}_\lambda$. Later, as part of our calculations of the energy, we are led to further subdivide the case of weak convergence to "weak-long" and "weak- short" cases. Further discussion of the physical interpretation of strong and weak convergence is given in Sections 7-9. Note that our terminology is slightly nonstandard because of the factor λ^{-3} in (4.4).

An alternative way to consider convergence of the $\mathbf{d}_\lambda$ is to assume the existence of a *background field* $\mathbf{m}_b \in L^2(\mathbb{R}^3, \mathbb{R}^3)$ and then to define $\mathbf{d}_\lambda$ by

$$\mathbf{d}_\lambda(\mathbf{x}) := \int_{\mathbf{x}+\lambda U} \mathbf{m}_b(\mathbf{z}) d\mathbf{z}. \tag{4.8}$$

More generally, we could allow the background field to vary with λ. The following shows that the notions of strong and weak convergence defined above in terms of the standard extension correspond to assuming the existence of background fields that converge, respectively, strongly and weakly in the conventional sense.

Proposition 4.1. *Let* $\mathbf{m}_\lambda$, $\lambda \in (0, 1]$, *be a sequence of vector fields in* $L^2(\mathbb{R}^3, \mathbb{R}^3)$ *with*

$$\mathbf{m}_\lambda \to \mathbf{m} \quad \textit{in} \quad L^2(\mathbb{R}^3, \mathbb{R}^3) \tag{4.9}$$

(respectively, $\mathbf{m}_\lambda \rightharpoonup \mathbf{m}$ *in* $L^2(\mathbb{R}^3, \mathbb{R}^3)$*). Let* $\mathbf{d}_\lambda : L_\lambda \to \mathbb{R}^3$ *be defined by*

$$\mathbf{d}_\lambda(\mathbf{x}) = \int_{\mathbf{x}+\lambda U} \mathbf{m}_\lambda(\mathbf{z})\mathrm{d}\mathbf{z}. \tag{4.10}$$

Then $\mathbf{d}_\lambda$ *converges strongly* (resp. weakly) *to* $\mathbf{m}$ *in the sense of* (4.5) (resp. (4.6)). *Moreover, if*

$$\lim_{\rho\to\infty} \sup_{\lambda\in(0,1]_{\mathbb{R}^3-B(\rho)}} \int |\mathbf{m}_\lambda|^2 \,\mathrm{d}\mathbf{x} = 0 \tag{4.11}$$

then the standard extension of $\mathbf{d}_\lambda$ *satisfies* (4.7).

Conversely, if $\mathbf{d}_\lambda : L_\lambda \to \mathbb{R}$ *is a strongly* (resp. weakly) *convergent sequence in the sense of* (4.5) (resp. (4.6)), *then there is a strongly* (resp. weakly) *convergent background field* $\mathbf{m}_\lambda \in L^2(\mathbb{R}^3, \mathbb{R}^3)$ *so that* $\mathbf{d}_\lambda$ *is given by* (4.10). *If* $\mathbf{d}_\lambda$ *also satisfies* (4.7) *then the sequence* $\mathbf{m}_\lambda$ *can be chosen to satisfy* (4.11).

Proof. The converse is obvious as we may choose $\mathbf{m}_\lambda = \mathbf{d}_\lambda^{\sim}$. To prove the first part, we first note that if $\mathbf{g} \in L^2$ and $\mathbf{r}_\lambda : L_\lambda \to \mathbb{R}^3$ is defined by the formula

$$\mathbf{r}_\lambda(\mathbf{x}) := \int_{\mathbf{x}+\lambda U} \mathbf{g}(\mathbf{z})\mathrm{d}\mathbf{z}, \tag{4.12}$$

then the standard extension of $\mathbf{r}_\lambda$ belongs to L^2 and satisfies

$$||\mathbf{r}_\lambda^{\sim}|| \leq ||\mathbf{g}||. \tag{4.13}$$

The result (4.13) follows from the calculation

$$||\mathbf{r}_\lambda^{\sim}||^2 = \lambda^{-3} \sum_{\mathbf{x}\in L_\lambda} |\mathbf{r}_\lambda(\mathbf{x})|^2 \leq \sum_{\mathbf{x}\in L_\lambda} \int_{\mathbf{x}+\lambda U} |\mathbf{g}(\mathbf{z})|^2 \,\mathrm{d}\mathbf{z} = ||\mathbf{g}||^2, \tag{4.14}$$

where the inequality follows from Jensen's inequality. We assume that $\mathbf{m}_\lambda \to \mathbf{m}$ in L^2 and that $\mathbf{d}_\lambda$ is defined by (4.10). Let $\delta > 0$ be given. By the density of C_0^∞ in L^2 there is a $\hat{\mathbf{m}} \in C_0^\infty(\mathbb{R}^3, \mathbb{R}^3)$ such that

$$\|\hat{\mathbf{m}} - \mathbf{m}\| < \delta. \tag{4.15}$$

Let $\hat{\mathbf{d}}_\lambda : L_\lambda \to \mathbb{R}^3$ be given by

$$\hat{\mathbf{d}}_\lambda(\mathbf{x}) := \int_{\mathbf{x}+\lambda U} \hat{\mathbf{m}}(\mathbf{y})\mathrm{d}\mathbf{y} \tag{4.16}$$

with standard extension $\hat{\mathbf{d}}_\lambda^{\sim}$. We have

$$\|\mathbf{d}_\lambda^{\sim} - \mathbf{m}\| \leq \|\mathbf{d}_\lambda^{\sim} - \hat{\mathbf{d}}_\lambda^{\sim}\| + \|\hat{\mathbf{d}}_\lambda^{\sim} - \hat{\mathbf{m}}\| + \|\hat{\mathbf{m}} - \mathbf{m}\|. \tag{4.17}$$

By the linearity of standard extensions and the result (4.12), (4.13), the first term on the right hand side of (4.17) satisfies

$$\|\mathbf{d}_\lambda^{\sim} - \hat{\mathbf{d}}_\lambda^{\sim}\| \leq \|\mathbf{m}_\lambda - \hat{\mathbf{m}}\| \leq \|\mathbf{m}_\lambda - \mathbf{m}\| + \|\mathbf{m} - \hat{\mathbf{m}}\|. \tag{4.18}$$

To estimate the second term on the right hand side of (4.17), we note that for $\mathbf{y} \in \mathbf{x} + \lambda U$

$$|\hat{\mathbf{d}}_\lambda^{\sim}(\mathbf{y}) - \hat{\mathbf{m}}(\mathbf{y})| \leq \lambda^{-3} \int_{\mathbf{x}+\lambda U} |\hat{\mathbf{m}}(\mathbf{z}) - \hat{\mathbf{m}}(\mathbf{y})| d\mathbf{z} \leq \quad \hat{c}_\delta \lambda, \tag{4.19}$$

where $\hat{c}_\delta$ depends upon the choice of $\hat{\mathbf{m}}$. Therefore, since $\hat{\mathbf{d}}_\lambda^{\sim}$ and $\hat{\mathbf{m}}$ vanish outside a bounded set, $\hat{\mathbf{d}}_\lambda^{\sim} \to \hat{\mathbf{m}}$ uniformly as $\lambda \to 0$. Combining (4.15), (4.18) and (4.19) with (4.17) we get

$$\limsup_{\lambda \to 0} \|\mathbf{d}_\lambda^{\sim} - \mathbf{m}\| \leq 2\delta, \tag{4.20}$$

and so by the arbitrariness of δ we get $\mathbf{d}_\lambda^{\sim} \to \mathbf{m}$ in $L^2(\mathbb{R}^3, \mathbb{R}^3)$.

Next we consider the case of weak convergence. By the result just proved, it is only necessary to consider the case $\mathbf{m}_\lambda \rightharpoonup 0$. Let $\psi \in C_0^\infty(\mathbb{R}^3, \mathbb{R})$. We introduce the *corner map*, $\mathbf{x}_\lambda(\mathbf{y}) := \mathbf{x}$ for $\mathbf{y} \in \mathbf{x} + \lambda U$, $\mathbf{x} \in L_\lambda$, and write

$$\begin{aligned}
\int_{\mathbb{R}^3} \mathbf{d}_\lambda^{\sim}(\mathbf{y})\psi(\mathbf{y})d\mathbf{y} &= \int_{\mathbb{R}^3} \lambda^{-3} \int_{\mathbf{x}_\lambda(\mathbf{y})+\lambda U} \mathbf{m}_\lambda(\mathbf{z})\psi(\mathbf{y})d\mathbf{z}d\mathbf{y} \\
&= \int_{\mathbb{R}^3} \lambda^{-3} \int_{\mathbf{x}_\lambda(\mathbf{y})+\lambda U} \mathbf{m}_\lambda(\mathbf{z})\psi(\mathbf{z})d\mathbf{z}d\mathbf{y} \\
&+ \int_{\mathbb{R}^3} \lambda^{-3} \int_{\mathbf{x}_\lambda(\mathbf{y})+\lambda U} \mathbf{m}_\lambda(\mathbf{z}) \ (\psi(\mathbf{y}) - \psi(\mathbf{z})) \ d\mathbf{z}d\mathbf{y}.
\end{aligned} \tag{4.21}$$

To estimate the second term in (4.21), we let A be a set containing supp $\psi + U$, and we let ν_λ be the number of lattice points of L_λ in A so that $\nu_\lambda \leq$ const. λ^{-3}. Two applications of Jensen's inequality gives

$$\begin{aligned}
&\left(\int_{\mathbb{R}^3} \lambda^{-3} \int_{\mathbf{x}_\lambda(\mathbf{y})+\lambda U} \mathbf{m}_\lambda(\mathbf{z})(\psi(\mathbf{y}) - \psi(\mathbf{z}))d\mathbf{z} \ d\mathbf{y} \right)^2, \\
&\leq \left(\int_{\mathbb{R}^3} \lambda^{-3} \int_{\mathbf{x}_\lambda(\mathbf{y})+\lambda U} |\mathbf{m}_\lambda(\mathbf{z})| c\lambda \ d\mathbf{z} \right)^2, \\
&\leq c^2\lambda^2\nu_\lambda^2 \left(\frac{1}{\nu_\lambda^2} \sum_{\mathbf{y} \in L_\lambda \cap A} \int_{\mathbf{x}_\lambda(\mathbf{y})+\lambda U} |\mathbf{m}_\lambda(\mathbf{z})| d\mathbf{z} \right)^2, \\
&\leq c^2\lambda^2\nu_\lambda \sum_{\mathbf{y} \in L_\lambda \cap A} \lambda^6 \left(\frac{1}{\lambda^3} \int_{\mathbf{x}_\lambda(\mathbf{y})+\lambda U} |\mathbf{m}_\lambda(\mathbf{z})| d\mathbf{z} \right)^2, \\
&\leq c^2\lambda^2(\nu_\lambda\lambda^3)\|\mathbf{m}_\lambda\|^2.
\end{aligned} \tag{4.22}$$

Also, the first integral on the right hand side of $(4.21)_2$ is

$$\int_{\mathbb{R}^3} \mathbf{m}_\lambda(\mathbf{z})\psi(\mathbf{z})d\mathbf{z} \to 0, \tag{4.23}$$

by assumption. Hence, $\mathbf{d}_\lambda^{\sim} \to 0$. To complete the proof, we note that (4.7) implies (4.11) because

$$\int_{\mathbb{R}^3 - B(\rho)} |\lambda^{-3} \int_{\mathbf{x}_\lambda(\mathbf{y}) + \lambda U} \mathbf{m}_\lambda(\mathbf{z}) d\mathbf{z}|^2 d\mathbf{y} \leq \int_{\mathbb{R}^3 - B(\rho')} |\mathbf{m}_\lambda(\mathbf{z})|^2 d\mathbf{z}. \tag{4.24}$$

where $\rho' = \rho + c_1\lambda$ for some $c_1 > 0$ (recall that meas $U = 1$). □

The choice of a Bravais lattice as the underlying lattice is not crucial for our calculations, although the nature of the results would be slightly altered in several cases. For example, there could be several unit cells corresponding to different sublattices or the unit cell could be a function of the lattice point. The important hypothesis for the calculations of the following sections is the hypothesis of strong (or weak) convergence of the appropriate standard extension (*cf.* (4.4)-(4.6)).

5 Field of a fine array of dipoles

In this section we derive formulae for various macroscopic fields under conditions of weak convergence of the dipole field $\mathbf{d}_\lambda$ as $\lambda \to 0$. The results of this section are insensitive as to whether weak or strong convergence is assumed.

For applications to micromagnetics and other theories the main case of interest is the case in which the dipole field has compact support. Hence, we assume the existence of a background field $\mathbf{m}_\lambda \to \mathbf{m}$ in L^1 with supp $\mathbf{m}_\lambda \subset \Omega$ such that

$$\mathbf{d}_\lambda(\mathbf{x}) = \int_{\mathbf{x} + \lambda U} \mathbf{m}_\lambda(\mathbf{z}) d\mathbf{z}, \quad \mathbf{x} \in L_\lambda. \tag{5.1}$$

If $\mathbf{G} : \mathbb{R}^3 \to M^{3\times 3}$ represents a field quantity associated with a single dipole, e.g., the field $\mathbf{K}$, magnetic induction $\mathbf{B}^{(r)}$, the Maxwell self-field $\mathbf{P}_{MS}^{(r)}$ or the Lorentz local field $\mathbf{P}_{LL}^{(r)}$, we define the corresponding field quantity for the array of dipoles by

$$\mathbf{g}_\lambda(\mathbf{x}) := \sum_{\mathbf{y} \in L_\lambda} \mathbf{G}(\mathbf{x} - \mathbf{y}) \mathbf{d}_\lambda(\mathbf{y}), \tag{5.2}$$

that is, by summing the field quantity at lattice points. Because of the issues raised in Section 2, we will view the field quantities for the array of dipoles as distributions. For quantities in $L^1(\mathbb{R}^3, M^{3\times 3})$ such as $\mathbf{G} = \mathbf{B}^{(r)}$ or $\mathbf{G} = \mathbf{P}_{MS}^{(r)}$ we define distributions in the obvious way,

$$\mathbf{g}_\lambda(\psi) := \int_{\mathbb{R}^3} \mathbf{g}(\mathbf{x}) \psi(\mathbf{x}) d\mathbf{x} = \sum_{\mathbf{y} \in L_\lambda} \int_{\mathbb{R}^3} \psi(\mathbf{x}) \mathbf{G}(\mathbf{x} - \mathbf{y}) \mathbf{d}_\lambda(\mathbf{y}) d\mathbf{x} \tag{5.3}$$

for $\psi \in C_0^\infty(\mathbb{R}^3, \mathbb{R})$. For quantities not in $L^1(\mathbb{R}^3, M^{3\times 3})$ we use special formulae. In particular, the *field* $\mathbf{h}_\lambda$ *of an array of dipoles* is given by the sum of (2.10) over lattice points

$$\mathbf{h}_\lambda(\psi) := \sum_{\mathbf{y} \in L_\lambda} \int_{\mathbb{R}^3} (\mathbf{u}(\mathbf{x} - \mathbf{y}) \cdot \mathbf{d}_\lambda(\mathbf{y})) \nabla \psi(\mathbf{x}) d\mathbf{x} \tag{5.4}$$

for all $\psi \in C_o^\infty(\mathbb{R}^3, \mathbb{R})$. In (5.4) $\mathbf{u}(\mathbf{x}) = \frac{\mathbf{x}}{4\pi|\mathbf{x}|^3}$ is the monopole field.

We are interested in the limits of field quantities for an array of dipoles as $\lambda \to 0$. These are obtained by use of the following result.

