Journal of Elasticity, Vol. 11, No. 2, April 1981

Alphen aan den Rijn
Printed in The Netherlands

# The elastica and the problem of the pure bending for a non-convex stored energy function

R. L. FOSDICK and R. D. JAMES

*Department of Aerospace Engineering and Mechanics, University of Minnesota, Minneapolis, Minnesota 55455, U.S.A.*

(Received February 19, 1980)

## 1. Introduction

In 1744 Euler [1] published his famous appendix on elastic curves in which he not only emphasized his energy method of minimization as a viable method for solving problems in the field of natural science, but in which he also obtained his now famous "Euler formula" for the least force required to buckle a long thin column. The general energy method has long since become orthodoxy and the specific theory of the elastica has been linearized, treated with imperfections, analyzed from the point of view of bifurcation theory, and relaxed to include extensibility, but apparently not studied in its original setting with a stored energy response function more general than its original quadratic form $W \propto (\text{curvature})^2$.

The present work is motivated by this latter observation, and, thus, it concerns the theory of the inextensible elastica for a general possibly non-convex stored energy function. For many of our specific results we concentrate on what is often considered to be the simplest problem on the bending of beams; that associated either with prescribed terminal slope angles or with prescribed terminal bending moments. This so-called "pure bending" problem is contained in almost every elementary text on strength of materials or deformable body mechanics and, in fact, is most often used to introduce the student to the subject of bending. Many books give purely geometric arguments based on symmetry in order to conclude that for this problem an originally straight isotropic, homogeneous beam must assume the bent shape of a circular arc.† We find this conclusion generally to be false.

† One popular text that is widely used in the teaching of undergraduate deformable body mechanics is that of Crandall, Dahl and Lardner [2]. On p. 418 of this book it is noted that for the problem of pure bending the deformation pattern can be fixed by symmetry arguments alone. The specific arguments, and the conclusion that originally straight longitudinal lines must become arcs of circles, are contained on pp 418–420.

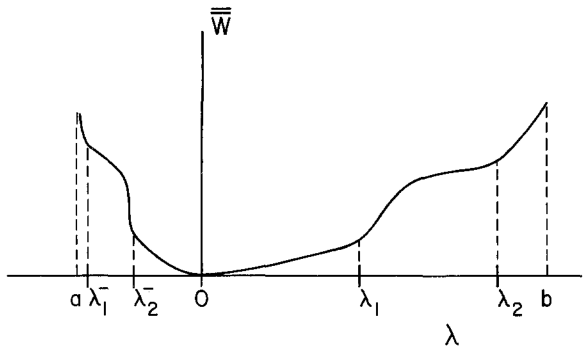


Figure 1. The stored energy response function $W = \bar{\bar{W}}(\lambda)$.

In Section 4 we completely and explicitly solve the pure bending problem, posed as a problem of minimization, for an inextensible, homogeneous, initially straight elastica whose graph of stored energy versus curvature is non-convex and like that given in Figure 1. Roughly, we denote by an Eulerian state, (see (4.2–4.4)) any configuration which minimizes the total stored energy of the elastica among a certain class of functions (including those with discontinuous curvatures) which take on prescribed terminal slope angles, and in Theorems 1, 2 and 3 we address the questions of existence and uniqueness for such states.

Theorem 1 is a standard result in the calculus of variations and is central to this paper in that it lists two necessary conditions for the existence of Eulerian states; the integrated form of the Euler equation and the Weierstrass condition. In Theorems 2 and 3 we explicitly determine the Eulerian states and their degree of uniqueness. Theorem 3, in particular, contains the novel conclusion that for a certain open set of prescribed terminal slope angles an Eulerian state must consist of smoothly connected circular patches, each patch having one of either two distinct radii. While the total *length* of the patches of each kind is uniquely determined, the particular arrangement of the distribution of the patches themselves is not; i.e., we have uniqueness up to a rearrangement. We believe that these kinds of equilibrium configurations are related to those that are observed even in a casual experiment on the pure bending of a steel pocket measuring tape.

Eulerian states have a property not shared by weak Eulerian states (cf. (4.6)). A weak Eulerian state†, by definition, minimizes the total stored energy relative to those functions whose derivatives lie uniformly close to the derivatives of the weak Eulerian state, and whose terminal slope angles agree with those of the weak Eulerian state. Like the Eulerian states, these weak Eulerian states satisfy a restricted form of the Weierstrass condition.

The distinction between Eulerian and weak Eulerian states is illuminated by the first integral, or "energy integral", of the Euler equation. The Euler equation has an energy integral, at least locally, for weak Eulerian states; that is, an integral of the Euler equation is constant on each interval of smoothness of the curvature, but the

† Weak Eulerian states are analogous to the 'metastable configurations' of Ericksen [3].

constant generally changes from interval to interval. We show that for Eulerian states, however, the energy integral is constant – the same constant – for every point on the elastica.

Another distinction, which we discuss in Section 4, concerns the dissipative nature of weak Eulerian states and the conservative nature of Eulerian states. For any cycle of prescribed terminal slope angles and corresponding sequence of Eulerian states, the net work of the loading device must be zero. However, for some cycles there exists corresponding sequences of weak Eulerian states for which the net work of the loading device is positive.

In Theorem 4 we show how to interpret Eulerian states so as to solve the pure bending problem when terminal bending moments are prescribed. Here, we are concerned with the minimization of the total potential energy of the elastica. In a discussion following Theorem 4 we show, among other things, that for certain distinct applied bending moments a configuration of minimum potential energy is highly non-unique, not even unique up to a rearrangement. In fact, any configuration which consists of combinations of smoothly connected patches of circular arcs having two distinct and specific radii will do, possibly all one radius or all the other, or *any* combination of the two. Figure 3 illustrates the configurations of minimum total potential energy that are possible in this problem.

As introductory to the problem just discussed concerning Eulerian states, in Sections 2 and 3 we describe the kinematic and constitutive properties of an inextensible elastic curve in $\mathbb{E}^2$. We view the elastica as a one dimensional elastic solid of second grade, so that relative to an arbitrary, but fixed, reference configuration the stored energy response function per unit length is defined on a certain domain which consists of pairs of first and second deformation gradients. These deformation gradients are shown to have simple representations in terms of the reference and present curvatures and the corresponding tangent vectors and, thus, our constitutive assumption is equivalent to a certain Cosserat theory of rods. We give a complete discussion of the restriction of frame indifference and the notions of material symmetry and homogeneity. In this theory the reference configuration of an elastica may possess no material symmetry, or it may be either isotropic or enantiomorphic; and these properties may be possessed either locally or globally. If isotropic, it is necessary that the reference configuration be straight. Moreover, regardless of its symmetry, if the reference configuration of an elastica is homogeneous then its reference form must be either circular or straight. The detailed restrictions on the stored energy response function are given in Section 3. For our study of Eulerian states in Section 4, we assume that the elastica is homogeneous and straight in a natural reference configuration, but it need not be isotropic.

## 2. The elastica: kinematic notions

In the elementary theory of the elastica it is assumed that every conceivable configuration of the body must lie in a fixed plane which we shall denote as $\mathbb{E}^2$, a

2-dimensional Euclidean point space. Thus, by a *placement* of the elastica we shall mean a mapping $\boldsymbol{\chi}(r)$ of a fixed real interval $r \in [0, R]$ into a curve $\mathscr{C}_\chi \subset \mathbb{E}^2$. To be definite, we shall assume throughout this work that a placement is continuously differentiable, twice piecewise differentiable, and non-singular in the sense that $d\boldsymbol{\chi}(r)/dr \neq 0$ for all $r \in [0, R]$. If we let $\boldsymbol{\kappa}(\cdot):[0, R] \to \mathscr{C}_\kappa \subset \mathbb{E}^2$ be a *reference placement,* then the *deformation* of $\mathscr{C}_\kappa$ into $\mathscr{C}_\chi$ is described by a mapping $\boldsymbol{\chi}_\kappa(\cdot):\mathscr{C}_\kappa \to \mathscr{C}_\chi \subset \mathbb{E}^2$ which satisfies the condition

$$\boldsymbol{\chi}(r) = \boldsymbol{\chi}_\kappa(\boldsymbol{\kappa}(r)), \qquad r \in [0, R]. \tag{2.1}$$

For the reference placement $\boldsymbol{\kappa}(\cdot)$, the arc length in $\mathscr{C}_\kappa$ is defined according to

$$s = s_\kappa(r) \equiv \int_0^r \left| \frac{d\boldsymbol{\kappa}(\xi)}{d\xi} \right| d\xi, \tag{2.2}$$

where $s_\kappa(R) = L$ is the total length of $\mathscr{C}_\kappa$. Since the placement $\boldsymbol{\kappa}(\cdot)$ is non-singular, $s_\kappa(\cdot)$ is invertible, and this allows us to parametrize the reference placement in terms of arc length:

$$\boldsymbol{\kappa}(s_\kappa^{-1}(\cdot)):[0, L] \to \mathscr{C}_\kappa \subset \mathbb{E}^2. \tag{2.3}$$

Further, it readily follows that

$$\boldsymbol{\alpha}_\kappa(s) \equiv \frac{d}{ds} \boldsymbol{\kappa}(s_\kappa^{-1}(s)) \tag{2.4}$$

represents that *unit tangent vector* to $\mathscr{C}_\kappa$ at $\boldsymbol{\kappa}(r)$ which has the direction of increasing arc length. This unit vector forms a basis for the 1-dimensional *tangent vector space* to $\mathscr{C}_\kappa$ at $\boldsymbol{\kappa}(r)$ which we shall refer to as $\mathscr{I}_\kappa(s)$. Owing to the previously mentioned smoothness requirements of any placement, we see that $\boldsymbol{\alpha}_\kappa(\cdot) \in C^0[0, L] \cap P^1[0, L]$,† and if we let $\boldsymbol{\alpha}_\kappa^\perp(s)$ denote that *unit normal vector* to $\mathscr{C}_\kappa$ at $\boldsymbol{\kappa}(r)$ which is generated as a counterclockwise 90° rotation of $\boldsymbol{\alpha}_\kappa(s)$ so that the ordered pair $(\boldsymbol{\alpha}_\kappa(s), \boldsymbol{\alpha}_\kappa^\perp(s))$ is orthonormal and right handed, it also follows that $\boldsymbol{\alpha}_\kappa^\perp(\cdot) \in C^0[0, L] \cap P^1[0, L]$. At any point $s \in [0, L]$ where $\boldsymbol{\alpha}_\kappa(\cdot)$ is differentiable we must have

$$\frac{d}{ds} \boldsymbol{\alpha}_\kappa(s) = \lambda_\kappa(s) \boldsymbol{\alpha}_\kappa^\perp(s), \tag{2.5}$$

where the scalar $\lambda_\kappa(s)$ is the *curvature* of $\mathscr{C}_\kappa$ at $\boldsymbol{\kappa}(r)$; clearly $\lambda_\kappa(\cdot) \in P^0[0, L]$. If we let $(\boldsymbol{i}_1, \boldsymbol{i}_2)$ denote a fixed right-handed orthonormal basis for the translation space $V$ which is associated with $\mathbb{E}^2$, and let $\theta_\kappa(s)$ represent the angle between $\boldsymbol{i}_1$ and $\boldsymbol{\alpha}_\kappa(s)$, measured from $\boldsymbol{i}_1$, with positive taken as counterclockwise, then it follows that

$$\begin{aligned} \boldsymbol{\alpha}_\kappa(s) &= \boldsymbol{i}_1 \cos \theta_\kappa(s) + \boldsymbol{i}_2 \sin \theta_\kappa(s), \\ \boldsymbol{\alpha}_\kappa^\perp(s) &= -\boldsymbol{i}_1 \sin \theta_\kappa(s) + \boldsymbol{i}_2 \cos \theta_\kappa(s), \end{aligned} \tag{2.6}$$

† Throughout this work, $P^m[0, L]$ denotes the set of $m$ times piecewise differentiable functions on the interval $[0, L]$.

and we have $\theta_\kappa(\cdot) \in C^0[0, L] \cap P^1[0, L]$. With (2.5) we then see that

$$\lambda_\kappa(s) = \boldsymbol{\alpha}_\kappa^\perp(s) \cdot \frac{d}{ds} \boldsymbol{\alpha}_\kappa(s) = \frac{d}{ds} \theta_\kappa(s). \tag{2.7}$$

Since the placement $\boldsymbol{\chi}(\cdot)$ is differentiable, the *gradient of the deformation* $\boldsymbol{\chi}_\kappa(\cdot)$ is well defined and at $\boldsymbol{\kappa}(r) \in \mathscr{C}_\kappa$ is equal to that *unique* linear transformation $\boldsymbol{F}_\kappa(s): \mathscr{I}_\kappa(s) \to V$ which satisfies

$$\frac{d}{ds} \boldsymbol{\chi}(s_\kappa^{-1}(s)) = \boldsymbol{F}_\kappa(s) \boldsymbol{\alpha}_\kappa(s). \tag{2.8}$$

It is common practice to write $\boldsymbol{F}_\kappa \equiv \nabla \boldsymbol{\chi}_\kappa$.

As above in (2.2), we may analogously introduce the arc length in $\mathscr{C}_\chi$ through the definition

$$s_\chi(r) \equiv \int_0^r \left| \frac{d\boldsymbol{\chi}(\xi)}{d\xi} \right| d\xi. \tag{2.9}$$

Thus, since the chain rule and (2.2) yields

$$\begin{aligned} \frac{d}{ds} \boldsymbol{\chi}(s_\kappa^{-1}(s)) &= \frac{d}{dr} \boldsymbol{\chi}(r) \frac{d}{ds} s_\kappa^{-1}(s) \\ &= \frac{d}{dr} \boldsymbol{\chi}(r) \Big/ \left| \frac{d}{dr} \boldsymbol{\kappa}(r) \right|, \end{aligned} \tag{2.10}$$

we see by (2.8) that

$$s_\chi(r) = \int_0^r \left| \frac{d}{d\xi} \boldsymbol{\kappa}(\xi) \right| \left| \boldsymbol{F}_\kappa(s_\kappa(\xi)) \boldsymbol{\alpha}_\kappa(s_\kappa(\xi)) \right| d\xi. \tag{2.11}$$

Hence, with the *assumption of inextensibility*, i.e.,

$$s_\kappa(r) = s_\chi(r) \qquad \forall r \in [0, R], \tag{2.12}$$

we see from (2.2) and (2.11) that

$$|\boldsymbol{F}_\kappa(s) \boldsymbol{\alpha}_\kappa(s)| = 1 \qquad \forall s \in [0, L], \tag{2.13}$$

and we may conclude that

$$\boldsymbol{\alpha}_\chi(s) \equiv \frac{d}{ds} \boldsymbol{\chi}(s_\kappa^{-1}(s)) = \boldsymbol{F}_\kappa(s) \boldsymbol{\alpha}_\kappa(s) \tag{2.14}$$

is a unit tangent vector to $\mathscr{C}_\chi$ at $\boldsymbol{\chi}(r)$. Consequently, we see that the range of $\boldsymbol{F}_\kappa(s)$ is $\mathscr{I}_\chi(s)$, the 1-dimensional tangent vector space to $\mathscr{C}_\chi$ at $\boldsymbol{\chi}(r)$, and so we have the representation

$$\boldsymbol{F}_\kappa(s) = \boldsymbol{\alpha}_\chi(s) \otimes \boldsymbol{\alpha}_\kappa(s). \tag{2.15}$$

Since $\boldsymbol{F}_\kappa(\cdot) \in C^0[0, L] \cap P^1[0, L]$, then whenever it is differentiable we have

$$\frac{d}{ds} \boldsymbol{F}_\kappa(s) = \lambda_\chi(s) \boldsymbol{\alpha}_\chi^\perp(s) \otimes \boldsymbol{\alpha}_\kappa(s) + \lambda_\kappa(s) \boldsymbol{\alpha}_\chi(s) \otimes \boldsymbol{\alpha}_\kappa^\perp(s), \tag{2.16}$$

where we have used (2.5) and its analogue for $\mathscr{C}_\chi$, i.e.,

$$\frac{d}{ds}\boldsymbol{\alpha}_\chi(s)=\lambda_\chi(s)\boldsymbol{\alpha}_\chi^\perp(s). \tag{2.17}$$

Here, $\lambda_\chi(s)$ is the curvature of $\mathscr{C}_\chi$ at $\boldsymbol{\chi}(r)$, and, by analogy to (2.7), we have the relation

$$\lambda_\chi(s)=\frac{d}{ds}\theta_\chi(s), \tag{2.18}$$

where $\theta_\chi(s)$ represents the angle between $\boldsymbol{i}_1$ and $\boldsymbol{\alpha}_\chi(s)$, measured from $\boldsymbol{i}_1$, with positive taken as counterclockwise.