Proposition 5.1. *Let $T^{(k)}$ be a one- parameter family of distributions and suppose that*

$$T^{(k)} \to T \quad \text{in} \quad D'. \tag{5.5}$$

Suppose $\mathbf{m}_\lambda \rightharpoonup \mathbf{m}$ in L^1 with supp $\mathbf{m}_\lambda \subset \Omega$ and let

$$\mathbf{d}_\lambda(\mathbf{x}) := \int_{\mathbf{x}+\lambda U} \mathbf{m}_\lambda(\mathbf{y})d\mathbf{y}, \ \mathbf{x} \in L_\lambda, \tag{5.6}$$

where L_λ and U are defined by (4.2) *and* (4.3). *Let $k_\lambda \to \infty$ as $\lambda \to 0$. Define the distribution*

$$T_\lambda(\psi) := \sum_{\mathbf{y}\in L_\lambda} T^{(k_\lambda)}(\psi(\mathbf{y}+\cdot))\mathbf{d}_\lambda(\mathbf{y}). \tag{5.7}$$

Then

$$T_\lambda \to \hat{T} \text{ in } D' \text{ as } \lambda \to 0, \tag{5.8}$$

where (cf. (2.5))

$$\hat{T}(\psi) := T(\mathbf{m} \diamond \psi), \quad \psi \in C_o^\infty(\mathbb{R}^3) \tag{5.9}$$

and

$$(\mathbf{m} \diamond \psi)(\mathbf{x}) = \int_{\mathbb{R}^3} \mathbf{m}(\mathbf{z}-\mathbf{x})\psi(\mathbf{z})d\mathbf{z}. \tag{5.10}$$

Proof. Let $\psi \in C_o^\infty(\mathbb{R}^3)$ and let T_λ be given by (5.8). By linearity we have,

$$T_\lambda(\psi) = T^{(k_\lambda)}(\phi_\lambda), \tag{5.11}$$

where

$$\phi_\lambda(\mathbf{x}) = \sum_{\mathbf{y}\in L_\lambda} \psi(\mathbf{x}+\mathbf{y})\mathbf{d}_\lambda(\mathbf{y}). \tag{5.12}$$

Note that since $\mathbf{d}_\lambda$ has compact support on Ω, $\phi_\lambda \in C_o^\infty(\mathbb{R}^3, \mathbb{R}^3)$ for each $\lambda > 0$. Hence, it suffices to show that

$$\phi_\lambda \to \mathbf{m} \diamond \psi \quad \text{in} \quad C_o^\infty(\mathbb{R}^3, \mathbb{R}^3). \tag{5.13}$$

Furthermore, since

$$\frac{\partial^\alpha \phi_\lambda(\mathbf{x})}{\partial \mathrm{x}^\alpha} = \sum_{\mathbf{y}\in L_\lambda} \frac{\partial^\alpha \psi(\mathbf{x}+\mathbf{y})}{\partial \mathrm{x}^\alpha}\mathbf{d}_\lambda(\mathbf{y}), \tag{5.14}$$

it suffices to verify uniform convergence in (5.13). By (5.6) and (5.10) we have,

$$
\begin{aligned}
|\phi_\lambda(\mathbf{x}) - (\mathbf{m} \diamond \psi)(\mathbf{x})| &= \left| \sum_{\mathbf{y} \in L_\lambda} \psi(\mathbf{x}+\mathbf{y}) \mathbf{d}_\lambda(\mathbf{y}) - \int_{\mathbb{R}^3} \psi(\mathbf{z}) \mathbf{m}(\mathbf{z}-\mathbf{x}) d\mathbf{z} \right|, \\
&= \left| \sum_{\mathbf{y} \in L_\lambda} \left\{ \psi(\mathbf{x}+\mathbf{y}) \int_{\mathbf{y}+\lambda U} \mathbf{m}_\lambda(\mathbf{z}) d\mathbf{z} - \int_{\mathbf{y}+\lambda U} \psi(\mathbf{x}+\mathbf{w}) \mathbf{m}(\mathbf{w}) d\mathbf{w} \right\} \right|, \\
&= |\mathbf{a}_\lambda(\mathbf{x}) + \mathbf{b}_\lambda(\mathbf{x})|,
\end{aligned}
\tag{5.15}
$$

where

$$
\begin{aligned}
\mathbf{a}_\lambda(\mathbf{x}) &:= \sum_{\mathbf{y} \in L_\lambda} \left\{ \psi(\mathbf{x}+\mathbf{y}) \int_{y+\lambda U} (\mathbf{m}_\lambda(\mathbf{z}) - \mathbf{m}(\mathbf{z})) \, d\mathbf{z} \right\}, \\
\mathbf{b}_\lambda(\mathbf{x}) &:= \sum_{\mathbf{y} \in L_\lambda} \int_{y+\lambda U} (\psi(\mathbf{x}+\mathbf{y}) - \psi(\mathbf{x}+\mathbf{z})) \, \mathbf{m}(\mathbf{z}) d\mathbf{z}.
\end{aligned}
\tag{5.16}
$$

Clearly

$$
|\mathbf{b}_\lambda| \leq \sup_{\mathbf{v} \in \mathbb{R}^3} \sup_{\mathbf{w} \in \lambda U} |\psi(\mathbf{v}+\mathbf{w}) - \psi(\mathbf{v})| \, ||\mathbf{m}||_{L^1}, \tag{5.17}
$$

so $|\mathbf{b}_\lambda| \to 0$ as $\lambda \to 0$ uniformly. To analyze $\mathbf{a}_\lambda$, we first recall the corner map $\mathbf{x}_\lambda$ introduced just before (4.21). For $\mathbf{x}, \mathbf{z} \in \mathbb{R}^3$ we define $\psi^{\sim}(\mathbf{x}, \mathbf{z}) := \psi(\mathbf{x}+\mathbf{x}_\lambda(\mathbf{z}))$. Then $\mathbf{a}_\lambda$ can be written

$$
\begin{aligned}
\mathbf{a}_\lambda(\mathbf{x}) &= \sum_{\mathbf{y} \in L_\lambda} \int_{\mathbf{y}+\lambda U} \psi^{\sim}(\mathbf{x}, \mathbf{z}) (\mathbf{m}_\lambda(\mathbf{z}) - \mathbf{m}(\mathbf{z})) d\mathbf{z}, \\
&= \int_{\mathbb{R}^3} \psi^{\sim}(\mathbf{x}, \mathbf{z}) (\mathbf{m}_\lambda(\mathbf{z}) - \mathbf{m}(\mathbf{z})) d\mathbf{z}.
\end{aligned}
\tag{5.18}
$$

Hence,

$$
\begin{aligned}
|\mathbf{a}_\lambda(\mathbf{x})| &= \left| \int_{\mathbb{R}^3} (\psi^{\sim}(\mathbf{x}, \mathbf{z}) - \psi(\mathbf{x}+\mathbf{z})) (\mathbf{m}_\lambda(\mathbf{z}) - \mathbf{m}(\mathbf{z})) \, d\mathbf{z} \right. \\
&\quad \left. + \int_{\mathbb{R}^3} \psi(\mathbf{x}+\mathbf{z}) (\mathbf{m}_\lambda(\mathbf{z}) - \mathbf{m}(\mathbf{z})) \, d\mathbf{z} \right| \\
&\leq \int_{\mathbb{R}^3} |\psi^{\sim}(\mathbf{x}, \mathbf{z}) - \psi(\mathbf{x}+\mathbf{z})| \, |\mathbf{m}_\lambda(\mathbf{z}) - \mathbf{m}(\mathbf{z})| \, d\mathbf{z} \\
&\quad + \left| \int_{\mathbb{R}^3} \psi(\mathbf{x}+\mathbf{z}) (\mathbf{m}_\lambda(\mathbf{z}) - \mathbf{m}(\mathbf{z})) \, d\mathbf{z} \right|.
\end{aligned}
\tag{5.19}
$$

The first term of $(5.19)_2$ is bounded by

$$
\lambda (\sup_{\mathbb{R}^3} \nabla\psi) \operatorname{diam} U \, ||\mathbf{m}_\lambda - \mathbf{m}||_{L^1}. \tag{5.20}
$$

Since any weakly convergent sequence in L^1 has the property that its L^1 norm is uniformly bounded, the first term in $(5.19)_2$ tends to zero uniformly as $\lambda \to 0$. The second term in $(5.19)_2$ tends to zero pointwise by the definition of weak convergence. Since the gradient of this term (inside the absolute value signs) is uniformly bounded, the second term in $(5.19)_2$ tends to zero uniformly. Combining these results, we conclude that the left hand side of (5.15) tends to zero uniformly as $\lambda \to 0$, completing the proof. □

Using Propositions 5.1 and 3.1, we can find the limits of field quantities. This is done below.

1. *Magnetic field.* Here, we take $T^{(k)}$ of Proposition 5.1 to have the form $T^{(i)} = \mathbf{K}^T$ (independent of i) and observe that formula (5.7) gives $T_\lambda = \mathbf{h}_\lambda$, where $\mathbf{h}_\lambda$ is defined by (5.4). Applying Proposition 5.1, we get

$$\mathbf{h}_\lambda \to \mathbf{h} \quad \text{in} \quad D' \tag{5.21}$$

where

$$\mathbf{h} = \int_{\mathbb{R}^3} \left\{ \int_{\mathbb{R}^3} \mathbf{u}(\mathbf{z}-\mathbf{x}) \cdot \mathbf{m}(\mathbf{x}) \mathrm{d}\mathbf{x} \right\} \nabla\psi(\mathbf{z}) \mathrm{d}\mathbf{z}, \tag{5.22}$$

which can be written as $-D(\mathbf{u} * \mathbf{m})$.

2. *Magnetic induction.* The precursor of the magnetic induction $\mathbf{B}^{(r)}$ defined by (3.3) involves the additional parameter r. This suggests two ways to define the macroscopic magnetic induction. We could first fix r, apply Proposition 5.1 to $\mathbf{B}^{(r)}$, and then take the limit as $r \to 0$. In this case, more and more dipoles would enter the region rR as $\lambda \to 0$. Alternatively, we could let r depend on λ with, say, r_λ having a certain decay rate as $\lambda \to 0$; if the decay rate is sufficiently fast then $r_\lambda R$ would only contain one dipole for all $\lambda > 0$. In fact, the same macroscopic magnetic induction is obtained in both cases.

To see this, we first apply Proposition (5.1) to $T^{(k)} = \mathbf{B}^{(r)}$ (independent of k) with r fixed. The formula (5.7) for the induction $\mathbf{b}_\lambda^{(r)}$ is then

$$\mathbf{b}_\lambda^{(r)}(\psi) := \sum_{\mathbf{y} \in L_\lambda} \int_{\mathbb{R}^3} \mathbf{B}^{(r)}(\mathbf{x}-\mathbf{y}) \mathbf{d}_\lambda(\mathbf{y}) \psi(\mathbf{x}) \mathrm{d}\mathbf{x}, \quad \psi \in C_0^\infty(\mathbb{R}^3). \tag{5.23}$$

Note that $\mathbf{b}_\lambda^{(r)}$ is divergence-free, because $\mathbf{B}^1$ is divergence-free. Applying Proposition (5.1) and using the definition (3.3) of $\mathbf{B}^{(r)}$, we get that

$$\mathbf{b}_\lambda^{(r)} \to \mathbf{b}^{(r)} \quad \text{in} \quad D' \text{ as } \lambda \to 0, \tag{5.24}$$

where

$$\mathbf{b}^{(r)}(\psi) := \int_{\mathbb{R}^3} \psi(\mathbf{z}) \int_{\mathbb{R}^3} \mathbf{B}^{(r)}(\mathbf{z}-\mathbf{x}) \mathbf{m}(\mathbf{x}) \mathrm{d}\mathbf{x} \mathrm{d}\mathbf{z}. \tag{5.25}$$

By (3.4) $\mathbf{b}^{(r)}$ can be written

$$\mathbf{b}^{(r)}(\psi) = \mathbf{b}_1^{(r)}(\psi) + \mathbf{b}_2^{(r)}(\psi), \tag{5.26}$$

where

$$\mathbf{b}_1^{(r)}(\psi) := \int_{\mathbb{R}^3} \mathbf{m}(\mathbf{x}) \int_{\mathbf{x}+(\mathbb{R}^3-rR)} \mathbf{K}(\mathbf{z}-\mathbf{x})\psi(\mathbf{z})\mathrm{d}\mathbf{z}\mathrm{d}\mathbf{x},$$

$$\mathbf{b}_2^{(r)}(\psi) := \int_{\mathbb{R}^3} \mathbf{m}(\mathbf{x}) \int_{\mathbf{x}+rR} \frac{1}{r^3}\mathbf{K}_*\left(\frac{\mathbf{z}-\mathbf{x}}{r}\right)\psi(\mathbf{z})\mathrm{d}\mathbf{z}\mathrm{d}\mathbf{x}. \tag{5.27}$$

It is easily checked by calculations analogous to those in (3.14) that

$$\mathbf{b}^{(r)} \rightharpoonup \mathbf{h}+\mathbf{m} \quad \text{in} \quad D', \tag{5.28}$$

where $\mathbf{h}$ is defined by (5.22) and $\mathbf{m}$ is the magnetization. Thus, if we let $\mathbf{b}$ denote the limit of $\mathbf{b}^{(r)}$ as $r \to 0$, we recover the standard formula of electromagnetism,

$$\mathbf{b} = \mathbf{h}+\mathbf{m}. \tag{5.29}$$

The distribution $\mathbf{b}$ inherits the property of being divergence-free.

Alternatively, we can begin again and let r depend on λ; say $r_\lambda \to 0$ as $\lambda \to 0$. Then we apply Proposition 5.1 to the distribution $T^{(k)} = \mathbf{B}^{(\frac{1}{k})}$. Letting $k_\lambda = (r_\lambda)^{-1}$ we observe that (5.7) is a formula for $\mathbf{b}_\lambda^{(r_\lambda)}$ (*cf.* (5.23)). Let $\mathbf{b}_\lambda = \mathbf{b}_\lambda^{(r_\lambda)}$. Proposition 5.1 now allows us to evaluate the limit of $\mathbf{b}_\lambda$ as $\lambda \to 0$ in D'. Applying Proposition 5.1, we get the same result as (5.29), *i.e.*,

$$\mathbf{b}_\lambda \rightharpoonup \mathbf{h}+\mathbf{m} \quad \text{in} \quad D'. \tag{5.30}$$

3.*Maxwell self-field.* Again we have two alternatives for defining the Maxwell self-field. Here, we shall just treat the case $r_\lambda \to 0$ as $\lambda \to 0$, and remark that the alternative procedure ($r = r_0$, $\lambda \to 0$, then $r_0 \to 0$) gives the same result. We take $T^{(k)}$ of Proposition 5.1 to have the form $T^{(k)} = \mathbf{P}_{MS}^{(\frac{1}{k})}$ and let $k_\lambda = (r_\lambda)^{-1}$. The formula (5.7) gives

$$\mathbf{p}_{MS}^{(\lambda)}(\psi) := \sum_{\mathbf{y}\in L_\lambda} \int_{\mathbb{R}^3} \mathbf{P}_{MS}^{(r_\lambda)}(\mathbf{x}-\mathbf{y})\mathbf{d}_\lambda(\mathbf{y})\psi(\mathbf{x})\mathrm{d}\mathbf{x} \tag{5.31}$$

for all $\psi \in C_0^\infty(\mathbb{R}^3)$. Applying Proposition 5.1, we get

$$\mathbf{p}_{MS}^{(\lambda)} \rightharpoonup \mathbf{h}+\frac{1}{3}\mathbf{m} \quad \text{in} \quad D'. \tag{5.32}$$

With regard to the description of macroscopic fields, the Maxwell self-field does not offer any particular advantages. For example, it is not generally divergence-free. However, it plays a major role in calculations of energy. The main conclusion of this section is that $\mathbf{m}$ is interpreted as the magnetization under conditions of either weak or strong convergence.

6 Energy of a fine array of dipoles without oscillations

In this section we calculate the total energy of a fine array of dipoles in the case of strong convergence of the background field. This calculation differs from the previous one in that here we are simply calculating the limit of a sequence of numbers. We say that the background field is without oscillations under the conditions of strong convergence. Throughout this section we fix $r > 0$.

The energy of a lattice of dipoles is calculated in the standard way by summing the work done in bringing each dipole from ∞ to a point on the lattice (see Brown [1962, 1963]). This gives the expression

$$e_\lambda = \frac{-1}{2} \sum_{\substack{\mathbf{x},\mathbf{y}\in L_\lambda \\ \mathbf{x}\neq\mathbf{y}}} \mathbf{d}_\lambda(\mathbf{x}) \cdot \mathbf{K}(\mathbf{x}-\mathbf{y})\mathbf{d}_\lambda(\mathbf{y}). \tag{6.1}$$

We assume the existence of a background field $\mathbf{m}_\lambda$ with

$$\mathbf{d}_\lambda(\mathbf{x}) = \int_{\mathbf{x}+\lambda U} \mathbf{m}_\lambda(\mathbf{z})d\mathbf{z}, \tag{6.2}$$

(see Section 4).

For now we assume $\mathbf{m}_\lambda \in C_0^\infty(\mathbb{R}^3, \mathbb{R}^3)$ so that the expression for e_λ is well-defined. Later (Lemma 6.1) we will show that (6.1) makes sense for $\mathbf{m}_\lambda \in L^2(\mathbb{R}^3, \mathbb{R}^3)$. Introducing characteristic functions, we can write (6.1) in the form

$$e_\lambda = \frac{-1}{2} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \mathbf{m}_\lambda(\mathbf{x}) \cdot \mathbf{K}_\lambda(\mathbf{x},\mathbf{y})\mathbf{m}_\lambda(\mathbf{y})d\mathbf{x}d\mathbf{y}, \tag{6.3}$$

where

$$\mathbf{K}_\lambda(\mathbf{x},\mathbf{y}) := \sum_{\substack{\mathbf{v},\mathbf{w}\in L_\lambda \\ \mathbf{v}\neq\mathbf{w}}} \chi_{\mathbf{v}+\lambda U}(\mathbf{x})\mathbf{K}(\mathbf{v}-\mathbf{w})\chi_{\mathbf{w}+\lambda U}(\mathbf{y}). \tag{6.4}$$

Using the fact that characteristic functions have the scaling property $\chi_{\lambda D}(\mathbf{x}) = \chi_D(\frac{\mathbf{x}}{\lambda})$, $\lambda > 0$, and using the definition (2.1) of $\mathbf{K}$, we get the scaling

$$\mathbf{K}_\lambda(\mathbf{x},\mathbf{y}) = \frac{1}{\lambda^3}\mathbf{K}_1(\frac{\mathbf{x}}{\lambda}, \frac{\mathbf{y}}{\lambda}). \tag{6.5}$$

The formula for e_λ can be written

$$e_\lambda = \frac{-1}{2} \langle \mathbf{m}_\lambda, \mathbf{T}_\lambda \mathbf{m}_\lambda \rangle_{L^2} \tag{6.6}$$

where

$$(\mathbf{T}_\lambda \mathbf{m})(\mathbf{x}) := \int_{\mathbb{R}^3} \mathbf{K}_\lambda(\mathbf{x},\mathbf{y})\mathbf{m}(\mathbf{y})d\mathbf{y}. \tag{6.7}$$

The map $\mathbf{T}_\lambda$ is a kind of discretized version of the Maxwell self-field, as can be seen by placing the expression (6.4) into (6.7) and by comparing with (5.31) and (3.8).

We now show that $\mathbf{T}_\lambda$ extends in a natural way from a map from $C_0^\infty \to L^2$ to a map from $L^2 \to L^2$. Once this has been shown, it is clear from (6.6) that e_λ is well-defined for any $\mathbf{m}_\lambda \in L^2$.