If $\boldsymbol{F}_\kappa(\cdot)$ is differentiable at $s=s_\kappa(r)$, the *second gradient of the deformation* at $\boldsymbol{\kappa}(r)\in\mathscr{C}_\kappa$ is defined as that unique linear transformation $\boldsymbol{M}_\kappa(s):\mathscr{I}_\kappa(s)\to \mathrm{Lin}\,(V,V)$ which is such that

$$\frac{d}{ds}\boldsymbol{F}_\kappa(s)=\boldsymbol{M}_\kappa(s)\boldsymbol{\alpha}_\kappa(s). \tag{2.19}$$

Here, Lin $(V, V)$ denotes the set of linear transformations of $V$ into $V$, and we note that it is common practice to write $\boldsymbol{M}_\kappa=\nabla\nabla\boldsymbol{\chi}_\kappa=\nabla\boldsymbol{F}_\kappa$. Because of (2.16), it follows from this definition of $\boldsymbol{M}_\kappa$ that we have the representation

$$\boldsymbol{M}_\kappa(s)=[\lambda_\chi(s)\boldsymbol{\alpha}_\chi^\perp(s)\otimes\boldsymbol{\alpha}_\kappa(s)+\lambda_\kappa(s)\boldsymbol{\alpha}_\chi(s)\otimes\boldsymbol{\alpha}_\kappa^\perp(s)]\otimes\boldsymbol{\alpha}_\kappa(s). \tag{2.20}$$

Finally, due to the a priori smoothness requirements for any placement we observe that $\boldsymbol{\alpha}_\chi(\cdot)\in C^0[0,L]\cap P^1[0,L]$, $\lambda_\chi(\cdot)\in P^0[0,L]$, $\theta_\chi(\cdot)\in C^0[0,L]\cap P^1[0,L]$ and $\boldsymbol{M}_\kappa(\cdot)\in P^0[0,L]$.

## 3. The elastica: constitutive theory

In this section we shall pursue the notion that for pure mechanics an elastica is constitutively characterized by the prescription of a stored energy density response function, this function being given per unit length of a reference placement and being dependent upon first and second deformation gradients relative to this reference placement. Of course, the form of this constitutive response function will itself be dependent upon the particular reference placement, and in general the elastica may be inhomogeneous. Thus, relative to the reference placement $\boldsymbol{\kappa}(\cdot):[0,R]\to\mathscr{C}_\kappa$ we assume that

$$W=W_\kappa(\boldsymbol{F}_\kappa(s),\boldsymbol{M}_\kappa(s);s), \tag{3.1}$$

where at each $s\in[0,L]$ the *stored energy response function* $W_\kappa(\cdot,\cdot;s)$ is defined on the domain

$$\begin{aligned}\mathscr{D}_\kappa^1(s)\times\mathscr{D}_\kappa^2(s)\equiv\{(\boldsymbol{F},\boldsymbol{M})\mid \boldsymbol{F}&=\boldsymbol{\alpha}\otimes\boldsymbol{\alpha}_\kappa(s)\ \textit{and}\\ \boldsymbol{M}&=[\lambda\boldsymbol{\alpha}^\perp\otimes\boldsymbol{\alpha}_\kappa(s)+\lambda_\kappa(s)\boldsymbol{\alpha}\otimes\boldsymbol{\alpha}_\kappa^\perp(s)]\otimes\boldsymbol{\alpha}_\kappa(s)\ \textit{for}\\ &\textit{all unit vectors } \boldsymbol{\alpha}\in V \textit{ and all } \lambda\in\mathscr{I}_s\}.\end{aligned} \tag{3.2}$$

Here, $\boldsymbol{\alpha}^{\perp} \in V$ is generated as a counterclockwise $90°$ rotation of $\boldsymbol{\alpha}$ so that the ordered pair $(\boldsymbol{\alpha}, \boldsymbol{\alpha}^{\perp})$ is orthonormal and right handed, and $\mathcal{I}_s \subseteq \mathbb{R}$ denotes an open interval dependent on $s$ and typically including zero. Thus, given the reference placement $\boldsymbol{\kappa}(\cdot)$, it may happen that the value of the stored energy response function is undefined for other placements which require large values of the curvature.

Evidently, the above constitutive assumption is equivalent to a certain Cosserat theory of rods. That is, given a reference placement $\boldsymbol{\kappa}(\cdot)$ we see from (3.2) that essentially the domain of $W_\kappa(\cdot, \cdot; s)$ is determined by the two independent variables $\lambda$ and $\boldsymbol{\alpha}$. Thus, using (3.2) we may define

$$\bar{W}_\kappa(\lambda, \boldsymbol{\alpha}; s) \equiv W_\kappa(\boldsymbol{F}, \boldsymbol{M}; s) \tag{3.3}$$

for all $(\boldsymbol{F}, \boldsymbol{M}) \in \mathcal{D}^1_\kappa(s) \times \mathcal{D}^2_\kappa(s)$, and by (3.1) we may write

$$W = \bar{W}_\kappa(\lambda_\chi(s), \boldsymbol{\alpha}_\chi(s); s). \tag{3.4}$$

The reference placement $\boldsymbol{\kappa}(\cdot)$ is said to be *natural* if $\bar{W}_\kappa(\lambda_\kappa(s), \boldsymbol{\alpha}_\kappa(s); s) = \text{const.} \leq \bar{W}(\lambda, \boldsymbol{\alpha}; s)$ for all $s \in [0, L]$, all $\lambda \in \mathcal{I}_s$, and all unit vectors $\boldsymbol{\alpha} \in V$. Clearly, by (2.17) it is possible to replace the dependence on $\lambda_\chi(s)$ and $\boldsymbol{\alpha}_\chi(s)$ in (3.4) by a dependence on the two vectors $(d/ds)\boldsymbol{\alpha}_\chi(s)$ and $\boldsymbol{\alpha}_\chi(s)$, and, in so doing, we illustrate the noted connection to Cosserat theory. We shall not pursue this connection any further here.

For the reference placement $\boldsymbol{\kappa}(\cdot)$, the response function $W_\kappa$ is subject to the restriction of frame indifference. This requires that the two values of $W_\kappa$, as computed at $s \in [0, L]$ first for any placement $\boldsymbol{\chi}(\cdot): [0, R] \to \mathcal{C}_\chi$ such that $[\boldsymbol{F}_\kappa(s), \boldsymbol{M}_\kappa(s)] \in \mathcal{D}^1_\kappa(s) \times \mathcal{D}^2_\kappa(s)$ and then for the corresponding placement $\boldsymbol{\chi}^*(\cdot): [0, R] \to \mathcal{C}_{\chi^*}$ that is induced as an arbitrary proper rigid transformation of $\mathcal{C}_\chi$ in $\mathbb{E}^2$, be equal. Since for such a transformation we have

$$\boldsymbol{\alpha}_{\chi^*}(s) = \boldsymbol{Q}\boldsymbol{\alpha}_\chi(s), \qquad \lambda_{\chi^*}(s) = \lambda_\chi(s) \tag{3.5}$$

for all $s \in [0, L]$, where $\boldsymbol{Q}$ is any assigned proper orthogonal linear transformation of $V$ into $V$, and since for any fixed $s$ the placement $\boldsymbol{\chi}(\cdot)$ can be chosen so that $\lambda_\chi(s)$ is any number in $\mathcal{I}_s$ while $\boldsymbol{\alpha}_\chi(s)$ is any unit vector in $V$, it follows that at each $s \in [0, L]$

$$\bar{W}_\kappa(\lambda, \boldsymbol{\alpha}; s) = \bar{W}_\kappa(\lambda, \boldsymbol{Q}\boldsymbol{\alpha}; s), \tag{3.6}$$

for all such proper orthogonal $\boldsymbol{Q}$, all $\lambda \in \mathcal{I}_s$, and all unit vectors $\boldsymbol{\alpha} \in V$. Thus, in a standard way we have, for such $\lambda, \boldsymbol{\alpha}$, and $s$, the necessary and sufficient condition

$$\bar{W}_\kappa(\lambda, \boldsymbol{\alpha}; s) = \bar{\bar{W}}_\kappa(\lambda; s), \tag{3.7}$$

where $\bar{\bar{W}}_\kappa(\cdot; s)$ is defined on the open interval $\mathcal{I}_s$.

Had we applied the condition of frame indifference for arbitrary orthogonal $\boldsymbol{Q}$ of $V$ into $V$, proper or improper, then of course we still would have had

$$\boldsymbol{\alpha}_{\chi^*}(s) = \boldsymbol{Q}\boldsymbol{\alpha}_\chi(s). \tag{3.8}$$

However, since

$$\boldsymbol{\alpha}^{\perp}_\chi(s) = \boldsymbol{R}\boldsymbol{\alpha}_\chi(s) \quad \text{and} \quad \boldsymbol{\alpha}^{\perp}_{\chi^*}(s) = \boldsymbol{R}\boldsymbol{\alpha}_{\chi^*}(s),$$

where $\boldsymbol{R}$ is a 90° counterclockwise proper rotation of $V$ into $V$, it follows that

$$\boldsymbol{\alpha}_{\chi^*}^{\perp}(s) = \boldsymbol{RQR}^{\mathrm{T}}\boldsymbol{\alpha}_{\chi}^{\perp}(s). \tag{3.9}$$

Moreover, with the identity $\boldsymbol{RQR}^{\mathrm{T}} = (\det \boldsymbol{Q})\ \boldsymbol{Q}$ we see that

$$\boldsymbol{\alpha}_{\chi^*}^{\perp}(s) = (\det \boldsymbol{Q})\boldsymbol{Q}\boldsymbol{\alpha}_{\chi}^{\perp}(s), \tag{3.10}$$

so that $\boldsymbol{\alpha}_{\chi}^{\perp}(s)$ transforms as an axial vector. In addition, since

$$\frac{d}{ds}\boldsymbol{\alpha}_{\chi}(s) = \lambda_{\chi}(s)\boldsymbol{\alpha}_{\chi}^{\perp}(s) \quad \text{and} \quad \frac{d}{ds}\boldsymbol{\alpha}_{\chi^*}(s) = \lambda_{\chi^*}(s)\boldsymbol{\alpha}_{\chi^*}^{\perp}(s), \tag{3.11}$$

we see, by applying (3.8) and (3.10), that

$$\lambda_{\chi^*}(s)(\det \boldsymbol{Q})\boldsymbol{Q}\boldsymbol{\alpha}_{\chi}^{\perp}(s) = \lambda_{\chi}(s)\boldsymbol{Q}\boldsymbol{\alpha}_{\chi}^{\perp}(s), \tag{3.12}$$

so that

$$\lambda_{\chi^*}(s) = (\det \boldsymbol{Q})\lambda_{\chi}(s), \tag{3.13}$$

which shows that $\lambda_{\chi}(s)$ transforms as an axial scalar. With this transformation established, it is clear that while the condition of frame indifference when applied for arbitrary orthogonal $\boldsymbol{Q}$ would again yield the result (3.7), it would, in addition, imply that the interval $\mathcal{I}_s$ be centered at zero and that $\bar{\bar{W}}_{\kappa}(\cdot; s)$ be an even function on $\mathcal{I}_s$. We disagree with this conclusion. Rather we believe that the evenness of $\bar{\bar{W}}_{\kappa}(\cdot; s)$ should be the result of an argument of *material symmetry*, that $\bar{\bar{W}}_{\kappa}(\cdot; s)$ need not be even for certain thin elastic rods and ribbons which ought to be covered by a general theory of the elastica, and that the requirement of frame indifference should include only proper orthogonal transformations.

We turn now to the question of material symmetry. Roughly, within the subject of pure mechanics, to determine the material symmetries that a body may possess one must characterize that set of reference placements of the body relative to which "equal deformation" corresponds to "equal response". While it is usual practice that for solid bodies the admissible reference placements are restricted to be certain rigidly displaced copies of one another, we shall, at the outset, not make such a restriction here since we believe that the idea of material symmetry should contain a broader notion.

To initiate a more detailed treatment of material symmetry let us first recall that our basic constitutive assumptions, (3.1) and (3.2), required the existence of a frame indifferent stored energy response function $W_{\kappa}(\cdot, \cdot; s)$ on the domain $\mathcal{D}_{\kappa}^{1}(s) \times \mathcal{D}_{\kappa}^{2}(s)$ defined for each reference placement $\boldsymbol{\kappa}(\cdot)$. By applying this assumption to the reference placement $\boldsymbol{\mu}(\cdot):[0, R] \to \mathscr{C}_{\mu}$, and by requiring that for any permissible placement $\boldsymbol{\chi}(\cdot):[0, R] \to \mathscr{C}_{\chi}$ the values of the stored energy at each $s \in [0, L]$ are equal when computed relative to either $\boldsymbol{\kappa}(\cdot)$ or $\boldsymbol{\mu}(\cdot)$, we see that

$$\bar{\bar{W}}_{\kappa}(\lambda; s) = \bar{\bar{W}}_{\mu}(\lambda; s) \tag{3.14}$$

for all $\lambda \in \mathcal{I}_s$, $s \in [0, L]$, and for all reference placements $\boldsymbol{\kappa}(\cdot)$ and $\boldsymbol{\mu}(\cdot)$. Thus, $\bar{\bar{W}}_{\kappa}$ *is* *independent of the reference placement* and, henceforth, we may drop this dependence and write

$$\bar{\bar{\bar{W}}}(\lambda; s) \equiv \bar{\bar{\bar{W}}}_\kappa(\lambda; s). \tag{3.15}$$

Now, suppose that the reference placements $\boldsymbol{\kappa}(\cdot)$ and $\boldsymbol{\mu}(\cdot)$ are such that at $s_0 \in [0, L]$ the domains of the stored energy functions $W_\kappa(\cdot, \cdot; s_0)$ and $W_\mu(\cdot, \cdot; s_0)$ are equal, i.e.,

$$\mathscr{D}^1_\kappa(s_0) \times \mathscr{D}^2_\kappa(s_0) = \mathscr{D}^1_\mu(s_0) \times \mathscr{D}^2_\mu(s_0), \tag{3.16}$$

and consider any two placements of the elastica, $\boldsymbol{\chi}(\cdot): [0, R] \to \mathscr{C}_\chi$ and $\acute{\boldsymbol{\chi}}(\cdot): [0, R] \to \mathscr{C}_{\acute{\chi}}$, such that the deformations $\boldsymbol{\chi}_\kappa(\cdot): \mathscr{C}_\kappa \to \mathscr{C}_\chi$ and $\acute{\boldsymbol{\chi}}_\mu(\cdot): \mathscr{C}_\mu \to \mathscr{C}_{\acute{\chi}}$ satisfy the requirements

$$\boldsymbol{F}_\kappa(s_0) = \acute{\boldsymbol{F}}_\mu(s_0), \qquad \boldsymbol{M}_\kappa(s_0) = \acute{\boldsymbol{M}}_\mu(s_0). \tag{3.17}$$

We say that the reference placements $\boldsymbol{\kappa}(\cdot)$ and $\boldsymbol{\mu}(\cdot)$ are *peers* at $s_0 \in [0, L]$ if

$$W_\kappa(\boldsymbol{F}_\kappa(s_0), \boldsymbol{M}_\kappa(s_0); s_0) = W_\mu(\acute{\boldsymbol{F}}_\mu(s_0), \acute{\boldsymbol{M}}_\mu(s_0); s_0). \tag{3.18}$$

Whereas (3.17) defines the notion of "equal deformation" mentioned earlier, (3.18) may be interpreted as the meaning of "equal response".