Proposition 6.1. *The map* $\mathbf{T}_\lambda : \mathbf{m} \to \int_{\mathbb{R}^3} \mathbf{K}_\lambda(\cdot, \mathbf{y})\mathbf{m}(\mathbf{y})d\mathbf{y}$ *defined on* $C_0^\infty(\mathbb{R}^3, \mathbb{R}^3)$ *extends to a bounded linear map from* $L^2(\mathbb{R}^3, \mathbb{R}^3)$ *to* $L^2(\mathbb{R}^3, \mathbb{R}^3)$. *Moreover,*

$$\|\mathbf{T}_\lambda\|_{\mathcal{L}(L^2,L^2)} = \|\mathbf{T}_1\|_{\mathcal{L}(L^2,L^2)} \quad \forall \lambda > 0. \tag{6.8}$$

Proof. Assuming $\mathbf{T}_1$ is bounded linear map from L^2 to L^2, (6.8) follows from the scaling (6.5). That is, if we let $(\mathbf{S}_\lambda \mathbf{f})(\mathbf{x}) := \lambda^{\frac{3}{2}}\mathbf{f}(\lambda\mathbf{x})$ we note first that $\mathbf{S}_\lambda$ is an isometry from $L^2(\mathbb{R}^3, \mathbb{R}^3)$ to $L^2(\mathbb{R}^3, \mathbb{R}^3)$. It then follows from (6.5) that

$$(\mathbf{T}_\lambda \mathbf{m})(\mathbf{x}) = \mathbf{T}_1(\lambda^{-\frac{3}{2}}\mathbf{S}_\lambda \mathbf{m})(\frac{\mathbf{x}}{\lambda}) = (\mathbf{S}_\lambda^{-1}\mathbf{T}_1\mathbf{S}_\lambda \mathbf{m})(\mathbf{x}). \tag{6.9}$$

Hence, it is sufficient to show that $\mathbf{T}_1$ extends to a bounded linear map from L^2 to L^2. According to the argument presented at the end of Section 2, it is sufficient to show that for all $\mathbf{m} \in C_0^\infty(\mathbb{R}^3, \mathbb{R}^3)$

$$\|\mathbf{T}_1 \mathbf{m}\| \leq c\|\mathbf{m}\| \tag{6.10}$$

for some constant $c > 0$. Recalling the definition of the Maxwell self-field, equation (3.8), we let

$$\mathbf{L}^{(r)}(\mathbf{x}, \mathbf{y}) := \mathbf{K}_1(\mathbf{x}, \mathbf{y}) - \mathbf{P}_{MS}^{(r)}(\mathbf{x} - \mathbf{y}). \tag{6.11}$$

We claim that there is a function $g^{(r)} \in L^1(\mathbb{R}^3, \mathbb{R})$ such that

$$|\mathbf{L}^{(r)}(\mathbf{a}, \mathbf{b})| \leq g^{(r)}(|\mathbf{a} - \mathbf{b}|). \tag{6.12}$$

To prove (6.12), we first observe that both $|\mathbf{K}_1|$ and $|\mathbf{P}_{MS}^{(r)}|$ are bounded for each $r > 0$. The latter is obvious from the definition (3.8); the former becomes obvious when we write $\mathbf{K}_1$ in terms of the corner map $\mathbf{x}_\lambda(\mathbf{y})$ introduced just before (4.21) (specialized to $\lambda = 1$), *viz.*,

$$\mathbf{K}_1(\mathbf{a}, \mathbf{b}) = \begin{cases} 0, & \mathbf{a} \text{ and } \mathbf{b} \text{ in the same unit cell of } L_1, \\ \mathbf{K}(\mathbf{x}_1(\mathbf{a}) - \mathbf{x}_1(\mathbf{b})), & \text{otherwise} \end{cases} \tag{6.13}$$

Hence, $|\mathbf{L}^{(r)}(\mathbf{a} - \mathbf{b})|$ is bounded for $|\mathbf{a} - \mathbf{b}|$ bounded. Therefore, to establish the estimate (6.12) it is only necessary to consider $\mathbf{a}$ and $\mathbf{b}$ satisfying $|\mathbf{b} - \mathbf{a}| > (\text{diam}U + r)$. In this case,

$$|\mathbf{L}^{(r)}(\mathbf{a}, \mathbf{b})| = |\mathbf{K}(\mathbf{x}_1(\mathbf{a}) - \mathbf{x}_1(\mathbf{b})) - \mathbf{K}(\mathbf{a} - \mathbf{b})| \leq \frac{c_1}{c_2 + |a - b|^4}, \tag{6.14}$$

where $c_1 > 0$ and $c_2 > 0$ depend only on r and $\text{diam}U$. This establishes (6.12).

Now we combine (6.12) with a standard result on convolution (see *e.g.*, Brezis [1983, Theorem IV.15, p. 66]) to get that

$$\int_{\mathbb{R}^3} \mathbf{L}^{(r)}(\mathbf{x}, \mathbf{y})\mathbf{m}(\mathbf{y})d\mathbf{y}$$

defines a bounded linear map on $C_0^\infty(\mathbb{R}^3, \mathbb{R}^3)$. The boundedness of the remaining part of $\mathbf{T}_1\mathbf{m}$,

$$\| \int_{\mathbb{R}^3} \mathbf{P}_{MS}^{(r)}(\mathbf{x}-\mathbf{y})\mathbf{m}(\mathbf{y})d\mathbf{y}\| \leq c\|\mathbf{m}\|, \tag{6.15}$$

follows by applying a standard result on singular integrals (Stein [1970, Chapt. II, Section 4]). To apply this result, we only need to observe that for any $r > 0$

$$\int_{\partial B(r)} \mathbf{P}_{MS}^{(r)}(\mathbf{x})d\mathbf{x} = 0,$$

which follows from (3.18) and the definition of $\mathbf{P}_{MS}^{(r)}$.

The proof is completed by using the argument at the end of Section 2, (2.11)-(2.13).□

Our main result of this section is a precise version of the expression (*cf.* Brown [1963]) for the energy of a lattice of dipoles under conditions of strong convergence. Recall that $\text{meas}U = 1$.

Theorem 6.2. *Let*

$$\mathbf{d}_\lambda(\mathbf{x}) = \int_{\mathbf{x}+\lambda U} \mathbf{m}_\lambda(\mathbf{z})d\mathbf{z} \tag{6.16}$$

and suppose $\mathbf{m}_\lambda \to \mathbf{m}$ *in* $L^2(\mathbb{R}^3, \mathbb{R}^3)$. *Let* $\mathbf{T}_\lambda$ *be given by* (6.7) *and let* e_λ *be given by* (6.1). *Then*

$$\mathbf{T}_\lambda\mathbf{m}_\lambda \to \mathbf{K}*\mathbf{m} + \left(\frac{1}{3}\mathbf{1} + \mathbf{S}\right)\mathbf{m} \quad \text{in} \quad L^2(\mathbb{R}^3, \mathbb{R}^3), \tag{6.17}$$

and

$$e_\lambda \to \frac{-1}{2}\left(\langle \mathbf{m}, \mathbf{K}*\mathbf{m}\rangle + \frac{1}{3}\|\mathbf{m}\|^2 + \langle \mathbf{m}, \mathbf{S}\mathbf{m}\rangle\right), \tag{6.18}$$

where

$$\mathbf{S} := \lim_{\rho\to\infty} \sum_{\mathbf{y}\in(L_1\cap B(\rho))\setminus\{\mathbf{o}\}} \mathbf{K}(\mathbf{y}). \tag{6.19}$$

Remark 6.3. The term $\mathbf{K}*\mathbf{m}$ in (6.17) is well-defined for $\mathbf{m} \in L^2(\mathbb{R}^3, \mathbb{R}^3)$ by the argument given at the end of Section 2, equations (2.11) - (2.13) with $X = L^2(\mathbb{R}^3, \mathbb{R}^3)$. To use this argument we must show that (*cf.* (2.11))

$$\|\mathbf{K}*\mathbf{f}\| \leq c\|\mathbf{f}\| \quad \text{for all} \quad \mathbf{f} \in C_0^\infty(\mathbb{R}^3, \mathbb{R}^3). \tag{6.20}$$

The bound (6.20) follows immediately by taking the limit of (6.15) as $r \to 0$, noting that by Stein [1970, *loc. cit.*] c in (6.20) is independent of r, and then by using $(3.13)_2$. Alternatively, the property (6.20) follows from properties of the

Newtonian potential and Poisson's equation. For $\mathbf{f} \in C_0^\infty(\mathbb{R}^3, \mathbb{R}^3)$ we can write (in components)

$$K_{ij} * f_j = \frac{-\partial^2}{\partial x_i \partial x_j} \int_{\mathbb{R}^3} \frac{-1}{4\pi|\mathbf{x}-\mathbf{y}|} f_j(\mathbf{y}) d\mathbf{y} \tag{6.21}$$

(*cf.* Rudin [1991], Theorem 6.30). Let g_j be the integral on the right hand side of (6.21). Then (*cf.* Gilbarg and Trudinger [1977], Chapter 4) g_j is the Newtonian potential of f_j so, with $\mathbf{f} \in C_0^\infty$,

$$\Delta g_j = f_j \quad \text{on} \quad \mathbb{R}^3,$$

$$K_{ij} * f_j = -g_{j,ij}. \tag{6.22}$$

By the theory of elliptic equations (e.g., Gilbarg and Trudinger [1977], Chapter 8.3), $\mathbf{g}$ satisfies

$$\|\text{grad div}\mathbf{g}\| \leq c\|\mathbf{f}\|, \tag{6.23}$$

which establishes (6.20). Alternatively, the inequality (6.23) follows by the use of Fourier transforms, as below in Example 7.5.

Proof of Theorem 6.2 According to Lemma 6.1, it suffices to consider $\mathbf{m}_\lambda = \mathbf{m}$ (independent of λ) with $\mathbf{m} \in C_0^\infty(\mathbb{R}^3, \mathbb{R}^3)$. That is,

$$\mathbf{T}_\lambda \mathbf{m}_\lambda = \mathbf{T}_\lambda \mathbf{m} + \mathbf{T}_\lambda(\mathbf{m} - \mathbf{m}_\lambda) \tag{6.24}$$

and second term on the right goes to zero by Lemma 6.1 while the first term on the right equals $\lim_{k\to\infty} \mathbf{T}_\lambda \mathbf{m}^k$ for $\mathbf{m}^k \to \mathbf{m}$ in L^2, $\mathbf{m}^k \in C_0^\infty$. As in the proof of Lemma 6.1, we define $\mathbf{L}^{(r)}$ by (6.11) and write

$$\begin{aligned} \mathbf{K}_\lambda(\mathbf{x}, \mathbf{y}) = &\ \lambda^{-3} \mathbf{P}_{MS}^{(r)}\left(\frac{\mathbf{x}-\mathbf{y}}{\lambda}\right) + \lambda^{-3}\mathbf{L}^{(r)}\left(\frac{\mathbf{x}}{\lambda}, \frac{\mathbf{y}}{\lambda}\right), \\ = &\ \mathbf{P}_{MS}^{(r\lambda)}(\mathbf{x}-\mathbf{y}) + \lambda^{-3}\mathbf{L}^{(r)}\left(\frac{\mathbf{x}}{\lambda}, \frac{\mathbf{y}}{\lambda}\right). \end{aligned} \tag{6.25}$$

The limiting behavior of $\mathbf{P}_{MS}^{(r\lambda)}$ as $\lambda \to 0$ is known from $(3.13)_2$, so it remains to study $\mathbf{L}^{(r)}$.

We first note that by the argument between (6.12) and (6.14) we have for any $R > 0$

$$\begin{aligned} &\int_{|\mathbf{x}-\mathbf{y}|\geq R\lambda} \lambda^{-3} \left|\mathbf{L}^{(r)}\left(\frac{\mathbf{x}}{\lambda}, \frac{\mathbf{y}}{\lambda}\right)\right| d\mathbf{y}, \\ &\leq c \int_{|\mathbf{x}-\mathbf{y}|\geq R\lambda} \frac{\lambda}{\lambda^4 + |\mathbf{x}-\mathbf{y}|^4} d\mathbf{y} \leq c \min\left(\frac{1}{R}, 1\right). \end{aligned} \tag{6.26}$$

We then write

$$\begin{aligned}
&\lambda^{-3}\int_{\mathbb{R}^3}\mathbf{L}^{(r)}\left(\frac{\mathbf{x}}{\lambda},\frac{\mathbf{y}}{\lambda}\right)\mathbf{m}(\mathbf{y})\,d\mathbf{y}\\
&=\int_{\mathbb{R}^3}\lambda^{-3}\mathbf{L}^{(r)}\left(\frac{\mathbf{x}}{\lambda},\frac{\mathbf{y}}{\lambda}\right)d\mathbf{y}\,\mathbf{m}(\mathbf{x})\\
&-\int_{|\mathbf{x}-\mathbf{y}|\geq R\lambda}\lambda^{-3}\mathbf{L}^{(r)}\left(\frac{\mathbf{x}}{\lambda},\frac{\mathbf{y}}{\lambda}\right)d\mathbf{y}\,\mathbf{m}(\mathbf{x})\\
&+\int_{|\mathbf{x}-\mathbf{y}|\leq R\lambda}\lambda^{-3}\mathbf{L}^{(r)}\left(\frac{\mathbf{x}}{\lambda},\frac{\mathbf{y}}{\lambda}\right)(\mathbf{m}(\mathbf{y})-\mathbf{m}(\mathbf{x}))\,d\mathbf{y}\\
&+\int_{|\mathbf{x}-\mathbf{y}|\geq R\lambda}\left(\lambda^{-3}\mathbf{L}^{(r)}\left(\frac{\mathbf{x}}{\lambda},\frac{\mathbf{y}}{\lambda}\right)\right)\mathbf{m}(\mathbf{y})\,d\mathbf{y}.
\end{aligned}\tag{6.27}$$

Consider the limit $\lambda\to 0$ with R fixed. In view of (6.26) the third term on the right hand side of (6.27) goes to zero while the second and fourth are bounded by $\frac{1}{R}$ as $R\to\infty$. Note that

$$\int_{\mathbb{R}^3}\lambda^{-3}\mathbf{L}^{(r)}\left(\frac{\mathbf{x}}{\lambda},\frac{\mathbf{y}}{\lambda}\right)d\mathbf{y}=\int_{\mathbb{R}^3}\mathbf{L}^{(r)}\left(\frac{\mathbf{x}}{\lambda},\mathbf{y}\right)d\mathbf{y}.$$

We will see momentarily that the term on the right is independent of $\mathbf{x}$. Thus taking the limit $\lambda\to 0$ and the $R\to\infty$ in (6.27), we obtain

$$\lim_{\lambda\to 0}\lambda^{-3}\int_{\mathbb{R}^3}\mathbf{L}^{(r)}\left(\frac{\mathbf{x}}{\lambda},\frac{\mathbf{y}}{\lambda}\right)\mathbf{m}(\mathbf{y})\,d\mathbf{y}=\mathbf{m}(\mathbf{x})\int_{\mathbb{R}^3}\mathbf{L}^{(r)}(0,\mathbf{y})d\mathbf{y}\,\mathbf{m}(\mathbf{x})\tag{6.28}$$

We have used

Lemma 6.4. *Let* $\mathbf{L}^{(r)}$ *be defined by* (6.11). *For all* $\mathbf{x}\in\mathbb{R}^3$,

$$\int_{\mathbb{R}^3}\mathbf{L}^{(r)}(\mathbf{x},\mathbf{y})d\mathbf{y}=\int_{\mathbb{R}^3}\mathbf{L}^{(r)}(0,\mathbf{y})d\mathbf{y}.\tag{6.29}$$

Proof. First note that by the argument between (6.12) and (6.14) the separate integrals in (6.29) exist. Letting $B(\rho)$ be the ball of radius ρ and recalling the corner map $\mathbf{x}_1(\mathbf{x})$ for the lattice L_1 (see (6.13)), we write

$$\begin{aligned}
\int_{\mathbf{x}_1(\mathbf{x})+B(\rho)}\mathbf{L}^{(r)}(\mathbf{x},\mathbf{y})d\mathbf{y}&=\int_{\mathbf{x}_1(\mathbf{x})+B(\rho)}\mathbf{K}_1(\mathbf{x},\mathbf{y})d\mathbf{y}+\int_{\mathbf{x}_1(\mathbf{x})+B(\rho)}\mathbf{P}^{(r)}_{MS}(\mathbf{x}-\mathbf{y})d\mathbf{y},\\
&=\int_{B(\rho)}\mathbf{K}_1(\mathbf{x},\mathbf{x}_1(\mathbf{x})+\mathbf{z})d\mathbf{z}+\int_{B(\rho)}\mathbf{P}^{(r)}_{MS}(\mathbf{x}-\mathbf{x}_1(\mathbf{x})-\mathbf{z})d\mathbf{z}.
\end{aligned}\tag{6.30}$$

By (6.13) $\mathbf{K}_1(\mathbf{x}+\mathbf{z}, \mathbf{y}+\mathbf{z}) = \mathbf{K}_1(\mathbf{x}, \mathbf{y})$ for any $\mathbf{z} \in L_1$. Hence the integrand of the first integral on the right hand side of (6.30) is

$$\begin{aligned}\mathbf{K}_1(\mathbf{x}, \mathbf{x}_1(\mathbf{x}) + \mathbf{z}) &= \mathbf{K}_1(\mathbf{x} - \mathbf{x}_1(\mathbf{x}), \mathbf{z}) \\ &= \sum_{\mathbf{w} \in L_1 \setminus \{\mathbf{o}\}} \mathbf{K}(-\mathbf{w}) \chi_{\mathbf{w}+U}(\mathbf{z}) = \mathbf{K}_1(0, \mathbf{z}).\end{aligned} \tag{6.31}$$

Turning attention to the second integral in (6.30), we put $\mathbf{y} = \mathbf{x} - \mathbf{x}_1(\mathbf{x})$ and note that by spherical symmetry (*cf.* $(6.15)_{ff}$)

$$\int_{B(\rho)} \mathbf{P}_{MS}^{(r)}(\mathbf{y} - \mathbf{z}) d\mathbf{z} = \int_{B(\rho)} \mathbf{P}_{MS}^{(r)}(\mathbf{y} - \mathbf{z}) - \mathbf{P}_{MS}^{(r)}(-\mathbf{z}) d\mathbf{z}. \tag{6.32}$$

The integrand on the right hand side of (6.32) is in L^1, so we can pass to the limit $\rho \to \infty$ in (6.32) and (6.30). To complete the proof of Lemma 6.4, we observe that

$$\begin{aligned} T :&= \int_{\mathbb{R}^3} \mathbf{P}_{MS}^{(r)}(\mathbf{y} - \mathbf{z}) - \mathbf{P}_{MS}^{(r)}(-\mathbf{z}) d\mathbf{z} \\ &= \int_{\mathbb{R}^3} \mathbf{P}_{MS}^{(r)}\left(\frac{1}{2}\mathbf{y} + \mathbf{w}\right) - \mathbf{P}_{MS}^{(r)}\left(-\frac{1}{2}\mathbf{y} + \mathbf{w}\right) d\mathbf{w} \\ &= \int_{\mathbb{R}^3} \mathbf{P}_{MS}^{(r)}\left(-\frac{1}{2}\mathbf{y} - \mathbf{w}\right) - \mathbf{P}_{MS}^{(r)}\left(\frac{1}{2}\mathbf{y} - \mathbf{w}\right) d\mathbf{w} \\ &= -T. \end{aligned} \tag{6.33}$$

Hence, $T = 0$. □

Continuation of the proof of Theorem 6.2.

With $\mathbf{m} \in C_0^\infty(\mathbb{R}^3, \mathbb{R}^3)$ we have by $(3.12)_2$

$$\left(\mathbf{P}_{MS}^{(r\lambda)} * \mathbf{m}\right)(\mathbf{x}) \to (\mathbf{K} * \mathbf{m})(\mathbf{x}) + \frac{1}{3}\mathbf{m}(\mathbf{x}) \quad \text{as} \quad \lambda \to 0 \quad \text{pointwise.} \tag{6.34}$$

Collecting the results (6.28) and (6.34) and keeping in mind Proposition 6.1, we get (6.17) and (6.18) with (see below for further explanation)

$$\begin{aligned} \mathbf{S} &= \int_{\mathbb{R}^3} \mathbf{L}^{(r)}(0, \mathbf{y}) d\mathbf{y} \\ &= \lim_{\rho \to \infty} \int_{B(\rho)} \mathbf{L}^{(r)}(0, \mathbf{y}) d\mathbf{y} \\ &= \lim_{\rho \to \infty} \int_{B(\rho)} \mathbf{K}_1(0, \mathbf{y}) - \mathbf{P}_{MS}^{(r)}(-\mathbf{y}) \quad d\mathbf{y} \\ &= \lim_{\rho \to \infty} \int_{B(\rho)} \mathbf{K}_1(0, \mathbf{y}) d\mathbf{y} \\ &= \lim_{\rho \to \infty} \sum_{\substack{\mathbf{y} \in L_1 \setminus \{0\} \\ \cap B(\rho)}} \mathbf{K}(\mathbf{y}). \end{aligned} \tag{6.35}$$

Here the term involving $\mathbf{P}^{(r)}_{MS}(-\mathbf{y})$ drops out of $(6.35)_3$ by spherical symmetry. The passage from $(6.35)_4$ to $(6.35)_5$ follows by noting that the measure of the spherical annulus of width $2(\text{diam}U)$ and radius ρ is proportional to ρ^2 plus lower order terms in ρ; since $\mathbf{K}$ decays as ρ^{-3}, the difference between the sums $(6.35)_4$ and $(6.35)_5$ tends to zero as $\rho \to \infty$. □

Remark 6.4. The lattice sum $\mathbf{S}$ vanishes when the lattice has cubic symmetry (e.g., BCC, FCC, simple cubic). To see this we first recall that the *point group* of L_1 is the set

$$\mathbf{P}_1 = \{\mathbf{Q} \in O(3) : \mathbf{Q}L_1 = L_1\}.$$

By (6.19), if $\mathbf{Q} \in \mathbf{P}_1$,

$$\mathbf{S} = \lim_{\rho\to\infty} \sum_{\substack{\mathbf{y}\in L_1\setminus\{0\} \\ \cap B(\rho)}} \mathbf{K}(\mathbf{Qy}) = \mathbf{QSQ}^T. \tag{6.36}$$

Hence, $\mathbf{QSQ}^T = \mathbf{S}$ for all $\mathbf{Q} \in P_1$. If P_1 is a cubic point group, then the only $\mathbf{S}$ satisfying this identity is $\mathbf{S} = \alpha\mathbf{1}$, $\alpha \in \mathbb{R}$. By taking the trace of (6.36) and noting that $tr\mathbf{K} = \mathbf{0}$, we find that in the cubic case $\mathbf{S} = \mathbf{0}$.

7 Energy of a fine array of dipoles with long wave oscillations (weak-long)

As discussed above in Section 4, oscillations of the background field $\mathbf{m}_\lambda$ on the scale of the lattice are modeled by weak convergence $\mathbf{m}_\lambda \rightharpoonup \mathbf{m}$ (in L^2). It is expected, however, that the notion of "oscillations on the scale of the lattice" and the notion of weak convergence do not precisely coincide. For example, a background field of the form

$$\mathbf{m}_\lambda(\mathbf{x}) = \psi(\mathbf{x})\mathbf{m}_\#(\mathbf{x}/\lambda^\alpha) \tag{7.1}$$

with $0 < \alpha < 1$, ψ having compact support, and $\mathbf{m}_\#$ periodic on $\mathbb{R}^3$ has the property that it converges weakly in L^2 (but not strongly). However, a typical wavelength of the oscillations of (7.1) is $const.\lambda^\alpha$ which for $0 < \alpha < 1$ is much larger than the lattice spacing λ in the sense that $const.\lambda^\alpha/\lambda \to \infty$ as $\lambda \to 0$. For this example it might be expected that the macroscopic operator $\mathbf{w} \to \langle \mathbf{w}, \mathbf{K} * \mathbf{w}\rangle + \frac{1}{3}\|\mathbf{w}\|^2 + \langle \mathbf{w}, \mathbf{Sw}\rangle$ plays some role, and this in fact turns out to be true (we calculate the energy of the sequence $\tilde{\mathbf{m}}_\lambda$ given by (7.1) at the end of this section).

To quantify this idea, we introduce the quantity $R_\rho(\mathbf{w})$ defined by

$$R_\rho(\mathbf{w}) := \sup_{|\mathbf{z}|\le\rho} \|\mathbf{w}(\cdot + \mathbf{z}) - \mathbf{w}(\cdot)\|, \quad \mathbf{w} \in L^2. \tag{7.2}$$

$R_\rho(\mathbf{w})$ measures the correlation between $\mathbf{w}(\cdot)$ and it translates $\mathbf{w}(\cdot + \mathbf{z})$, $|\mathbf{z}| \le \rho$; the connection with correlations is seen from the alternative form of (7.2)

obtained by expanding the square,

$$\frac{1}{2}R_\rho(\mathbf{w})^2 = \|\mathbf{w}\|^2 - \inf_{|\mathbf{z}|\le\rho} \int_{\mathbb{R}^3} \mathbf{w}(\mathbf{x}+\mathbf{z})\cdot\mathbf{w}(\mathbf{x})d\mathbf{x}. \tag{7.3}$$

It is easy to verify that the sequence $\tilde{\mathbf{m}}_\lambda$ given by (7.1) is well correlated with itself on a scale of length λ in the sense that $R_\lambda(\tilde{\mathbf{m}}_\lambda) \to 0$ as $\lambda \to 0$. It follows from the definition that for $\mathbf{w} \in L^2$, $R_\sigma(\mathbf{w})$ has the properties

1. $R_\sigma(\mathbf{w}) \to 0$ as $\sigma \to 0$,
2. $R_{k\sigma}(\mathbf{w}) \le kR_\sigma(\mathbf{w}), \quad k = 1, 2, 3\ldots,$ (7.4)
3. $R_\sigma(\mathbf{w}) \le 2\|\mathbf{w}\|,$
4. $\|\rho_\delta * \mathbf{w} - \mathbf{w}\| \le R_\delta(\mathbf{w})$, where $\rho_\delta(\mathbf{x}) = \frac{1}{\delta^3}\rho(\frac{\mathbf{x}}{\delta})$ is the standard mollifier.

If $\alpha = 1$ in (7.1), then $R_\lambda(\mathbf{m}_\lambda)$ stays bounded away from zero as $\lambda \to 0$, except in trivial cases. Such a sequence models oscillations that are truly on the scale of the lattice. To reflect this distinction, we use the terminology ***weak − long*** to describe background fields $\mathbf{m}_\lambda \rightharpoonup \mathbf{m}$ in L^2 such that $R_\lambda(\mathbf{m}_\lambda) \to 0$ as $\lambda \to 0$ and ***weak − short*** to describe the case $\inf_{\lambda>0} R_\lambda(\mathbf{m}_\lambda) > 0$.

We now show that in the weak-long case the limiting energy is obtained by evaluating the macroscopic operator on the sequence.

Proposition 7.1. *Let* $\mathbf{m}_\lambda \rightharpoonup \mathbf{m}$ *in* $L^2(\mathbb{R}^3, \mathbb{R}^3)$, *define*

$$\mathbf{d}_\lambda(\mathbf{x}) = \int_{\mathbf{x}+\lambda U} \mathbf{m}_\lambda(\mathbf{z})d\mathbf{z}, \tag{7.5}$$

and suppose

$$R_\lambda(\mathbf{m}_\lambda) \to 0 \text{ as } \lambda \to 0. \tag{7.6}$$

Let the energy e_λ *be given by (6.1) and let* $\mathbf{T}_0$ *be defined by*

$$\mathbf{T}_0\mathbf{w} := \mathbf{K} * \mathbf{w} + \left(\frac{1}{3}\mathbf{1} + \mathbf{S}\right)\mathbf{w}, \quad \mathbf{w} \in L^2(\mathbb{R}^3, \mathbb{R}^3). \tag{7.7}$$

(*cf.* (6.17)). Then,

$$\lim_{\lambda\to 0} e_\lambda = \lim_{\lambda\to 0} \frac{-1}{2}\langle \mathbf{m}_\lambda, \mathbf{T}_0\mathbf{m}_\lambda\rangle. \tag{7.8}$$

Remark 7.2. Proposition (7.1) shows that in the weak-long case the energy is only influenced by the configuration of the lattice through the lattice sum $\mathbf{S}$ (*cf.* (6.19)), like the case of strong convergence. However, in contrast to the case of strong convergence, it is necessary to know more about the dipoles than the macroscopic magnetization in order to calculate the weak-long energy, *i.e.*, it is not generally possible to pass to the limit in (7.8).

Proof of Proposition 7.1 The quantity to be calculated is

$$e_\lambda = \frac{-1}{2}\langle \mathbf{m}_\lambda, \mathbf{T}_\lambda\mathbf{m}_\lambda\rangle \tag{7.9}$$

and the goal of the proof is to show that for λ small we can essentially replace $\mathbf{T}_\lambda$ by $\mathbf{T}_o$ in (7.9). As before, we write $\mathbf{T}_\lambda$ in the form (6.7) and split $\mathbf{K}_\lambda$ into the Maxwell self-kernel and the remaining local part (*cf.*(6.25)):

$$\mathbf{K}_\lambda(\mathbf{x},\mathbf{y}) = \mathbf{P}^{(\lambda)}_{MS}(\mathbf{x}-\mathbf{y}) + \lambda^{-3}\mathbf{L}\left(\frac{\mathbf{x}}{\lambda},\frac{\mathbf{y}}{\lambda}\right). \tag{7.10}$$

Here we have put $r = 1$ and defined $\mathbf{L} := \mathbf{L}^{(1)}$. The two terms in (7.10) are treated by different methods.

1. *Maxwell self-energy.* We write

$$\begin{aligned}(\mathbf{P}^{(\lambda)}_{MS} * \mathbf{m}_\lambda)(\mathbf{x}) &:= \int_{\mathbb{R}^3} \mathbf{P}^{(\lambda)}_{MS}(\mathbf{x}-\mathbf{y})\mathbf{m}_\lambda(\mathbf{y})d\mathbf{y},\\ (\mathbf{P}^{o}_{MS} * \mathbf{m}_\lambda)(\mathbf{x}) &:= (\mathbf{K} * \mathbf{m}_\lambda)(\mathbf{x}) + \frac{1}{3}\mathbf{m}(\mathbf{x}), \quad \mathbf{x}\in\mathbb{R}^3.\end{aligned} \tag{7.11}$$

Since

$$\begin{aligned}\|\mathbf{P}^{(\lambda)}_{MS} * \mathbf{m}_\lambda - \mathbf{P}^{o} * \mathbf{m}_\lambda\| &\le \|\mathbf{P}^{(\lambda)}_{MS} * \mathbf{m}_\lambda - \mathbf{P}^{(\varepsilon)}_{MS} * \mathbf{m}_\lambda\|\\ &\quad + \|\mathbf{P}^{(\varepsilon)}_{MS} * \mathbf{m}_\lambda - \mathbf{P}^{o} * \mathbf{m}_\lambda\|,\end{aligned} \tag{7.12}$$

the first part of the proof will be complete if we can show that for $0 < \varepsilon < \lambda$,

$$\|\mathbf{P}^{(\lambda)}_{MS} * \mathbf{m}_\lambda - \mathbf{P}^{(\varepsilon)}_{MS} * \mathbf{m}_\lambda\| \le cR_\lambda(\mathbf{m}_\lambda) \tag{7.13}$$

with c independent of λ and ε. That is, once (7.13) has been established, we fix λ and let $\varepsilon \to 0$ in (7.12), using $(6.34)_{ff}$; this yields

$$\|\mathbf{P}^{(\lambda)}_{MS} * \mathbf{m}_\lambda - \mathbf{P}^{o} * \mathbf{m}_\lambda\| < cR_\lambda(\mathbf{m}_\lambda) \tag{7.14}$$

which in turn yields

$$|\langle \mathbf{m}_\lambda, \mathbf{P}^{(\lambda)}_{MS}\mathbf{m}_\lambda\rangle - \langle \mathbf{m}_\lambda, \mathbf{P}^{o}_{MS}\mathbf{m}_\lambda\rangle| \to 0 \text{ as } \lambda \to 0. \tag{7.15}$$

To establish (7.12), we let $0 < \varepsilon < \lambda$ and write

$$\mathbf{P}^{\lambda,\varepsilon}(\mathbf{x}) = \mathbf{P}^{(\lambda)}_{MS}(\mathbf{x}) - \mathbf{P}^{(\varepsilon)}_{MS}(\mathbf{x}) \tag{7.16}$$

and observe that the left hand side of (7.13) satisfies

$$\|\mathbf{P}^{\lambda,\varepsilon} * \mathbf{m}_\lambda\| \le \|\mathbf{P}^{\lambda,\varepsilon} * (\rho_\lambda * \mathbf{m}_\lambda)\| + \|\mathbf{P}^{\lambda,\varepsilon} * (\mathbf{m}_\lambda - \rho_\lambda * \mathbf{m}_\lambda)\|. \tag{7.17}$$

By (6.15) and $(7.4)_4$ we have for the second term on the right hand side of (7.17),

$$\|\mathbf{P}^{\lambda,\varepsilon} * (\mathbf{m}_\lambda - \rho_\lambda * \mathbf{m}_\lambda)\| \le c\|\mathbf{m}_\lambda - \rho_\lambda * \mathbf{m}_\lambda\| \le cR_\lambda(\mathbf{m}_\lambda). \tag{7.18}$$

It remains to examine the first term on the right hand side of (7.17). Noting that

$$\int_{B(\rho)} \mathbf{P}^{\lambda,\varepsilon}(\mathbf{z})d\mathbf{z} = 0 \tag{7.19}$$

for every $\rho > 0$ and that $\mathbf{P}^{\lambda,\varepsilon}$ has bounded support, we write

$$\begin{aligned}\mathbf{P}^{\lambda,\varepsilon} * (\rho_\lambda * \mathbf{m}_\lambda)(\mathbf{x}) &= \int_{\mathbb{R}^3} \mathbf{P}^{\lambda,\varepsilon}(\mathbf{x}-\mathbf{y})(\rho_\lambda * \mathbf{m}_\lambda)(\mathbf{y})d\mathbf{y},\\ &= \int_{\mathbb{R}^3} \mathbf{P}^{\lambda,\varepsilon}(\mathbf{x}-\mathbf{y})\left[(\rho_\lambda * \mathbf{m}_\lambda)(\mathbf{y}) - (\rho_\lambda * \mathbf{m}_\lambda)(\mathbf{x})\right] d\mathbf{y}, \qquad (7.20)\\ &= \int_{\mathbb{R}^3} \hat{\mathbf{P}}^{\lambda,\varepsilon}(\mathbf{x}-\mathbf{y})(\mathbf{m}_\lambda(\mathbf{y}) - \mathbf{m}_\lambda(\mathbf{x}))d\mathbf{y},\end{aligned}$$

where

$$\hat{\mathbf{P}}^{\lambda,\varepsilon}(\mathbf{x}) := \int_{\mathbb{R}^3} \mathbf{P}^{\lambda,\varepsilon}(\mathbf{x}-\mathbf{z})(\rho_\lambda(\mathbf{z}) - \rho_\lambda(\mathbf{x}))d\mathbf{z} \qquad (7.21)$$

and we have made use of Fubini's Theorem. A straightforward calculation shows that $\hat{\mathbf{P}}^{\lambda,\varepsilon}$ has a uniformly bounded L^1 norm

$$\|\hat{\mathbf{P}}^{\lambda,\varepsilon}\|_{L^1} < c, \qquad (7.22)$$

and is supported on a ball of radius 2λ. (The mollifier was introduced to make the new kernel have these properties.) Hence, we can change variables in $(7.20)_3$ and write

$$\|\mathbf{P}^{\lambda,\varepsilon} * (\rho_\lambda * \mathbf{m}_\lambda)\| \le \int_{\mathbb{R}^3} |\int_{\mathbb{R}^3} |\mathbf{m}_\lambda(\mathbf{x}+\mathbf{z}) - \mathbf{m}_\lambda(\mathbf{x})|\hat{\mathbf{P}}^{\lambda,\varepsilon}(-\mathbf{z})|d\mathbf{z}|^2 d\mathbf{x}. \qquad (7.23)$$

Applying Jensen's inequality with probability measure $d\mu = |\hat{\mathbf{P}}^{\lambda,\varepsilon}(-\mathbf{z})|d\mathbf{z}/\|\hat{\mathbf{P}}^{\lambda,\varepsilon}\|_{L^1}$ to (7.23) we get

$$\|\mathbf{P}^{\lambda,\varepsilon} * (\rho_\lambda * \mathbf{m}_\lambda)\| \le \|\hat{\mathbf{P}}^{\lambda,\varepsilon}\|_{L^1}^2 R_{2\lambda}(\mathbf{m}_\lambda) \le 2c^2 R_\lambda(\mathbf{m}_\lambda), \qquad (7.24)$$

with establishes (7.13).