In order to characterize more specifically the peers at $s_0 \in [0, L]$ it is first essential to determine the equivalence class of reference placements for which (3.16) holds, since it is only in this class that peers are possible. Thus, we have the following

*Lemma* 1. *A necessary and sufficient condition that* (3.16) *holds at* $s_0 \in [0, L]$ *is either*

(i) $\boldsymbol{\alpha}_\mu(s_0) = \boldsymbol{\alpha}_\kappa(s_0)$ *and* $\lambda_\mu(s_0) = \lambda_\kappa(s_0)$, $\qquad (3.19)_1$

*or*

(ii) $\boldsymbol{\alpha}_\mu(s_0) = -\boldsymbol{\alpha}_\kappa(s_0)$ *and* $\lambda_\mu(s_0) = -\lambda_\kappa(s_0)$, $\qquad (3.19)_2$

*where in case* (ii) *the interval* $\mathscr{I}_{s_0}$ *in* (3.2) *must be symmetric about zero.*

*Proof.* For a given arbitrary reference placement $\boldsymbol{\kappa}(\cdot)$, suppose that $(\boldsymbol{F}, \boldsymbol{M}) \in \mathscr{D}^1_\kappa(s_0) \times \mathscr{D}^2_\kappa(s_0)$, so that at $s_0 \in [0, L]$ the representation (3.2) holds for some unit vector $\boldsymbol{\alpha} \in V$ and some $\lambda \in \mathscr{I}_{s_0}$. Then, if $(\boldsymbol{F}, \boldsymbol{M})$ is also to belong to $\mathscr{D}^1_\mu(S_0) \times \mathscr{D}^2_\mu(s_0)$ for reference placement $\boldsymbol{\mu}(\cdot)$, it follows that there must exist a unit vector $\acute{\boldsymbol{\alpha}} \in V$ and $\acute{\lambda} \in \mathscr{I}_{s_0}$ such that

$$\begin{aligned}
&\acute{\boldsymbol{\alpha}} \otimes \boldsymbol{\alpha}_\mu(s_0) = \boldsymbol{\alpha} \otimes \boldsymbol{\alpha}_\kappa(s_0),\\
&[\acute{\lambda}\acute{\boldsymbol{\alpha}}^\perp \otimes \boldsymbol{\alpha}_\mu(s_0) + \lambda_\mu(s_0)\acute{\boldsymbol{\alpha}} \otimes \boldsymbol{\alpha}^\perp_\mu(s_0)] \otimes \boldsymbol{\alpha}_\mu(s_0)\\
&\quad = [\lambda\boldsymbol{\alpha}^\perp \otimes \boldsymbol{\alpha}_\kappa(s_0) + \lambda_\kappa(s_0)\boldsymbol{\alpha} \otimes \boldsymbol{\alpha}^\perp_\kappa(s_0)] \otimes \boldsymbol{\alpha}_\kappa(s_0).
\end{aligned} \tag{3.20}$$

Thus, from $(3.20)_1$ we not only obtain $\boldsymbol{\alpha}_\mu(s_0) = \pm\boldsymbol{\alpha}_\kappa(s_0)$, but also, correspondingly, $\acute{\boldsymbol{\alpha}} = \pm\boldsymbol{\alpha}$. Further, by placing these results into $(3.20)_2$ it readily follows that $\lambda_\mu(s_0) = \pm\lambda_\kappa(s_0)$ must hold, and we determine $\acute{\lambda} = \pm\lambda$. Clearly, in the case of the negative sign, $\acute{\lambda}$ will belong to $\mathscr{I}_{s_0}$ for all $\lambda \in \mathscr{I}_{s_0}$ if and only if $\mathscr{I}_{s_0}$ is centered at zero.

Conversely, suppose, for the above given $\boldsymbol{\kappa}(\cdot)$ and any $(\boldsymbol{F}, \boldsymbol{M}) \in \mathscr{D}^1_\kappa(s_0) \times \mathscr{D}^2_\kappa(s_0)$, we consider a reference placement $\boldsymbol{\mu}(\cdot)$ such that (3.19) holds. Then, in either case we have

$$\begin{aligned} \boldsymbol{F} &= \pm \boldsymbol{\alpha} \otimes \boldsymbol{\alpha}_\mu(s_0), \\ \boldsymbol{M} &= [\lambda \boldsymbol{\alpha}^\perp \otimes \boldsymbol{\alpha}_\mu(s_0) \pm \lambda_\mu(s_0) \boldsymbol{\alpha} \otimes \boldsymbol{\alpha}^\perp_\mu(s_0)] \otimes \boldsymbol{\alpha}_\mu(s_0), \end{aligned} \tag{3.21}$$

and we see that $(\boldsymbol{F}, \boldsymbol{M}) \in \mathscr{D}^1_\mu(s_0) \times \mathscr{D}^2_\mu(s_0)$, provided, of course, when the negative sign holds the interval $\mathscr{I}_{s_0}$ is centered at zero.

Because of this lemma, an elastica can have peers at $s_0$ which fall into at most two possible categories; we refer to these categories as the *peer groups* (under the operation of multiplication) $\{1\}$ and $\{1, -1\}$.

The group $\{1\}$ is trivial in that it indicates that for any given $\boldsymbol{\kappa}(\cdot)$, all other corresponding peers at $s_0$ must satisfy $(3.19)_1$. That is, the set of all peers associated with this group consists of those reference placements that have the same unit tangent-normal pairs and the same curvatures as $s_0$. *An elastica which possesses* $\{1\}$ *as its sole peer group at* $s_0$ *is said to possess no material symmetry at* $s_0$.

The group $\{1, -1\}$ is *non-trivial* and is associated with reference placements $\boldsymbol{\kappa}(\cdot)$ and $\boldsymbol{\mu}(\cdot)$ which satisfy $(3.19)_2$. In addition, the interval $\mathscr{I}_{s_0}$ must be centered at zero. Thus, with the aid of (3.18), (3.17), (3.15), (3.7) and (3.3) it readily follows that *the stored energy function for an elastica which possesses* $\{1, -1\}$ *as its peer group at* $s_0$ *must satisfy*

$$\bar{\bar{W}}(\lambda; s_0) = \bar{\bar{W}}(-\lambda; s_0) \tag{3.22}$$

*for all* $\lambda \in \mathscr{I}_{s_0}$, *where* $\mathscr{I}_{s_0}$ *is centered at zero.*

We shall now show that the peers at $s_0$ which are associated with the non-trivial peer group $\{1, -1\}$ fall naturally into one of two possible classes. To this end, suppose that the reference placements $\boldsymbol{\kappa}(\cdot) : [0, R] \to \mathscr{C}_\kappa$ and $\boldsymbol{\mu}(\cdot) : [0, R] \to \mathscr{C}_\mu$ are two such peers at $s_0$. If these reference placements may be related by a transformation of the form

$$\boldsymbol{\alpha}_\kappa(s) = \boldsymbol{Q}(s) \boldsymbol{\alpha}_\mu(s) \tag*{$(3.23)_1$}$$

for all $s \in [0, L]$, where $\boldsymbol{Q}(s)$ is an orthogonal linear transformation of $V$ into $V$ with

$$\frac{d}{ds} \boldsymbol{Q}(s)|_{s=s_0} = 0, \tag*{$(3.23)_2$}$$

then we say that the *transformation is rigid at* $s_0$, a *rotation* if $\det \boldsymbol{Q}(s_0) = +1$, and a *reflection* if $\det \boldsymbol{Q}(s_0) = -1$. The motivation for this nomenclature lies in the fact that under such a transformation we have, similar to (3.8)–(3.13),

$$\lambda_\kappa(s_0) = \lambda_\mu(s_0) \det \boldsymbol{Q}(s_0), \tag*{$(3.23)_3$}$$

so that the curvatures $\lambda_\kappa(s_0)$ and $\lambda_\mu(s_0)$ are equal in magnitude.