2. *Lorentz local energy.* We now estimate the term $\lambda^{-3}\mathbf{L}(\mathbf{x}/\lambda, \mathbf{y}/\lambda)$ of (7.10). We write

$$\begin{aligned}&\lambda^{-3}\mathbf{L}\int_{\mathbb{R}^3} \mathbf{L}(\frac{\mathbf{x}}{\lambda}, \frac{\mathbf{y}}{\lambda})\mathbf{m}_\lambda(\mathbf{y})d\mathbf{y}\\ &= \int_{\mathbb{R}^3} \mathbf{L}(\frac{\mathbf{x}}{\lambda}, \mathbf{z})d\mathbf{z}\ \mathbf{m}_\lambda(\mathbf{x}) + \lambda^{-3}\int_{\mathbb{R}^3} \mathbf{L}(\frac{\mathbf{x}}{\lambda}, \frac{\mathbf{y}}{\lambda})(\mathbf{m}_\lambda(\mathbf{y}) - \mathbf{m}_\lambda(\mathbf{x}))d\mathbf{y}.\end{aligned} \qquad (7.25)$$

By (6.35) and Lemma 6.4 the first term on the right hand side of (7.25) is

$$\mathbf{S}\mathbf{m}_\lambda(\mathbf{x}), \qquad (7.26)$$

where $\mathbf{S}$ is the lattice sum (6.19). Thus, to complete the proof of Proposition 7.1, it is sufficient to show that the second term on the right hand side of (7.25) tends to zero as $\lambda \to 0$. Recalling from (6.12) that

$$|\mathbf{L}(\mathbf{a}, \mathbf{b})| \le g(|\mathbf{a}-\mathbf{b}|), \quad g \in L^1, \qquad (7.27)$$

the L^2 norm of the second term on the right hand side of (7.25) satisfies

$$\begin{aligned}
&\|\lambda^{-3} \int_{\mathbb{R}^3} \mathbf{L}(\frac{\cdot}{\lambda}, \frac{\mathbf{y}}{\lambda})\mathbf{m}_\lambda(\mathbf{y}) - \mathbf{m}_\lambda(\cdot))d\mathbf{y}\| \\
&\leq \| \int_{\mathbb{R}^3} g(|\mathbf{z}|)(\mathbf{m}_\lambda(\cdot + \lambda\mathbf{z}) - \mathbf{m}_\lambda(\cdot))d\mathbf{z}\| \qquad (7.28) \\
&\leq \|g\|_{L^1} \int_{\mathbb{R}^3} g(|\mathbf{z}|) \int_{\mathbb{R}^3} |\mathbf{m}_\lambda(\mathbf{x} + \lambda\mathbf{z}) - \mathbf{m}_\lambda(\mathbf{x})|^2 \, d\mathbf{x}d\mathbf{z}.
\end{aligned}$$

Here, the last step follows by Jensen's inequality applied using the probability measure $d\mu = g/\|g\|_{L^1} d\mathbf{z}$. The right hand side of (7.28) is in turn bounded by

$$\|g\|_{L^1} \rho \int_{B(k)} g(|\mathbf{z}|)d\mathbf{z}R_\lambda(\mathbf{m}_\lambda) + 4 \int_{\mathbb{R}^3 - B(k)} g(|\mathbf{z}|)d\mathbf{z}\|\mathbf{m}_\lambda\|, \qquad (7.29)$$

where $k \geq 1$ is an integer and we have used the property $(7.4)_2$. In (7.29) we fix k and let $\lambda \to 0$; then we let $k \to \infty$. This completes the proof.

Remark 7.3. A simple but less general hypothesis on the background field $\mathbf{m}_\lambda$ than the condition $R_\lambda(\mathbf{m}_\lambda) \to 0$ is the condition

$$\mathbf{m}_\lambda(\mathbf{x}) - \mathbf{m}_\lambda(\mathbf{y}) < c_\lambda|\mathbf{x} - \mathbf{y}|, \quad \mathbf{x}, \mathbf{y} \in \mathbb{R}^3, \qquad (7.30)$$

where $\lambda c_\lambda \to 0$ as $\lambda \to 0$. This alternative hypothesis is sufficient to treat the case (7.1) and substantially simplifies the preceding proof but does not apply to background fields that have concentrations (see below).

While Proposition 7.1 simplifies the calculation of the energy in the weak-long case, it is still necessary to calculate the limit

$$\lim_{\lambda \to 0} \frac{-1}{2} \langle \mathbf{m}_\lambda, \mathbf{T}_0 \mathbf{m}_\lambda \rangle, \qquad (7.31)$$

where the macroscopic operator $\mathbf{T}_0$ is given by (7.7). Some examples are given below.

Example 7.4. To appreciate the distinction between (7.31) and the case of strong convergence, in which the limiting energy depends only on the magnetization (*i.e.*, the strong limit $\mathbf{m}$), consider the two sequences illustrated in Figure 1. These could be obtained by slicing a uniformly magnetized, hard magnetic material into equal slices with normals $\mathbf{e}$ and $\mathbf{f}$, respectively, and then rotating every other slice by 180° about $\mathbf{e}$.

Both sequences $\mathbf{m}_1^{(k)}$ and $\mathbf{m}_2^{(k)}$ obtained in this way converge weakly to zero in $L^2(\Omega, \mathbb{R}^3)$, and therefore have the same magnetization. Furthermore, both sequences have the same Young measure (see Ball [1989] or Evans [1990] for a definition of the Young measure and a summary of its properties) given by

$$\nu = \frac{1}{2}(\delta_{\mathbf{m}_1} + \delta_{-\mathbf{m}_1}), \qquad (7.32)$$

where $\delta_{\pm \mathbf{m}_1}$ is a Dirac mass at $\pm \mathbf{m}_1$. However, the two sequences have very different limiting energies. This can be appreciated in physical terms by observing that the slices in case 2 will tend to fly apart since the "north poles" are forced to be near each other, while the slices in case 1 will stay nicely stuck together. The limiting energy in case 1 is (James and Kinderlehrer [1990])

$$e_{1\lambda} \to e_1 := \frac{-1}{2}\left(\frac{1}{3}|\mathbf{m}_1|^2 + \mathbf{m}_1 \cdot \mathbf{S}\mathbf{m}_1\right) (\text{meas } \Omega) \text{ as } \lambda \to 0, \tag{7.33}$$

while the limiting energy in case 2 is given below.

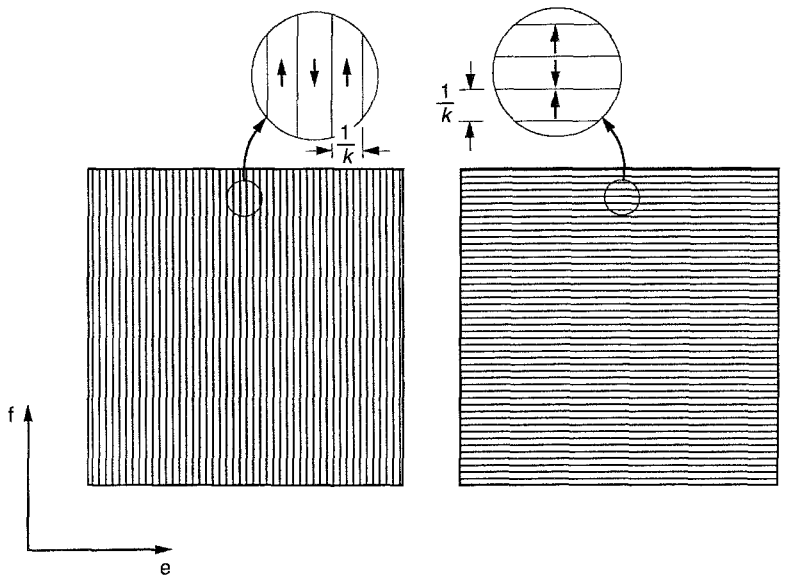

Fig. 1. Two sequences with the same megnetization and Young measure but different limiting energies. The energies are given, respectively, by (7.33) and (7.52).

The sequence illustrated in Figure 1 is in fact a minimizing sequence for the total energy of micromagnetics for a material that has easy axes $\pm \mathbf{m}$, (ibid.).

This example serves to illustrate limitations of the Young measure. (These limitations were also noticed by Rogers [1991]). The central question here is, "what information on a sequence of magnetizations is needed to calculate the limiting energy?" An answer in the weak-long case is given by the H-measure of L. Tartar [1990, 1992] (this has been developed simultaneously by P. Gérard [1990-91] under the name of microlocal defect measures); the H-measure of $\{\mathbf{m}^{(k)}\}$ determines the limiting energy (two sequences with the same H-measure have the same limiting energy).

There are alternative view points. That is, the limiting energy is in fact determined by the Young measure of the sequence of fields $\{\mathbf{h}^{(k)}\}$ corresponding to $\{\mathbf{m}^{(k)}\}$. With the latter point of view, the question is rephrased, "what Young measures arise from sequences of fields $\mathbf{h}^{(k)} = -\nabla u^{(k)}$ which solve the dif-

ferential equation $div(\mathbf{h}^k + \mathbf{m}^k\} = 0$?" This viewpoint is adopted by Pedregal [1992].

Example 7.5. (Oscillation of magnetization.) We now calculate the limiting energy for the sequence given in (7.1):

$$\hat{\mathbf{m}}_\lambda(\mathbf{x}) = \psi(\mathbf{x})\mathbf{m}_\#(\mathbf{x}/\lambda^\alpha), \quad 0 < \alpha < 1. \tag{7.34}$$

We assume first that $\hat{\psi} \in C_o^\infty(\mathbb{R}^3, \mathbb{R})$ and that $\mathbf{m}_\# \in C_\#^\infty(\mathbb{R}^3, \mathbb{R}^3)$, the set of smooth periodic functions with period cell $P = [0, 1]^3$, and that $\mathbf{m}_\#$ has mean zero

$$\int_P \mathbf{m}_\#(\mathbf{x})d\mathbf{x} = 0. \tag{7.35}$$

Here, $\hat{\psi}$ denotes the Fourier transform of ψ. If $\mathbf{m}_\#$ does not have mean zero, we can write

$$\mathbf{m}_\lambda(\mathbf{x}) = \psi(\mathbf{x})\left[\mathbf{m}_\#(\mathbf{x}/\lambda^\alpha) - \bar{\mathbf{m}}_\#\right] + \psi(\mathbf{x})\bar{\mathbf{m}}_\#, \tag{7.36}$$

where $\bar{\mathbf{m}}_\#$ is the left hand side of (7.35) and use the identity

$$\langle \mathbf{m}_1 + \mathbf{m}_2, \mathbf{K} * (\mathbf{m}_1 + \mathbf{m}_2)\rangle = \langle \mathbf{m}_1, \mathbf{K} * \mathbf{m}_1\rangle + 2\langle \mathbf{m}_1, \mathbf{K} * \mathbf{m}_2\rangle + \langle \mathbf{m}_2, \mathbf{K} * \mathbf{m}_2\rangle \tag{7.37}$$

to reduce this case to the case (7.35). Notice that in this case the cross term vanishes in the limit $\lambda \to 0$ and we simply add to the zero-mean result a classical term involving the magnetization $\psi(x)\bar{\mathbf{m}}_\#$. Further remarks in this direction are given in Section 9. The reasons for assuming that $\hat{\psi}$ (the Fourier transform of ψ) has compact support will be evident below.

We recall that for the sequence $\mathbf{m}_\lambda$ of (7.34) with $0 < \alpha < 1$, we have $R_\lambda(\mathbf{m}_\lambda) \to 0$. After applying Proposition 7.1, the presence of α in (7.34) is unnecessary and we can put $\delta = \lambda^\alpha$ and write $\mathbf{m}_\delta(\mathbf{x}) := \psi(\mathbf{x})\mathbf{m}_\#(\mathbf{x}/\delta)$. Omitting the local contributions to the energy, we wish to calculate

$$e_\delta := \frac{-1}{2}\langle \mathbf{m}_\delta, \mathbf{K} * \mathbf{m}_\delta\rangle. \tag{7.38}$$

Following Tartar [1990; Section 2] and referring to Remark 6.3, we take the Fourier transform of $(6.22)_{1,2}$ and observe that for any $\hat{\mathbf{f}} \in C_o^\infty(\mathbb{R}^3, \mathbb{R}^3)$ we have

$$\widehat{\mathbf{K} * \mathbf{f}(\boldsymbol{\xi})} = \frac{-\boldsymbol{\xi} \otimes \boldsymbol{\xi}}{|\boldsymbol{\xi}|^2}\hat{\mathbf{f}}(\xi) = \frac{-\boldsymbol{\xi} \cdot \hat{\mathbf{f}}(\boldsymbol{\xi})}{|\boldsymbol{\xi}|^2}\boldsymbol{\xi}, \tag{7.39}$$

where the hat denotes the Fourier transform, *i.e.*,

$$\hat{\mathbf{f}}(\boldsymbol{\xi}) := \int_{\mathbb{R}^3} e^{-2\pi i\boldsymbol{\xi}\cdot\mathbf{x}}\mathbf{f}(\mathbf{x})d\mathbf{x}. \tag{7.40}$$

We expand $\mathbf{m}_\#$ in a Fourier series,

$$\mathbf{m}_\#(\mathbf{x}) = \sum_{\substack{\mathbf{k}\in\mathbb{Z}^3 \\ \mathbf{k}\neq\mathbf{0}}} \hat{\mathbf{m}}_\#(\mathbf{k})e^{2\pi ik\cdot x}, \tag{7.41}$$

and note that

$$\hat{\mathbf{m}}_\delta(\boldsymbol{\xi}) = (\psi(\cdot)\mathbf{m}_\#(\cdot/\delta))\hat{}\,(\boldsymbol{\xi}) = \sum_{\substack{\mathbf{k}\in\mathbb{Z}^3\\ \mathbf{k}\neq\mathbf{0}}} \hat{\mathbf{m}}_\#(\mathbf{k})\hat{\psi}(\boldsymbol{\xi} - \delta^{-1}\mathbf{k}). \tag{7.42}$$

In (7.41) $\hat{\mathbf{m}}_\#(\mathbf{k}), \mathbf{k} \in \mathbb{Z}^3$, denotes the Fourier transform on $C^\infty_\#$ of $\hat{\mathbf{m}}_\#$, *i.e.*, the Fourier transform on periodic functions.

Now we use Parseval's Theorem to calculate the energy e_δ, ***viz.***,

$$\begin{aligned} e_\delta &= \frac{-1}{2}\int_{\mathbb{R}^3} \mathbf{m}_\delta \cdot \overline{\mathbf{K} * \mathbf{m}_\delta} d\mathbf{x}, \\ &= \frac{-c}{2}\int_{\mathbb{R}^3} \hat{\mathbf{m}}_\delta(\boldsymbol{\xi}) \cdot \overline{\widehat{\mathbf{K} * \mathbf{m}_\delta}}(\boldsymbol{\xi}) d\boldsymbol{\xi}, \\ &= \frac{c}{2}\int_{\mathbb{R}^3} \hat{\mathbf{m}}_\delta(\boldsymbol{\xi}) \cdot \frac{\boldsymbol{\xi}\otimes\boldsymbol{\xi}}{|\boldsymbol{\xi}|^2}\overline{\hat{\mathbf{m}}}_\delta(\boldsymbol{\xi}) d\boldsymbol{\xi}, \end{aligned} \tag{7.43}$$

where the overbar here denotes complex conjugate and c is the appropriate constant for Parseval's Theorem. We introduce (7.42) into $(7.43)_3$ and note that for δ sufficiently small the supports of $\hat{\psi}(\cdot - \delta^{-1}\mathbf{k})$ are disjoint because $\hat{\psi}$ has compact support by assumption. Therefore, we get

$$\begin{aligned} e_\delta &= \frac{c}{2}\int_{\mathbb{R}^3} \left(\sum_{\mathbf{k}\neq\mathbf{0}} \hat{\mathbf{m}}_\#(\mathbf{k})\hat{\psi}(\boldsymbol{\xi} - \delta^{-1}\mathbf{k})\right) \cdot \frac{\boldsymbol{\xi}\otimes\boldsymbol{\xi}}{|\boldsymbol{\xi}|^2} \overline{\left(\sum_{\mathbf{s}\neq\mathbf{0}} \hat{\mathbf{m}}_\#(\mathbf{s})\hat{\psi}(\boldsymbol{\xi} - \delta^{-1}\mathbf{s})\right)} d\boldsymbol{\xi}\,, \\ &= \frac{c}{2}\sum_{\mathbf{k}\neq\mathbf{0}} (\hat{\mathbf{m}}_\#(\mathbf{k}) \otimes \overline{\hat{\mathbf{m}}_\#(\mathbf{k})}) \cdot \int_{\mathbb{R}^3} \hat{\psi}(\boldsymbol{\xi} - \delta^{-1}\mathbf{k})\overline{\hat{\psi}(\boldsymbol{\xi} - \delta^{-1}\mathbf{k})}\frac{\boldsymbol{\xi}\otimes\boldsymbol{\xi}}{|\boldsymbol{\xi}|^2} d\boldsymbol{\xi}\,, \\ &= \frac{c}{2}\sum_{\mathbf{k}\neq\mathbf{0}} (\hat{\mathbf{m}}_\#(\mathbf{k}) \otimes \overline{\hat{\mathbf{m}}_\#(\mathbf{k})}) \cdot \int_{\mathbb{R}^3} \hat{\psi}(\mathbf{s})\overline{\hat{\psi}(\mathbf{s})}\frac{(\mathbf{s} + \delta^{-1}\mathbf{k})\otimes(\mathbf{s} + \delta^{-1}\mathbf{k})}{|\mathbf{s} + \delta^{-1}\mathbf{k}|^2} d\mathbf{s}\,. \end{aligned} \tag{7.44}$$

In the limit $\delta \to 0$, (7.44) becomes,

$$e_\delta \to \frac{1}{2}\left(\sum_{\mathbf{k}\neq\mathbf{0}} \hat{\mathbf{m}}_\#(\mathbf{k}) \cdot \frac{\mathbf{k}\otimes\mathbf{k}}{|\mathbf{k}|^2}\overline{\hat{\mathbf{m}}_\#(\mathbf{k})}\right)\|\psi\|^2, \tag{7.45}$$

which is one form of the result. A more convenient form follows by putting $\mathbf{f} = \mathbf{m}_\#$ in $(6.22)_{1,2}$ and expanding $\mathbf{g}$ in a Fourier series. We then see that a natural *definition* of $\mathbf{K} * \mathbf{m}_\#$ (for the *periodic* function $\mathbf{m}_\#$) is

$$(\mathbf{K} * \mathbf{m}_\#)(\mathbf{x}) := -\sum_{\substack{\mathbf{k}\in\mathbb{Z}^3\\ \mathbf{k}\neq\mathbf{0}}} \frac{\mathbf{k}\otimes\mathbf{k}}{|\mathbf{k}|^2}\hat{\mathbf{m}}_\#(\mathbf{k})e^{2\pi i\mathbf{k}\cdot\mathbf{x}}. \tag{7.46}$$

Now by direct calculation we have

$$\int_P \mathbf{m}_\# \cdot (\mathbf{K} * \mathbf{m}_\#) d\mathbf{x} = -\sum_{\mathbf{k}\neq\mathbf{0}} \hat{\mathbf{m}}_\#(\mathbf{k}) \cdot \frac{\mathbf{k}\otimes\mathbf{k}}{|\mathbf{k}|^2}\overline{\hat{\mathbf{m}}_\#(\mathbf{k})}, \tag{7.47}$$

where we have used that the volume of P is 1. Hence, we can write (7.45) in the alternative form

$$e_\delta \to \frac{-1}{2}\|\psi\|^2 \int_P \mathbf{m}_\# \cdot (\mathbf{K} * \mathbf{m}_\#) d\mathbf{x}. \tag{7.48}$$

Further comments on the interpretation of this limiting energy are given in Section 9.