Now, since $(3.19)_2$ must hold for the peers $\boldsymbol{\kappa}(\cdot)$ and $\boldsymbol{\mu}(\cdot)$ at $s_0$, we have that $\boldsymbol{\alpha}_\kappa(s_0) = -\boldsymbol{\alpha}_\mu(s_0)$ and either (i) $\lambda_\kappa(s_0) = \lambda_\mu(s_0) = 0$, or (ii) $\lambda_\kappa(s_0) = -\lambda_\mu(s_0) \neq 0$. In the former case it is straightforward to show that the orthogonal transformation

$$\boldsymbol{Q}(s) \equiv \boldsymbol{\alpha}_\kappa(s) \otimes \boldsymbol{\alpha}_\mu(s) + \boldsymbol{\alpha}_\kappa^\perp(s) \otimes \boldsymbol{\alpha}_\mu^\perp(s) \tag{3.24}$$

is compatible with (3.23) and corresponds to a rigid rotation of 180° at $s_0$ in $\mathbb{E}^2$. Therefore, in case (i) we say that the elastica is *isotropic at* $s_0$ and that $\boldsymbol{\kappa}(\cdot)$ and $\boldsymbol{\mu}(\cdot)$ are *undistorted* placements at $s_0$. In this case the peers at $s_0$ may be transformed into one another by a rigid rotation at $s_0$ which is a main ingredient in the classical notion of material symmetry. If $\boldsymbol{\kappa}(\cdot)$ and $\boldsymbol{\mu}(\cdot)$ are peers associated with $\{1, -1\}$ at *all* $s \in [0, L]$ with $\lambda_\kappa(s) = \lambda_\mu(s) = 0$, then the elastica is said to be simply *isotropic* and the reference configurations $\mathscr{C}_\kappa$ and $\mathscr{C}_\mu$ are straight.

In the case (ii) above it is clear from $(3.23)_3$ that there can be no relation of the form (3.23) where the orthogonal transformation is a rigid rotation at $s_0$. However, in this case we may take

$$\boldsymbol{Q}(s) \equiv \boldsymbol{\alpha}_\kappa(s) \otimes \boldsymbol{\alpha}_\mu(s) - \boldsymbol{\alpha}_\kappa^\perp(s) \otimes \boldsymbol{\alpha}_\mu^\perp(s), \tag{3.25}$$

which is, in fact, compatible with (3.23) and represents a rigid reflection at $s_0$. Therefore, in case (ii) we call the peers at $s_0$ *enantiomorphs* and refer to the elastica as being *enantiomorphic at* $s_0$. Thus, while the peers at $s_0$ cannot be transformed into one another by a rigid rotation at $s_0$, they can by the application of an appropriate rigid reflection and so in a sense they may be considered to be local mirror images of one another in $\mathbb{E}^2$. Of course, they also may be considered to be relatively deformed placements of one another where the deformation $\mathscr{C}_\kappa \leftrightarrow \mathscr{C}_\mu$ is non-rigid. If the peers are enantiomorphs at all $s \in [0, L]$ then the elastica is said to be simply *enantiomorphic.* Clearly, then, the reference configurations $\mathscr{C}_\kappa$ and $\mathscr{C}_\mu$ are not straight.

Finally, to conclude this section, we consider the notion of *homogeneity.* Let $\boldsymbol{\kappa}(\cdot): [0, R] \to \mathscr{C}_\kappa$ be a fixed reference placement. Corresponding to any choice of points $s_1$ and $s_2$ in $[0, L]$, let $\boldsymbol{\mu}(\cdot): [0, R] \to \mathscr{C}_\mu$ be another reference placement such that $\mathscr{C}_\mu$ is a rigidly displaced copy of $\mathscr{C}_\kappa$ in $\mathbb{E}^2$ and such that

$$\boldsymbol{\alpha}_\mu(s_2) = \boldsymbol{\alpha}_\kappa(s_1). \tag{3.26}$$

We shall say that $\boldsymbol{\kappa}(\cdot)$ is *homogeneous* if

$$\mathscr{D}_\kappa^1(s_1) \times \mathscr{D}_\kappa^2(s_1) = \mathscr{D}_\mu^1(s_2) \times \mathscr{D}_\mu^2(s_2), \tag{3.27}$$

and if for any choice of $(\boldsymbol{F}, \boldsymbol{M})$ in this coincident domain,

$$W_\kappa(\boldsymbol{F}, \boldsymbol{M}; s_1) = W_\mu(\boldsymbol{F}, \boldsymbol{M}; s_2). \tag{3.28}$$

Although the restriction (3.27) at first appears severe, it is not really so. Equation (3.26) combined with the argument which leads to (3.14) shows that (3.27) is equivalent to the condition $\mathscr{I}_{s_1} = \mathscr{I}_{s_2}$. To a large extent this formal definition of homogeneity is motivated by the following two working assumptions: (i) if a reference placement is homogeneous then all other rigidly displaced copies of that placement also are homogeneous. (ii) If any two of these reference placements are arranged so that the tangent-normal pair at an arbitrary but given point of one is the same as the tangent-normal pair at an arbitrary but given point of the other, then the prescription of equal first and second deformation gradients at these points will yield equal values for the corresponding strain energy densities.

Now, in a manner completely analogous to our earlier analysis of (3.16), or more specifically (3.20), we find from (3.26) and (3.27) that

$$\lambda_\mu(s_2) = \lambda_\kappa(s_1). \tag{3.29}$$

Moreover, since $\mathscr{C}_\kappa$ and $\mathscr{C}_\mu$ are (proper) rigid copies of one another, so that $\lambda_\mu(s) = \lambda_\kappa(s)$ for all $s \in [0, L]$, we see that

$$\lambda_\kappa(s_1) = \lambda_\kappa(s_2). \tag{3.30}$$

Finally, because of (3.3), (3.7) and (3.15), we see that (3.28) is equivalent to the condition

$$\bar{\bar{W}}(\lambda; s_1) = \bar{\bar{W}}(\lambda; s_2) \tag{3.31}$$

for all $\lambda \in \mathscr{I} \equiv \mathscr{I}_{s_1} = \mathscr{I}_{s_2}$. Therefore, *if* $\boldsymbol{\kappa}(\cdot)$ *is a homogeneous reference placement, then* (i) $\lambda_\kappa(s) = \text{constant}$ *for all* $s \in [0, L]$, *i.e.,* $\mathscr{C}_\kappa$ *is either circular or straight, and* (ii) $\bar{\bar{W}}(\lambda; s_1) = \bar{\bar{W}}(\lambda; s_2)$ *for all* $s_1, s_2 \in [0, L]$ *and all* $\lambda \in \mathscr{I}$, *i.e.,* $\bar{\bar{W}}$ *does not depend explicitly on* $s$. Whence, for any placement $\boldsymbol{\chi}(\cdot): [0, R] \to \mathscr{C}_\chi$ we may write

$$W = \bar{\bar{W}}(\lambda_\chi(s)). \tag{3.32}$$

If in addition to the reference placement $\boldsymbol{\kappa}(\cdot)$ being homogeneous, the elastica is isotropic (enantiomorphic) at only one point $s \in [0, L]$, then it must be isotropic (enantiomorphic) at all points, $\mathscr{C}_\kappa$ must be straight (circular), $\bar{\bar{W}}(\cdot): \mathscr{I} \to \mathbb{R}$ must be an even function, and the interval $\mathscr{I}$ must be centered at zero.