The formula (7.48) for the limiting energy is easily generalized to the case of an arbitrary period cell by changing variables. Suppose that

$$\mathbf{m}_\#(\mathbf{x} + \nu^i \tilde{\mathbf{e}}_i) = \mathbf{m}_\#(\mathbf{x}), \quad \forall \{\nu^i\} \in (\mathbb{Z}^3)^3, \quad \mathbf{x} \in \mathbb{R}^3, \tag{7.49}$$

where $\{\tilde{\mathbf{e}}_i\}$ are linearly independent but not necessarily orthonormal vectors, and the unit cell for $\mathbf{m}_\#$ is now $\tilde{P} := \{\mathbf{x} \in \mathbb{R}^3 : \mathbf{x} = \lambda^i \tilde{\mathbf{e}}_i,\ \lambda^i \in [0, 1)\}$. We assume $\mathbf{m}_\#$ has zero mean on $\tilde{P}$. The same argument given just above works in this case, with $\mathbf{m}_\#$ having the Fourier series representation

$$\mathbf{m}_\#(\mathbf{x}) = \frac{1}{|\det \mathbf{L}|} \sum_{\substack{\mathbf{k} \in L^{-T}\mathbb{Z}^3 \\ \mathbf{k} \neq \mathbf{0}}} \hat{\mathbf{m}}_\#(\mathbf{k}) e^{2\pi i \mathbf{k} \cdot \mathbf{x}}. \tag{7.50}$$

where $\mathbf{L}([0, 1]^3) = \tilde{P}$. In the case of a general period,

$$e_\delta \to \frac{-1}{2|\det \mathbf{L}|}\|\psi\|^2 \int_{\tilde{P}} \mathbf{m}_\# \cdot (\mathbf{K} * \mathbf{m}_\#) d\mathbf{x}. \tag{7.51}$$

These results* enable us to calculate the energies e_1 and e_2, respectively, of the two sequences pictured in Figure 1, or what is more interesting, the limiting energy difference $e_2 - e_1$. From (7.46)-(7.48),

$$e_2 - e_1 = \frac{1}{2}(\text{vol}.\Omega)(\mathbf{m}_1 \cdot \mathbf{f})^2, \tag{7.52}$$

where $\mathbf{f}$ is a unit vector pictured in Figure 1, while as mentioned earlier, the nonlocal energy corresponding to the sequence $\mathbf{m}_1^{(k)}$ is zero. The expression (7.52) remains valid for any laminated domain structure, where $\mathbf{f}$ is normal to the layers, the magnetization is $\pm\mathbf{m}_1$ on alternate layers and the volume fraction is $\frac{1}{2}$.

Example 7.6. (Concentration of dipoles with no magnetization.) We can have a situation in which the dipoles accumulate at a point and create a concentration of energy but with no magnetization. For example, consider the sequence

$$\mathbf{m}_\lambda(\mathbf{x}) = \lambda^{-\frac{3}{2}\alpha} \mathbf{m}_0(\mathbf{x}/\lambda^\alpha), \quad 0 < \alpha < 1, \tag{7.53}$$

*It is necessary to slightly generalize the above calculation to allow ψ having the form of a characteristic function.

with $\mathbf{m}_0 \in L^2(\mathbb{R}^3, \mathbb{R}^3)$. The scaling has been chosen so that

$$\|\mathbf{m}_\lambda\| = \|\mathbf{m}_0\|, \tag{7.54}$$

but at the same time $\mathbf{m}_\lambda \rightharpoonup 0$ in L^2. According to our results of Section 5, we should regard the weak limit of $\mathbf{m}_\lambda$ as the magnetization and therefore this sequence has no magnetization. it is easily verified that for this sequence

$$R_\lambda(\mathbf{m}_\lambda) = R_{\lambda^{1-\alpha}}(\mathbf{m}_0) \to 0 \text{ as } \lambda \to 0, \tag{7.55}$$

so by Proposition 7.1 the limiting energy is given by

$$\begin{aligned} &\frac{-1}{2} \lim_{\lambda \to 0} \langle \mathbf{m}_\lambda, \mathbf{T}_0 \mathbf{m}_\lambda \rangle, \\ &= \frac{-1}{2} \left\{ \lim_{\lambda \to 0} \langle \mathbf{m}_\lambda, \mathbf{K} * \mathbf{m}_\lambda \rangle + \frac{1}{3} \|\mathbf{m}_0\|_{L^2}^2 + \langle \mathbf{m}_0, \mathbf{S}\mathbf{m}_0 \rangle \right\}, \end{aligned} \tag{7.56}$$

where $\mathbf{S}$ is the lattice sum given by (6.19). The first term in (7.48) is easily calculated by using Remark 6.3 and the definition of $\mathbf{K}$, leading to

$$\langle \mathbf{m}_\lambda, \mathbf{K} * \mathbf{m}_\lambda \rangle = \langle \mathbf{m}_0, \mathbf{K} * \mathbf{m}_0 \rangle. \tag{7.57}$$

We conclude that the limiting energy of the sequence (7.45) is the same as the energy of the magnetization $\mathbf{m}(\mathbf{x}) = \mathbf{m}_0$. However, the sequence itself has zero magnetization in the limit $\lambda \to 0$.

8 Energy of a fine array of dipoles with short wave oscillations (weak-short)

We now calculate the limiting energy of an array of dipoles with oscillations that are truly on the scale of the lattice. We consider a modulated periodic field of dipoles for simplicity. Let $L_\lambda = L(\lambda \mathbf{e}_i)$, $\lambda > 0$, be a given family of Bravais lattices as above, and consider a family of sublattices defined by

$$\hat{L}_\lambda := L(\lambda \hat{\mathbf{e}}_i), \quad \lambda > 0. \tag{8.1}$$

where

$$\hat{\mathbf{e}}_i = v_i^j \mathbf{e}_j, \quad v_i^j \in \mathbb{Z}, \quad \det v \neq 0,$$

the interesting case being the case $\det v \neq \pm 1$ so that $\hat{L}_\lambda$ is a proper sublattice of L_λ. We consider the dipole field $\mathbf{d}_\lambda : L_\lambda \to \mathbb{R}^3$ given by

$$\mathbf{d}_\lambda(\mathbf{x}) := \lambda^3 \mathbf{d}(\mathbf{x}, \frac{\mathbf{x}}{\lambda}) \tag{8.2}$$

where $\mathbf{d}(\cdot, \mathbf{y})$ is smooth for each $\mathbf{y} \in L_1$, $\mathbf{d}(\mathbf{x}, \cdot)$ is periodic on $\hat{L}_1$, *viz.*,

$$\mathbf{d}(\mathbf{x}, \mathbf{y} + \mathbf{z}) = \mathbf{d}(\mathbf{x}, \mathbf{y}) \quad \forall \mathbf{x} \in \mathbb{R}^3, \quad \mathbf{y} \in L_1, \quad \mathbf{z} \in \hat{L}_1, \tag{8.3}$$

and supp $\mathbf{d}(\cdot, \mathbf{y}) \subset \Omega$ for each $\mathbf{y} \in \mathbb{R}^3$. A prototype is the choice

$$\mathbf{d}(\mathbf{x}, \mathbf{y}) = \psi(\mathbf{x})\mathbf{d}(\mathbf{y}), \quad \psi \in C_o^\infty, \tag{8.4}$$

with $\mathbf{d}(\cdot)$ periodic on $\hat{L}_1$. Generally, we have weak convergence of $\lambda^{-3}\mathbf{d}_\lambda(\mathbf{x})$ and the correlation $R_\lambda(\mathbf{d}_\lambda)$ is bounded away from zero, so we are in the weak-short case. The *period cell* and *symmetric period cell* for the function $\mathbf{d}(\mathbf{x}, \cdot)$ are defined, respectively, by

$$\begin{aligned} P &:= \left\{\mathbf{x} \in \mathbb{R}^3 : \mathbf{x} = \lambda^i \hat{\mathbf{e}}_i, \quad 0 \le \lambda^i < 1\right\}, \\ \tilde{P} &:= \left\{\mathbf{x} \in \mathbb{R}^3 : \mathbf{x} = \lambda^i \hat{\mathbf{e}}_i, \quad -1 \le \lambda^i < 1\right\}, \end{aligned} \tag{8.5}$$

while $k\tilde{P}$, $k = 1, 2, 3, \ldots$, is defined by

$$k\tilde{P} := \left\{k\mathbf{x} \ : \ \mathbf{x} \in \tilde{P}\right\}. \tag{8.6}$$

These definitions are framed with the corresponding half-open intervals so that for any function $f : \mathbb{R}^3 \to \mathbb{R}^n$ with say, compact support,

$$\sum_{\mathbf{z} \in L_1 \cap k\tilde{P}} f(\mathbf{z}) = \sum_{\mathbf{x} \in \hat{L}_1 \cap k\tilde{P}} \sum_{\mathbf{y} \in L_1 \cap P} f(\mathbf{x} + \mathbf{y}), \tag{8.7}$$

and corresponding for sums over $L_1 \backslash k\tilde{P}$. We assume that

$$\sum_{\mathbf{y} \in L_1 \cap P} \mathbf{d}(\mathbf{x}, \mathbf{y}) = 0 \quad \forall\, \mathbf{x} \in \mathbb{R}^3, \tag{8.8}$$

without loss of generality. If (8.8) fails we can add and subtract a suitable smooth function of $\mathbf{x}$ to $\mathbf{d}(\mathbf{x}, \mathbf{y})$ and then use the method given in (7.37). Note that any set of the form $P' = \mathbf{y} + P$, $\mathbf{y} \in L_1$, also serves as a period cell for $\mathbf{d}(\mathbf{x}, \cdot)$ and (8.8) remains satisfied with P replaced by P'. Our goal is to calculate the limit as $\lambda \to 0$ of the energy

$$e_\lambda = \frac{-1}{2} \sum_{\mathbf{x} \in L_\lambda} \mathbf{h}_\lambda(\mathbf{x}) \cdot \mathbf{d}_\lambda(\mathbf{x}), \tag{8.9}$$

where

$$\mathbf{h}_\lambda(\mathbf{x}) := \sum_{\mathbf{y} \in L_\lambda \backslash \{x\}} \mathbf{K}(\mathbf{x} - \mathbf{y})\mathbf{d}_\lambda(\mathbf{y}). \tag{8.10}$$

Formally, we let

$$\tilde{\mathbf{h}}(\mathbf{x}, \mathbf{y}) := \lim_{k \to \infty} \sum_{\substack{\mathbf{z} \in L_1 \backslash \{0\} \\ \mathbf{z} \in k\tilde{P}}} \mathbf{K}(-\mathbf{z})\mathbf{d}(\mathbf{x}, \mathbf{y} + \mathbf{z}). \tag{8.11}$$

Lemma 8.1. *The sequence on the right hand side of (8.11) converges uniformly to a function* $\tilde{\mathbf{h}} : \mathbb{R}^3 \times L_1 \to \mathbb{R}^3$. *The function* $\tilde{\mathbf{h}}(\cdot, \mathbf{y})$ *is smooth for each* $\mathbf{y} \in L_1$, *while the function* $\tilde{\mathbf{h}}(\mathbf{x}, \cdot)$ *is periodic on* $\hat{L}_1$:

$$\tilde{\mathbf{h}}(\mathbf{x}, \mathbf{y} + \mathbf{z}) = \tilde{\mathbf{h}}(\mathbf{x}, \mathbf{y}) \quad \forall \mathbf{x} \in \mathbb{R}^3, \ \mathbf{y} \in L_1, \ \mathbf{z} \in \hat{L}_1. \tag{8.12}$$

Proof. If the sum converges for each $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^3 \times L_1$, the function $\tilde{\mathbf{h}}(\mathbf{x}, \cdot)$ is $\hat{L}_1$- periodic by (8.3). Using (8.7) and (8.8) we have for integers $k \geq j \geq 1$,

$$
\begin{aligned}
&|\sum_{\substack{\mathbf{z}\in L_1\setminus\{\mathbf{o}\}\\ \mathbf{z}\in k\tilde{P}}} \mathbf{K}(-\mathbf{z})\mathbf{d}(\mathbf{x}, \mathbf{y}+\mathbf{z}) - \sum_{\substack{\mathbf{z}\in L_1\setminus\{\mathbf{o}\}\\ \mathbf{z}\in j\tilde{P}}} \mathbf{K}(-\mathbf{z})\mathbf{d}(\mathbf{x}, \mathbf{y}+\mathbf{z})|,\\
&= |\sum_{\substack{\mathbf{v}\in L_1\setminus\{\mathbf{o}\}\\ \mathbf{v}\in k\tilde{P}\setminus j\tilde{P}}} \sum_{\mathbf{w}\in L_1\cap P} \mathbf{K}(-(\mathbf{v}+\mathbf{w}))\mathbf{d}(\mathbf{x}, \mathbf{y}+\mathbf{w})|,\\
&\leq c \sum_{\substack{\mathbf{v}\in L_1\setminus\{\mathbf{o}\}\\ \mathbf{v}\in k\tilde{P}\setminus j\tilde{P}}} \sup_{\mathbf{a},\mathbf{b}\in L_1\cap P} |\mathbf{K}(-(\mathbf{v}+\mathbf{a})) - \mathbf{K}(-(\mathbf{v}+\mathbf{b}))| \sup|\mathbf{d}|,\\
&\leq c' \sum_{\substack{\mathbf{v}\in L_1\\ \mathbf{v}\in k\tilde{P}\setminus j\tilde{P}}} |\mathbf{v}|^{-4}.
\end{aligned}
\tag{8.13}
$$

Hence the sequence on the right hand side of (8.11) is uniformly convergent. By the same argument, this sequence is also uniformly convergent when $\mathbf{d}(\mathbf{x}, \mathbf{y})$ is replaced by any of its $\mathbf{x}$-derivatives, so $\mathbf{h}(\cdot, \mathbf{y})$ is smooth. □

We now consider (8.10) and write $\mathbf{h}_\lambda$ in the form

$$
\begin{aligned}
\mathbf{h}_\lambda(\mathbf{x}) &= \sum_{\mathbf{z}\in L_\lambda\setminus\{\mathbf{o}\}} \lambda^3 \mathbf{K}(-\mathbf{z})\mathbf{d}(\mathbf{x}+\mathbf{z}, \frac{\mathbf{x}+\mathbf{z}}{\lambda}),\\
&= \sum_{\mathbf{w}\in L_1\setminus\{\mathbf{o}\}} \mathbf{K}(-\mathbf{w})\mathbf{d}(\mathbf{x}+\lambda\mathbf{w}, \frac{\mathbf{x}}{\lambda}+\mathbf{w}),
\end{aligned}
\tag{8.14}
$$

which can be decomposed into the two sums

$$
= \mathbf{a}_{\lambda,k}(\mathbf{x}) + \mathbf{b}_{\lambda,k}(\mathbf{x}), \quad k = 1, 2, \ldots, \tag{8.15}
$$

where

$$
\begin{aligned}
\mathbf{a}_{\lambda,k}(\mathbf{x}) &:= \sum_{\substack{\mathbf{w}\in L_1\setminus\{\mathbf{o}\}\\ \mathbf{w}\in k\tilde{P}}} \mathbf{K}(-\mathbf{w})\mathbf{d}(\mathbf{x}+\lambda\mathbf{w}, \frac{\mathbf{x}}{\lambda}+\mathbf{w}),\\
\mathbf{b}_{\lambda,k}(\mathbf{x}) &:= \sum_{\mathbf{w}\in L_1\setminus k\tilde{P}} \mathbf{K}(-\mathbf{w})\mathbf{d}(\mathbf{x}+\lambda\mathbf{w}, \frac{\mathbf{x}}{\lambda}+\mathbf{w}).
\end{aligned}
\tag{8.16}
$$

Proposition 8.2. *Let* $\mathbf{a}_{\lambda,k}$ *and* $\mathbf{b}_{\lambda,k}$ *be given by (8.16). There exists* $c > 0$ *such that for* $k \geq 1$ *and* $\lambda \leq 1$,

$$
|\mathbf{b}_{\lambda,k}(\mathbf{x})| \leq c\lambda(1 - \log\lambda) + o(1), \quad \mathbf{x} \in \mathbb{R}^3 \tag{8.17}
$$

while there is a $g : \mathbb{Z} \to \mathbb{R}$ *(generally unbounded) such that*

$$
|\mathbf{a}_{\lambda,k}(\mathbf{x}) - \tilde{\mathbf{h}}(\mathbf{x}, \frac{\mathbf{x}}{\lambda})| \leq \lambda g(k) + o(1), \quad \mathbf{x} \in L_1. \tag{8.18}
$$

Here, $o(1) \to 0$ *uniformly as* $k \to \infty$. *Hence, there is a* $k(\lambda) \to \infty$ ***as*** $\lambda \to 0$ *such that*

$$
\mathbf{a}_{\lambda,k(\lambda)}(\mathbf{x}) \to \tilde{\mathbf{h}}(\mathbf{x}, \frac{\mathbf{x}}{\lambda}), \quad \mathbf{b}_{\lambda,k(\lambda)}(\mathbf{x}) \to \mathbf{0}, \tag{8.19}
$$

uniformly in $\mathbf{x}$ *as* $\lambda \to 0$.

Proof. The condition (8.18) follows immediately from the observation

$$|\mathbf{a}_{\lambda,k}(\mathbf{x}) - \tilde{\mathbf{h}}(\mathbf{x}, \frac{\mathbf{x}}{\lambda})| \leq \sum_{\substack{\mathbf{z}\in L_1\setminus\{0\} \\ \mathbf{z}\in k\tilde{P}}} |\mathbf{K}(-\mathbf{z})|\ |\mathbf{d}(\mathbf{x}+\lambda\mathbf{z}, \frac{\mathbf{x}}{\lambda}+\mathbf{z}) - \mathbf{d}(\mathbf{x}, \frac{\mathbf{x}}{\lambda}+\mathbf{z})| + o(1),$$

$$\leq \lambda \sum_{\substack{\mathbf{z}\in L_1\setminus\{0\} \\ \mathbf{z}\in k\tilde{P}}} |\mathbf{K}(-\mathbf{z})|\ \sup|\mathbf{d}_{\mathbf{x}}||\mathbf{z}| + o(1). \tag{8.20}$$

Turning attention to $\mathbf{b}_{\lambda,k}(\mathbf{x})$, we use $(8.7)_{ff}$ to write

$$|\mathbf{b}_{\lambda,k}(\mathbf{x})| = |\sum_{\mathbf{z}\in\hat{L}_1\setminus k\tilde{P}} \sum_{\mathbf{v}\in L_1\cap P} \mathbf{K}(-(\mathbf{z}+\mathbf{v}))\mathbf{d}(\mathbf{x}+\lambda(\mathbf{z}+\mathbf{v}), \frac{\mathbf{x}}{\lambda}+\mathbf{z}+\mathbf{v})|,$$

$$= |\sum_{\mathbf{z}\in\hat{L}_1\setminus k\tilde{P}} \Big\{ \sum_{\mathbf{v}\in L_1\cap P} \mathbf{K}(-(\mathbf{z}+\mathbf{v}))\Big[\mathbf{d}(\mathbf{x}+\lambda(\mathbf{z}+\mathbf{v}), \frac{\mathbf{x}}{\lambda}+\mathbf{z}+\mathbf{v}) \tag{8.21}$$

$$- \mathbf{d}(\mathbf{x}+\lambda\mathbf{z}, \frac{\mathbf{x}}{\lambda}+\mathbf{z}+\mathbf{v})\Big] + \mathbf{K}(-(\mathbf{z}+\mathbf{v}))\ \mathbf{d}(\mathbf{x}+\lambda\mathbf{z}, \frac{\mathbf{x}}{\lambda}+\mathbf{z}+\mathbf{v})\Big\}.$$

Recalling that $\mathbf{d}(\cdot, \mathbf{y})$ is supported on Ω, we choose a sufficiently large ball B such that $\Omega \subset B$ and $-\lambda\tilde{P}+\Omega \subset B$, $0 < \lambda \leq 1$. We estimate the term in square brackets in $(8.21)_2$ terms of $\sup \mathbf{d}_{\mathbf{x}}$, whereas we use the method embodied in (8.13) to handle the remaining term. With $k \geq 1$ we get

$$|\mathbf{b}_{\lambda,k}(\mathbf{x})| \leq \sum_{\mathbf{z}\in\hat{L}_1\setminus k\tilde{P}} \left\{ \frac{\lambda c_1 \chi((-\mathbf{x}+B)/\lambda)(\mathbf{z})}{|\mathbf{z}|^3} + \frac{c_2}{|\mathbf{z}|^4} \right\}. \tag{8.22}$$

Here, c_1 depends only on $\sup \mathbf{d}_{\mathbf{x}}$ while c_2 depends only on P and $\sup \mathbf{d}$. Hence, if the radius of B is r_1 and $r_0 > 0$ is chosen so that $kB(o, r_0) \subset k\tilde{P}$, we have

$$|\mathbf{b}_{\lambda,k}(\mathbf{x})| \leq \lambda c'_1 \int_{\max(kr_0, (|\mathbf{x}|-r_1)/\lambda)}^{(|\mathbf{x}|+r_1)/\lambda} \rho^{-1} d\rho + o(1), \tag{8.23}$$

$$= \begin{cases} c'_1\lambda \log\left(\frac{|\mathbf{x}|+r_1}{\lambda k r_0}\right) + o(1), & |\mathbf{x}| \leq r_1 + \lambda k r_0, \\ c'_1\lambda \log\left(\frac{|\mathbf{x}|+r_1}{|\mathbf{x}|-r_1}\right) + o(1), & |\mathbf{x}| > r_1 + \lambda k r_0, \end{cases}$$

$$\leq c\lambda(1 - \log\lambda) + o(1). \qquad \square \tag{8.24}$$

Now we calculate the limiting energy. We decompose $\mathbf{h}_\lambda$ as in (8.14), (8.15) and evaluate k at $k(\lambda)$ given by Proposition 8.2. Using the fact that $\mathbf{d}(\cdot, \mathbf{y})$ is uniformly bounded and has fixed compact support, we get

$$e_\lambda = \frac{-1}{2} \sum_{\mathbf{x}\in L_\lambda} \lambda^3 \tilde{\mathbf{h}}(\mathbf{x}, \frac{\mathbf{x}}{\lambda}) \cdot \mathbf{d}\left(\mathbf{x}, \frac{\mathbf{x}}{\lambda}\right) + 0(\lambda). \tag{8.25}$$

Let

$$w(\mathbf{x}, \mathbf{y}) := -\tilde{\mathbf{h}}(\mathbf{x}, \mathbf{y}) \cdot \mathbf{d}(\mathbf{x}, \mathbf{y}), \quad \mathbf{x}, \mathbf{y} \in L_1, \tag{8.26}$$

and note that w is smooth, periodic in $\mathbf{y}$ on $\hat{L}_1$, and compactly support in $\mathbf{x}$ with fixed support Ω. Using again (8.7) (omitting $\cap k\tilde{P}$) and these properties of w, we have

$$\begin{aligned} e_\lambda &= \frac{1}{2} \sum_{\mathbf{z} \in \hat{L}_1} \sum_{\mathbf{v} \in L_1 \cap P} \lambda^3 \, w(\lambda(\mathbf{z} + \mathbf{v}), \mathbf{z} + \mathbf{v}) + 0(\lambda), \\ &= \frac{1}{2} \sum_{\mathbf{z} \in \hat{L}_1} \sum_{\mathbf{v} \in L_1 \cap P} \lambda^3 \, w(\lambda(\mathbf{z} + \mathbf{v}), \mathbf{v}) + 0(\lambda), \\ &= \frac{1}{2} \sum_{\mathbf{z} \in \hat{L}_1} \lambda^3 \sum_{\mathbf{v} \in L_1 \cap P} w(\lambda \mathbf{z}, \mathbf{v}) + 0(\lambda), \\ &= \frac{1}{2} \sum_{\mathbf{x} \in \hat{L}_1} \lambda^3 \sum_{\mathbf{v} \in L_1 \cap P} w(\mathbf{x}, \mathbf{v}) + 0(\lambda). \end{aligned} \tag{8.27}$$

Hence, we conclude that

$$e_\lambda \to \frac{1}{2} \int_\Omega \left(\frac{1}{\text{meas} P} \sum_{\mathbf{y} \in L_1 \cap P} w(\mathbf{x}, \mathbf{y}) \right) d\mathbf{x} \quad \text{as } \lambda \to 0, \tag{8.28}$$

where w is given by (8.26) and $\tilde{\mathbf{h}}$ is given by (8.11).

Remark 8.3. In the weak-short case we typically will have a remnant lattice sum in the expression for the limiting energy. This sum will depend on both the lattice and the configuration of dipoles on the lattice. As in the weak-long case, measurements of magnetization give no information about the weak-short contribution to the energy. Note that, because of the condition (8.8), the magnetization of the dipole field (8.2) vanishes (*cf.* Section 5).

The various parts of the calculation admit the following physical interpretation. Fixing $\mathbf{x}$, the vector $\tilde{\mathbf{h}}(\mathbf{x}, \mathbf{y})$ is calculated by assuming that an exactly periodic dipole field $\mathbf{d}(\mathbf{x}, \mathbf{y} + \cdot)$ is defined on a 1-lattice but with the dipole at $\mathbf{y}$ removed. $\tilde{\mathbf{h}}(\mathbf{x}, \mathbf{y})$ represents the field at $\mathbf{y}$ due to this periodic distribution of dipoles. See Section 9 for further remarks on the significance of the 1-lattice (*i.e.*, why "1"?). Hence, the integrand

$$\frac{-1}{2\text{meas} P} \sum_{\mathbf{y} \in L_1 \cap P} \tilde{\mathbf{h}}(\mathbf{x}, \mathbf{y}) \cdot \mathbf{d}(\mathbf{x}, \mathbf{y}) \tag{8.29}$$

represents the energy per unit volume on the 1-lattice. In effect, the assumptions on $\mathbf{d}$ permit a separation of the short-wave calculation, done with exactly periodic fields, and the long-wave calculation, which in the end is just an integration of the short-wave energy density. The most striking result is that the limiting energy is essentially local. That is, the expression on the right hand side of (8.28) is

additive over disjoint subsets of Ω, in contrast to the usual expression for the field energy of micromagnetics (*cf.* the first term on the right hand side of (6.18)). The locality arose partly from (8.8) and also from the separation of scales given in (8.2) (see the forthcoming paper of Firoozye [1993] for further analysis of the question of whether (6.18) is the only nonlocal term that is possible).

Example 8.4. An interesting special case of the preceeding calculation is given by equation (8.4),

$$\mathbf{d}_\lambda(\mathbf{x}) = \lambda\psi(\mathbf{x})\hat{\mathbf{d}}(\frac{\mathbf{x}}{\lambda}), \tag{8.30}$$

where $\psi \in C_o^\infty(\mathbb{R}^3)$ and $\hat{\mathbf{d}}$ is periodic on $\hat{L}_1$ with zero average. By specialization of the formulae to this case, we get

$$e_\lambda \to \frac{-1}{2}\|\psi\|^2 \sum_{\mathbf{x}\in L_1\cap P} \hat{\mathbf{h}}(\mathbf{x})\cdot\hat{\mathbf{d}}(\mathbf{x})/\text{meas}P \quad \text{as } \lambda \to 0, \tag{8.31}$$

where

$$\hat{\mathbf{h}}(\mathbf{x}) := \lim_{k\to\infty} \sum_{\substack{\mathbf{y}\in L_1-\{0\}\\ \mathbf{y}\in k\bar{P}}} \mathbf{K}(-\mathbf{y})\hat{\mathbf{d}}(\mathbf{x}+\mathbf{y}).$$

Example 8.5. (Concentration.) In the analysis in this section we have so far neglected concentrations. A typical example involving concentrations is

$$\mathbf{d}_\lambda(\mathbf{x}) = \lambda^{3/2}\mathbf{d}(\frac{\mathbf{x}}{\lambda}), \tag{8.32}$$

where, for example, $\mathbf{d}$ has compact support. With this assumption the corresponding magnetization vanishes, while the limiting energy is given by

$$e_\lambda \to \frac{-1}{2} \sum_{\substack{\mathbf{x},\mathbf{y}\in L_1\\ \mathbf{x}\neq\mathbf{y}}} \mathbf{d}(\mathbf{x})\cdot\mathbf{K}(\mathbf{x}-\mathbf{y})\mathbf{d}(\mathbf{y}) \quad \text{as } \lambda \to 0. \tag{8.33}$$

9 Discussion and application to continuum theory

In applying the foregoing results to a material with a given lattice spacing, it is necessary to adapt results for an infinitesimal lattice parameter to a finite lattice parameter. We would do this in the following way. Suppose the real lattice is represented by a Bravais lattice $L_a = L(\lambda_a\mathbf{e}_i)$ where $\mathbf{e}_i$ are fixed lattice vectors and λ_a is the *actual* lattice parameter. On this lattice is a certain dipole distribution $\mathbf{d}_a(\mathbf{x})$, $\mathbf{x} \in L_a$. It is necessary to embed this given dipole field $\mathbf{d}_a(\mathbf{x})$ in a one-parameter family of dipole fields $\mathbf{d}_\lambda(\mathbf{x})$ in a reasonable way. The criterion for "reasonable" is that the quantities of interest like the fields $\mathbf{h}_\lambda, \mathbf{b}_\lambda, \mathbf{m}_\lambda$ or the energy e_λ do not change much in the interval $\lambda \in (0, \lambda_a]$. For example, it would be disasterous to have $\mathbf{d}_\lambda$ apparently converging strongly

(*cf.* Section 4) for $\lambda \sim \lambda_a$ but then in fact develop lattice-scale oscillations for $\lambda \ll \lambda_a$. It is difficult to give clear-cut rules for this embedding; it may at first seem sensible to put

$$\mathbf{d}_\lambda(\mathbf{x}) = \mathbf{d}_a(\frac{\lambda_a}{\lambda}\mathbf{x}), \tag{9.1}$$

but this would imply that we are in the weak-short case, except for very special fields $\mathbf{d}_a(\mathbf{x})$. Also $\mathbf{d}_a(\mathbf{x})$ would have to be defined for large values of $\mathbf{x}$, which is problematic in the case that the actual crystal is finite. It is clearly undesirable to go to the extreme represented by (9.1) for many materials, and the successful theory of micromagnetics is really based on the assumption of strong convergence.

This issue could be clarified by looking at rates of convergence of the quantities $\mathbf{h}_\lambda$, $\mathbf{b}_\lambda$, $\mathbf{d}_\lambda$ and e_λ, and some of these rates of convergence are inherent in our calculations of Sections 5-8. In any case, whatever embedding is chosen, it must have the property

$$\mathbf{d}_{\lambda_a}(\mathbf{x}) = \mathbf{d}_a(\mathbf{x}), \quad \mathbf{x} \in L_{\lambda_a}. \tag{9.2}$$

As an intuitive guideline, we would expect that the weak-short results would be necessary for materials that exhibit large oscillations on the order of a few lattice spacings, as in antiferromagnetism or ferrimagnetism. For ordinary ferromagnetic materials with micron-sized domains we would think that the strong convergence results would be adequate, based on the success of micromagnetics. The weak-long results would seem to apply to intermediate scales. Our results (Proposition 7.1) show that the weak-long and strong cases are governed by the same continuum theory.

On the actual lattice L_a the dipoles have a positive separation distance. For this reason it is unimportant whether or not the actual local fields are singular at the lattice points of L_a. Our results are still valid as long as the actual local fields are represented well by the field $\mathbf{K}(\mathbf{x}-\mathbf{y})\mathbf{d}(\mathbf{y})$ at distances $|\mathbf{x}-\mathbf{y}| \geq \lambda_a$, and (9.3) below is satisfied.