## 4. The problem statement, structure and solution

There are two fundamental and intimately related problems in the theory of the inextensible homogeneous elastica which we shall be concerned with here. Both deal with the phenomenon of bending from an originally straight and natural reference placement. In the *primary problem* we wish to characterize, by means of a minimization procedure and up to a rigid transformation in $\mathbb{E}^2$, those placements of the elastica that result by bending each of its ends solely with a bending moment through angles whose difference is prescribed, while in the *secondary problem* we wish to establish the same characterization in the case where the equilibrated bending moments on the ends are prescribed.

We shall assume that the elastica has no particular material symmetry, that the reference placement $\boldsymbol{\kappa}(\cdot): [0, R] \to \mathscr{C}_\kappa$ is natural and homogeneous, that $\mathscr{C}_\kappa$ is straight, and that the stored energy response function $\bar{\bar{W}}(\cdot): \mathscr{I} \to \mathbb{R}$ is non-convex and like the graph pictured in Figure 1. The assumption of non-convexity not only makes the subject problems non-trivial, but it also leads to the description of certain interesting phenomena which are observed in practice. Specifically, for $\mathscr{I}$ we shall take the open interval $\mathscr{I} \equiv (a, b)$, where $a < 0 < b$, assume $\bar{\bar{W}}(\cdot)$ to be twice continuously differentiable on $\mathscr{I}$, take $\bar{\bar{W}}(\lambda)$ to be non-negative and to vanish only at $\lambda = 0$, and require $\bar{\bar{W}}(\cdot)$ to be super-convex† everywhere on its domain $\mathscr{I}$ except in the closed intervals $[\lambda_1^-, \lambda_2^-]$ and $[\lambda_1, \lambda_2]$, where $a < \lambda_1^- < \lambda_2^- < 0 < \lambda_1 < \lambda_2 < b$. At the points $\lambda_1^-, \lambda_2^-, \lambda_1$ and $\lambda_2$, $\bar{\bar{W}}(\cdot)$ is convex, and in the open intervals $(\lambda_1^-, \lambda_2^-)$ and $(\lambda_1, \lambda_2)$ $\bar{\bar{W}}(\cdot)$ is non-convex.

Since a deformation $\boldsymbol{\chi}_\kappa(\cdot): \mathscr{C}_\kappa \rightarrow \mathscr{C}_\chi$ in $\mathbb{E}^2$ is determined up to an arbitrary translation once a fixed unit vector $\boldsymbol{i}_1 \in V$ has been chosen and the angle $\theta_\chi(s)$ between $\boldsymbol{i}_1$ and the unit tangent vector $\boldsymbol{\alpha}_\chi(s) \in \mathscr{I}_\chi(s)$ has been specified for all $s \in [0, L]$, the *primary Eulerian problem for the elastica* may be expressed in its most elementary form as follows: *Determine, within a specified class of functions C, those functions* $\theta(\cdot): [0, L] \rightarrow \mathbb{R}$ *which minimize the total stored energy.*

$$E[\theta] \equiv \int_0^L W(\theta'(s))\, ds, \tag{4.1}$$

*at fixed total angle change* $\theta(L) - \theta(0) \equiv \Delta$. Here, $\theta'$ denotes the first derivative of $\theta$ which, by (2.18), represents the curvature, and due to our considerations in Section 2 the natural class of functions is

$$C \equiv \{\theta(\cdot): [0, L] \rightarrow \mathbb{R} \mid \theta(\cdot) \in C^0[0, L] \cap P^1[0, L], \theta'(\cdot) \in \mathscr{I} \text{ a.e.}\}. \tag{4.2}$$

It is convenient to have available the following

Definition. *Let* $\Delta \in \mathbb{R}$ *be given and let* $C(\Delta) \subset C$ *denote that subset of* $C$ *for which*

$$\int_0^L \theta'(s)\, ds = \Delta. \tag{4.3}$$

*Then,* $\tilde{\theta}(\cdot) \in C(\Delta)$ *is said to be an Eulerian state if*

$$E[\tilde{\theta}] \leq E[\theta] \tag{4.4}$$

*for all* $\theta(\cdot) \in C(\Delta)$.

From (4.3) it is clear that $\theta'(s)$ cannot belong to $\mathscr{I}$ for almost all $s \in [0, L]$ unless $\Delta$ is such that $(\Delta/L) \in \mathscr{I}$. Since no Eulerian state can exist if $\Delta/L$ is otherwise, we see, from this definition, that solving the primary Eulerian problem for the elastica is tantamount to determining the possible Eulerian states when $(\Delta/L) \in \mathscr{I}$. Toward this end, we record the following well known theorem from the calculus of variations concerning necessary conditions for the existence of an Eulerian state.

Theorem 1. *Let* $\theta(\cdot) \in C(\Delta)$ *be an Eulerian state. Then,*
(i) *there exists a constant* $\tilde{M} \in \mathbb{R}$ *such that*

$$\bar{\bar{W}}'(\tilde{\theta}'(s)) = \tilde{M}, \tag{4.5$_1$}$$

† A point $\lambda \in \mathscr{I}$ is a point of super-convexity for $\bar{\bar{W}}(\cdot)$ if the tangent line to $\bar{\bar{W}}(\cdot)$ at $\bar{\bar{W}}(\lambda)$ lies everywhere else below the graph of $\bar{\bar{W}}(\cdot)$.

*and*

$$\text{(ii)}\quad \bar{\bar{W}}(\tilde{\theta}'(s))+[\lambda-\tilde{\theta}'(s)]\bar{\bar{W}}'(\tilde{\theta}'(s))\leq\bar{\bar{W}}(\lambda), \qquad (4.5)_2$$

*both holding for all points $s\in[0,L]$ of smoothness of $\tilde{\theta}(\cdot)$ and the latter holding for all $\lambda\in\mathcal{I}$.*

The number $\tilde{M}$ in Theorem 1 is commonly called the *static bending moment* which is associated with the curvature $\tilde{\theta}'(s)$. In words, the second part of this theorem shows that the curvature of any Eulerian state can only assume values which correspond to points of convexity for $\bar{\bar{W}}$. Thus, as is readily seen from Figure 1, these values must lie in the domain $(\alpha, \lambda_1^-]\cup[\lambda_2^-, \lambda_1]\cup[\lambda_2, b)$. The first part of this theorem shows that the curvature of any one Eulerian state must, throughout the deformed elastica, assume values which correspond to points of common slope for $\bar{\bar{W}}$ – the common slope being the static bending moment. The equation $(4.5)_1$ is an integrated form of the Euler equation associated with the functional $E[\cdot]$, and $(4.5)_2$ is the Weierstrass condition. We note that the integrated Euler equation is satisfied by functions other than the Eulerian states. For example, if $E[\cdot]$ is minimized at $\tilde{\tilde{\theta}}(\cdot)\in C(\Delta)$ relative to those functions $\theta(\cdot)\in C(\Delta)$ which satisfy, for some $\varepsilon>0$,

$$|\theta'(s)-\tilde{\tilde{\theta}}'(s)|<\varepsilon \qquad (4.6)$$

almost everywhere, then $\tilde{\tilde{\theta}}(\cdot)$ also satisfies $(4.5)_1$. We shall refer to these functions $\tilde{\tilde{\theta}}(\cdot)$ as *weak Eulerian states.* A weak Eulerian state also satisfies a restricted form of the condition of convexity $(4.5)_2$; that is, if $s$ is a point of smoothness of the weak Eulerian state $\tilde{\tilde{\theta}}(\cdot)$, then $(4.5)_2$ holds at $s$ for all $\lambda$ such that $|\lambda-\tilde{\tilde{\theta}}'(s)|<\varepsilon$, $\varepsilon$ being the same constant as in (4.6).

The next two theorems answer completely the existence and uniqueness questions for the Eulerian states. The proofs of these theorems (see, e.g., [4] or [5]) make essential use of the conditions (4.5) and, in fact, show that these conditions are sufficient for the existence of Eulerian states.