Continuing our interpretation of the results, we focus on the energy e_λ. With a well-chosen embedding we will have that

$$e_{\lambda_a} \doteq \lim_{\lambda \to 0} e_\lambda. \tag{9.3}$$

At several places in our calculations, we were left with certain sums over the 1-lattice, notably in (6.19) (reappearing in (7.7)) and in (8.28). These remnant sums are repeated here:

$$\begin{aligned} &\mathbf{S} = \lim_{\rho \to \infty} \sum_{\mathbf{y} \in (L_1 \setminus \{0\}) \cap B(\rho)} \mathbf{K}(\mathbf{y}) \qquad (6.19)_r; \\ &\frac{1}{\text{meas}P} \sum_{\mathbf{y} \in L_1 \cap P} \tilde{\mathbf{h}}(\mathbf{x}, \mathbf{y}) \cdot \mathbf{d}(\mathbf{x}, \mathbf{y}) \qquad (8.28)_r, \end{aligned} \tag{9.4}$$

where

$$\tilde{\mathbf{h}} = \lim_{k\to\infty} \sum_{\substack{\mathbf{z}\in L_1\setminus\{0\}\\ \mathbf{z}\in kP}} \mathbf{K}(-\mathbf{z})\mathbf{d}(\mathbf{x}, \mathbf{y}+\mathbf{z}). \tag{9.5}$$

In fact, these sums are really just sums on L_a based on the *actual* dipole distribution $\mathbf{d}_a(\mathbf{x})$. This is clear in $(9.4)_1$ by just changing variables $\mathbf{y} = \mathbf{x}/\lambda_a$, *viz.*

$$\mathbf{S} = \lim_{\rho\to\infty} \sum_{\mathbf{x}\in(L_a\setminus\{0\})\cap B(\rho)} \mathbf{K}(\mathbf{x})\text{meas}U_a, \tag{9.6}$$

where U_a is the unit cell of $L_a, U_a := \lambda_a^3 U$. For the weak-short sums $(9.4)_2$, (9.5), we recall from (8.2) that

$$\mathbf{d}_{\lambda_a}(\mathbf{x}) + \lambda_a^3 \mathbf{d}(\mathbf{x}, \frac{\mathbf{x}}{\lambda_a}). \tag{9.7}$$

Hence, changing variables $\mathbf{z} = \mathbf{w}/\lambda_a$ in (9.5) and $\mathbf{v} = \mathbf{x}/\lambda_a$ in $(9.4)_2$ we get

$$\frac{1}{\text{meas}P_a} \sum_{\mathbf{v}\in L_a\cap P_a} \tilde{\mathbf{h}}_a(\mathbf{x}, \mathbf{v}) \cdot \lambda_a^3 \mathbf{d}(\mathbf{x}, \frac{\mathbf{v}}{\lambda_a}), \tag{9.8}$$

where

$$\tilde{\mathbf{h}}_a(\mathbf{x}, \mathbf{v}) := \lim_{k\to\infty} \sum_{\substack{\mathbf{w}\in L_a\setminus\{0\}\\ \mathbf{w}\in kP_a}} \mathbf{K}(-\mathbf{w})\lambda_a^3 \mathbf{d}(\mathbf{x}, \frac{\mathbf{v}+\mathbf{z}}{\lambda_a}),$$
$$P_a := \lambda_a^3 P. \tag{9.9}$$

Hence, these weak-short sums represent sums over the actual lattice of an exactly periodic dipole field $\lambda_a^3 \mathbf{d}(\mathbf{x}, \cdot/\lambda_a)$ that agrees with the actual field at $\mathbf{x}$. In conclusion, the remnant sums really represent lattice sums over the actual dipole field (or its weak-short periodic extension) and there is no physical significance to the "1" in 1-lattice.

The interpretations given above should be tempered with the caution that our calculations do not include fluctuations of the dipoles that normally occur at finite temperature, nor do they account for the detailed distributions of atomic spins. Despite these limitations, it can be stated that the successful continuum theories of elastic dielectrics (Toupin [1956]) and magnetic/magnetostrictive materials (Brown [1966]) are founded on lattice dipole calculations (under assumptions corresponding to our strong convergence). The way these calculations are used is to give definite forms to the field energy terms in the continuum theory, without trying to relate them quantitatively to the numerical values of atomic moments.

We now examine the weak-short results, which in principal suggest new continuum theories. It is clear from our results that there is a huge variety of ways that energy can be stored in weak-short oscillations or concentrations, without being reflected in the magnetization of the material. What we hope to get out of our calculations is a set of continuum variables (the "internal" or "hidden" variables of continuum theory) that parameterize the energy not

calculable from magnetization. In applications to specific materials, we expect that it will be sufficient to consider rather few dipole fields. Hence, we confine attention to some general remarks about (8.28) and then specialize.

The first point to notice about (8.28) is that it is entirely local, ***i.e.***, representable as an integral over Ω. Thus, under our hypothesis (8.2) the total weak-short energy of two disjoint pieces of material is independent of the distance between them, unlike the classical expression $\frac{-1}{2}\langle \mathbf{K} * \mathbf{m}, \mathbf{m}\rangle$ for the magnetostatic energy. It would therefore be natural to combine the weak-short energy together with the anisotropy energy of micromagnetics, as is commonly done with the terms

$$\frac{-1}{2}\left(\frac{1}{3}\|\mathbf{m}\|^2 + \langle \mathbf{m}, \mathbf{Sm}\rangle\right) \tag{9.10}$$

of (6.18). The danger in doing this is that the integrand of (8.28) does not depend on the dipoles through the magnetization alone. More than that, the effect of the dipoles and the lattice are intertwined in (8.28), so that the implications of the symmetry of the lattice and frame-indifference are not as in the usual expression for the anisotropy energy (see below).

To get a better understanding of the weak-short energy, we consider special cases. Many situations in ferrimagnetism or in antiferromagnetism can be described by having sublattices of the given lattice, each with a constant dipole field. For this purpose we consider a Bravais lattice L_1 and we assume it is written as the union of a finite number of disjoint Bravais sublattices:

$$L_1 = L^{(1)} \cup L^{(2)} \cup \ldots \cup L^{(n)}. \tag{9.11}$$

Let $\chi^{(i)} : \mathbb{R}^3 \to \mathbb{R}$ be the characteristic function of the i^{th} sublattice $L^{(i)}$. On the i^{th} sublattice we place a uniform dipole $\mathbf{d}^{(i)}$ so that the dipole field $\mathbf{d}(\mathbf{x})$ is

$$\mathbf{d}(\mathbf{x}) = \sum_i \chi^{(i)}(\mathbf{x})\mathbf{d}^{(i)}, \quad \mathbf{x} \in L_1. \tag{9.12}$$

Such a dipole field is periodic and a suitable lattice $\hat{L}_1$ and period cell P of the form (8.5) can always be chosen. In typical cases, such as in ferrimagnetism with or without an applied field or in antiferromagnetism with an applied field, we will not have (8.8) satisfied by $\mathbf{d}(\mathbf{x}, \mathbf{y}) = \mathbf{d}(\mathbf{y})$. Hence, we define

$$\hat{\mathbf{d}}^{(i)} := \mathbf{d}^{(i)} - \bar{\mathbf{d}}, \quad i = 1, \ldots, n; \quad \hat{\mathbf{d}}(x) := \sum_i \chi^{(i)}(\mathbf{x})\hat{\mathbf{d}}^{(i)}, \tag{9.13}$$

where

$$\bar{\mathbf{d}} := \frac{1}{\sigma}\sum_{L_1 \cap P} \mathbf{d}(\mathbf{x}) \tag{9.14}$$

is the average dipole moment over the period cell and σ is the total number of lattice points in $L_1 \cap P$. Then we have that

$$\sum_{\mathbf{x} \in L_1 \cap P} \hat{\mathbf{d}}(\mathbf{x}) = 0. \tag{9.15}$$

Let $\sigma^{(i)}$ be the number of lattice points of the i^{th} sublattice in $L_1 \cap P$,

$$\sigma^{(i)} = \sum_{\mathbf{x} \in L_1 \cap P} \chi^{(i)}, \quad i = 1, \ldots, n, \tag{9.16}$$

and let $\mu^{(i)} := \sigma^{(i)}/\sigma$ be the corresponding number fractions. With this notation $\Sigma \mu^{(i)} = 1$, and (9.15) can be expressed

$$\sum_i \mu^{(i)} \hat{\mathbf{d}}^{(i)} = 0. \tag{9.17}$$

In order to calculate the limiting energy (8.28), we first need to evaluate the expression (8.11). To this end, let

$$\begin{aligned} \tilde{\mathbf{h}}(\mathbf{y}) &:= \lim_{k \to \infty} \sum_{\substack{\mathbf{z} \in L_1 \setminus \{0\} \\ \mathbf{z} \in k\tilde{P}}} \mathbf{K}(-\mathbf{z}) \hat{\mathbf{d}}(\mathbf{y} + \mathbf{z}), \\ &= \lim_{k \to \infty} \sum_{\substack{\mathbf{z} \in L_1 \setminus \{0\} \\ \mathbf{z} \in k\tilde{P}}} \left\{ \mathbf{K}(-\mathbf{z}) \sum_i \chi^{(i)}(\mathbf{y} + \mathbf{z}) \hat{\mathbf{d}}^{(i)} \right\}. \end{aligned} \tag{9.18}$$

We now introduce (9.17) into (9.18) to get

$$\begin{aligned} \tilde{\mathbf{h}}(\mathbf{y}) &:= \lim_{k \to \infty} \sum_{\substack{\mathbf{z} \in L_1 \setminus \{0\} \\ \mathbf{z} \in k\tilde{P}}} \left\{ \mathbf{K}(-\mathbf{z}) \sum_i (\chi^{(i)}(\mathbf{y} + \mathbf{z}) - \mu^{(i)}) \hat{\mathbf{d}}^{(i)} \right\}, \\ &= \sum_i \mathbf{S}^{(i)}(\mathbf{y}) \hat{\mathbf{d}}^{(i)} \end{aligned} \tag{9.19}$$

where

$$\mathbf{S}^{(i)}(\mathbf{y}) = \lim_{k \to \infty} \sum_{\substack{\mathbf{z} \in L_1 \setminus \{0\} \\ \mathbf{z} \in k\tilde{P}}} \left\{ \mathbf{K}(-\mathbf{z}) (\chi^{(i)}(\mathbf{y} + \mathbf{z}) - \mu^{(i)}) \right\}. \tag{9.20}$$

The passage from (9.18) to (9.20) follows because each individual sum of (9.20) converges, which follows from Lemma 8.1 and the observation

$$\sum_{\mathbf{z} \in L_1 \cap P} (\chi^{(i)}(\mathbf{y} + \mathbf{z}) - \mu^{(i)}) = 0, \quad i = 1, \ldots, n, \ \mathbf{y} \in L_1 \tag{9.21}$$

Putting this expression for $\tilde{\mathbf{h}}$ in (8.28) we get

$$e_\lambda \to \frac{-1}{2} \int_\Omega \sum_{\mathbf{y} \in L_1 \cap P} \sum_{i,j} \chi^{(i)}(\mathbf{y}) \hat{\mathbf{d}}^{(i)} \cdot \mathbf{S}^{(j)}(\mathbf{y}) \hat{\mathbf{d}}^{(j)} d\mathbf{x} \text{ as } \lambda \to 0. \tag{9.22}$$

To simplify this expression, let

$$\mathbf{S}^{ij} = - \sum_{\mathbf{y} \in L - 1 \cap P} \chi^{(i)}(\mathbf{y}) \mathbf{S}^{(j)}(\mathbf{y}) = - \sum_{\mathbf{y} \in -L^{(i)} \cap P} \mathbf{S}^{(j)}(\mathbf{y}), \quad i, j, \in \{1, \ldots, n\}. \tag{9.23}$$

Then (9.22) assumes the compact form

$$e_\lambda \to \frac{1}{2} \int_\Omega \sum_{i,j} \hat{\mathbf{d}}^{(i)} \cdot \mathbf{S}^{ij} \hat{\mathbf{d}}^{(j)} d\mathbf{x}. \tag{9.24}$$

The expression (9.24) does not represent the total limiting energy because the constant dipole field $\bar{\mathbf{d}}$ has been subtracted from the given dipole field $\mathbf{d}(\mathbf{x})$, *i.e.*, $\mathbf{d}(\mathbf{x}) = \hat{\mathbf{d}}(\mathbf{x}) + \bar{\mathbf{d}}$. To treat this constant part, we note that if $\mathbf{m}_\lambda = \mathbf{m}_\lambda^{ws} + \mathbf{m}_\lambda^{s}$ with $\mathbf{m}_\lambda^{ws} \rightharpoonup 0$ in $L^2(\mathbb{R}^3, \mathbb{R}^3)$ and $\mathbf{m}^s \to \bar{\mathbf{m}}$ in $L^2(\mathbb{R}^3, \mathbb{R}^3)$, then

$$\begin{aligned}\lim_{\lambda\to 0} e_\lambda &= \lim_{\lambda\to 0}\langle \mathbf{m}_\lambda, \mathbf{T}_\lambda \mathbf{m}_\lambda\rangle,\\ &= \lim_{\lambda\to 0}[\langle \mathbf{m}_\lambda^s, \mathbf{T}_\lambda \mathbf{m}_\lambda^s\rangle + 2\langle \mathbf{m}_\lambda^{ws}, \mathbf{T}_\lambda \mathbf{m}_\lambda^s\rangle + \langle \mathbf{m}_\lambda^{ws}, \mathbf{T}_\lambda \mathbf{m}_\lambda^{ws}\rangle], \qquad (9.25)\\ &= \lim_{\lambda\to 0}[\langle \mathbf{m}_\lambda^s, \mathbf{T}_\lambda \mathbf{m}_\lambda^s\rangle + \langle \mathbf{m}_\lambda^{ws}, \mathbf{T}_\lambda \mathbf{m}_\lambda^{ws}\rangle].\end{aligned}$$

Here, we have used Proposition 6.1. Informally, we can say that there is no "interaction" between weak-short and strong convergence. (It can also be shown that there is no interaction between weak-short and weak-long convergence.) This shows that the constant part $\bar{\mathbf{d}}$ is expressed by the classical term (6.17) involving the magnetization. Allowing a general form for the magnetization as well as an amplitude $\psi(\mathbf{x})$ as in (8.4) we get as the final expression for the limiting energy

$$\begin{aligned}\frac{1}{2}\int_\Omega \Big\{ \psi(\mathbf{x})^2 \Big(\sum_{i,j=1}^{n} \hat{\mathbf{d}}^{(i)} \cdot \mathbf{S}^{ij}\hat{\mathbf{d}}^{(j)} \Big) - \frac{1}{3}|\mathbf{m}(x)|^2 - \mathbf{m}(\mathbf{x}) \cdot \mathbf{S}\mathbf{m}(\mathbf{x})\\ - \mathbf{m}(\mathbf{x}) \cdot (\mathbf{K} * \mathbf{m})(\mathbf{x}) \Big\} d\mathbf{x}\end{aligned} \qquad (9.26)$$

The key features of this energy are:

1. $\mathbf{S}$ depends only on the configuration of the lattice L_1 and, in particular, not on the dipoles. $\mathbf{S}^{ij}, i, j \in \{1, \ldots, n\}$ are n^2 trace-free 3×3 matrices that depend only on geometric relations among the n sublattices $L^1, \ldots, L^n \subset L_1$. In particular, $\mathbf{S}^{ij}$ is a lattice sum depending on the geometric relation between sublattices L^i and L^j. See (6.19), (9.20) and (9.23).
2. ψ is an amplitude function giving the absolute magnitude of the normalized sublattice dipole moments. ψ could be absorbed in $\hat{\mathbf{d}}^i$, $i = 1, \ldots, n$, resulting in position-dependent normalized dipole moments.
3. The $\hat{\mathbf{d}}^{(i)}$, $i = 1, \ldots, n$, which enter quadratically, are the normalized dipole moments. It is important to keep in mind that they are not the actual sublattice dipole moments; they have been normalized,

$$\sum_{i=1}^{n} \mu^{(i)}\hat{\mathbf{d}}^{(i)} = 0, \qquad (9.27)$$

by subtracting a suitable constant vector. Here the $\mu^{(i)}$ is the number fraction of sublattice $L^{(i)}$, see $(9.16)_{ff}$. The $\hat{\mathbf{d}}^{(i)}$ have the interpretation as internal variables.

4. The vector field $\mathbf{m}(\mathbf{x})$ is the magnetization, which represents the weak limit in L^2 of the dipole field, in the sense given in Section 4. We assume that $\mathbf{m}$ has compact support on $\Omega(\supset \text{supp}\psi)$ for obvious reasons. The field $\mathbf{m}(\mathbf{x})$ is interpreted as the magnetization even under the conditions of weak convergence by reasons given in Section 5.
5. The only nonlocal term is the last term of (9.26), $\langle \mathbf{m}, \mathbf{K} * \mathbf{m} \rangle$, the nonlocality arising from $\mathbf{K} * \mathbf{m}$. From Section 5 and Remark 6.3, the term $\mathbf{h} = \mathbf{K} * \mathbf{m}$ can be viewed as the weak solution of the equations

$$\left.\begin{aligned} \operatorname{div}(-\nabla\psi + \mathbf{m}) &= 0 \\ \mathbf{h} &= -\nabla\psi \end{aligned}\right\} \text{ on } \mathbb{R}^3. \tag{9.28}$$

This is the usual viewpoint in micromagnetics.
6. If the dipoles oscillate in a weak-long fashion (Section 7) the function $\mathbf{m}$ in (9.26) should be replaced by a sequence of magnetizations $\mathbf{m}^{(k)}$, $k = 1, 2, 3, \ldots$, and the limiting energy should then be calculated by taking the limit of (9.26) as $k \to \infty$.
7. Fixing the lattice L_1 and the sublattices $L^1, \ldots, L^n$, all the functions $\varphi, \mathbf{m}, \mathbf{d}^1, \ldots, \mathbf{d}^n$ in (9.26) are independent, except for the restriction (9.27). That is, there are dipole fields $\mathbf{d}_\lambda(\mathbf{x}), \mathbf{x} \in L_\lambda$, that give arbitrary functions $\varphi \in C_0^\infty$, $\mathbf{m} \in L^2$ and arbitrary vectors $\hat{\mathbf{d}}^{(1)}, \ldots, \hat{\mathbf{d}}^{(n)}$ subject only to (9.27). Therefore, it is sensible to regard these variables as independent variables in the continuum theory. The smoothness restriction on φ can surely be relaxed, but we do not pursue these refinements here.
8. The usual practice of micromagnetics, even in the magnetostrictive case, is to introduce a saturation hypothesis. This hypothesis is often motivated (Brown [1966]) by the assumption that the magnetic moment of each atom has a fixed magnitude, dependent only on temperature, regardless of the deformation. The usual micromagnetic form of this hypothesis can be derived from our formula (4.10), under the conditions of strong convergence, and is given by

$$|(\det \mathbf{F})\mathbf{m}(\mathbf{x})| = g(\theta), \quad \mathbf{x} \in \Omega$$

where $\mathbf{F}$ is the deformation gradient ($\mathbf{F} = \mathbf{1}$ in the rigid case), and $g(\theta)$ is a function of temperature (see James and Kinderlehrer [1992]). In either the weak-long or weak-short cases, the assumption that the magnetic moment of each atom has a fixed magnitude no longer leads to (9.29), as can be seen easily from the formulae above.

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