THEOREM 2. *Suppose*

$$\frac{\Delta}{L}\in(a, \lambda_1^-]\cup[\lambda_2^-, \lambda_1]\cup[\lambda_2, b). \qquad (4.7)$$

*Then, an Eulerian state $\tilde{\theta}(\cdot)\in C(\Delta)$ exists, and its curvature $\tilde{\theta}'(\cdot)$is unique and is given by*

$$\tilde{\theta}'(s)=\frac{\Delta}{L} \qquad (4.8)$$

*for all $s\in[0,L]$.*

For such values of the prescribed *nominal curvature* $\Delta/L$ as given by (4.7), this theorem shows that any placement which corresponds to an Eulerian state must have the form of a circular arc with radius equal to $L/\Delta$. The associated bending moment is given by $\tilde{M}=\bar{\bar{W}}'(\Delta/L)$.

For values of $\Delta/L$ that are complementary to (4.7) in $\mathcal{I}$ we have

THEOREM 3. *Suppose*

$$\frac{\Delta}{L} \in (\lambda_1^-, \lambda_2^-) \cup (\lambda_1, \lambda_2). \tag{4.9}$$

*Then, an Eulerian state $\tilde{\theta}(\cdot) \in C(\Delta)$ exists, and its curvature $\tilde{\theta}'(\cdot)$ is characterized as follows*:
(i) *If $\Delta/L \in (\lambda_1^-, \lambda_2^-)$ then*

$$\tilde{\theta}'(s) = \begin{cases} \lambda_1^- & \text{for} \quad s \in \mathcal{P}, \\ \lambda_2^- & \text{for} \quad s \in [0, L] - \mathcal{P}, \end{cases} \tag{4.10}$$

*where $\mathcal{P} \subset [0, L]$ is the union of any finite number of open intervals whose total length $l(\mathcal{P})$ is given by*

$$l(\mathcal{P}) = \left(\frac{\lambda_2^- - (\Delta/L)}{\lambda_2^- - \lambda_1^-}\right)L. \tag{4.11}$$

(ii) *If $\Delta/L \in (\lambda_1, \lambda_2)$ then*

$$\tilde{\theta}'(s) = \begin{cases} \lambda_1 & \text{for} \quad s \in \mathcal{P}, \\ \lambda_2 & \text{for} \quad s \in [0, L] - \mathcal{P}, \end{cases} \tag{4.12}$$

*where $\mathcal{P} \subset [0, L]$ is similar to that noted above, but has total length*

$$l(\mathcal{P}) = \left(\frac{\lambda_2 - (\Delta/L)}{\lambda_2 - \lambda_1}\right)L. \tag{4.13}$$

This theorem shows that for values of the prescribed nominal curvature $\Delta/L$ that correspond to the non-convex portion of the stored energy graph in Figure 1, any placement of an Eulerian state must be composed of circular patches. Each patch must have one of two distinct radii and be connected to adjacent patches with a continuously turning tangent vector. While the *total length* of that portion $\mathcal{P}$ of the placement which corresponds to the circular patches having curvature $\lambda_1^-$ in case (i) and $\lambda_1$ in case (ii) is uniquely determined, the particular *distribution* of these patches in the respective placements is not. This means that for such values of $\Delta/L$, the curvature of any Eulerian state is *unique up to a rearrangement* (cf., Dunn and Fosdick [6]). The associated bending moment is unique, however, and is given by

$$\tilde{M} = \begin{cases} \bar{\bar{W}}'(\lambda_1^-) = \bar{\bar{W}}'(\lambda_2^-) \text{ for the case (i),} \\ \bar{\bar{W}}'(\lambda_1) = \bar{\bar{W}}'(\lambda_2) \text{ for the case (ii).} \end{cases} \tag{4.14}$$

Our characterization of all possible Eulerian states is now complete, and in Figure 2b we summarize our results for applied bending moment $\tilde{M}$ versus assigned nominal curvature $\Delta/L$. In order to aid the interpretation, we have drawn in Figure 2a the first derivative of the stored energy, taken from Figure 1, as a function of curvature. It should be noted that Figure 2b is a copy of Figure 2a except for the intervals $(\lambda_1^-, \lambda_2^-)$ and $(\lambda_1, \lambda_2)$ wherein according to (4.14) the respective dotted lines are traced. Borrowing nomenclature from the subject of thermostatics, we may call these dotted lines *Maxwell lines* since they satisfy *Maxwell's Area Rule*. To be more

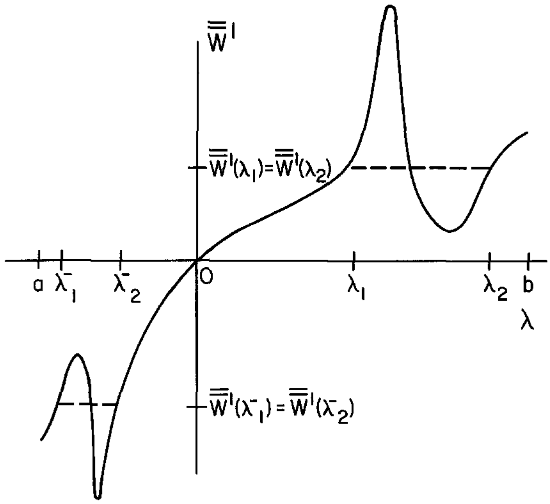


Figure 2a. The first derivative of the stored energy response function.

specific, consider, for example, the interval $(\lambda_1, \lambda_2)$, and note that

$$\int_{\lambda_1}^{\lambda_2} \bar{\bar{W}}'(\lambda)\, d\lambda = \bar{\bar{W}}(\lambda_2) - \bar{\bar{W}}(\lambda_1) \tag{4.15}$$

represents the area under the graph of $\bar{\bar{W}}'(\cdot)$ between $\lambda_1$ and $\lambda_2$. Note, also that the area under the dotted line in Figure 2a (or, equivalently the graph of $\tilde{M}$ in Figure 2b) between $\lambda_1$ and $\lambda_2$ is given by $(\lambda_2 - \lambda_1)\bar{\bar{W}}'(\lambda_1)$. The fact that these two areas must be equal is a direct consequence of the observation that $\lambda_1$ and $\lambda_2$ are two points of convexity for $\bar{\bar{W}}(\cdot)$ which share a common tangent line, so that

$$\bar{\bar{W}}(\lambda_2) = \bar{\bar{W}}(\lambda_1) + (\lambda_2 - \lambda_1)\bar{\bar{W}}'(\lambda_1). \tag{4.16}$$

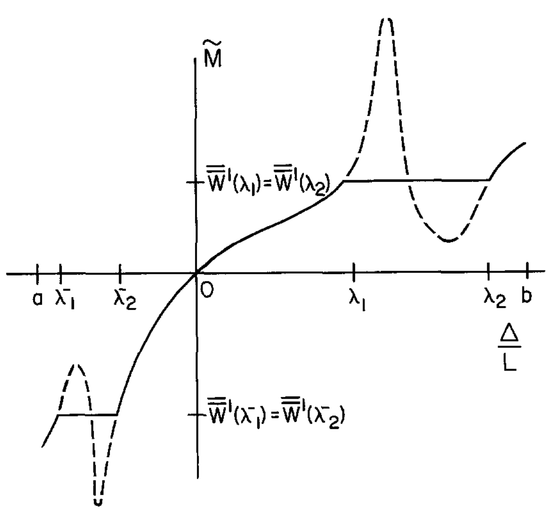


Figure 2b. Applied bending moment $\tilde{M}$ vrs. nominal curvature $\Delta/L$ for Eulerian states.

This equality of areas constitutes what is commonly called the Maxwell Area Rule. However, it is often equivalently stated in terms which equate to zero the signed area between the graph of $\bar{\bar{W}}'(\cdot)$ and the $\tilde{M}$ = constant line over the interval $(\lambda_1, \lambda_2)$. Of course, analogous remarks apply to the interval $(\lambda_1^-, \lambda_2^-)$.

Equation (4.16) has an interesting consequence with regard to the analysis of the Euler equation for non-smooth Eulerian states via the "energy integral". For the present problem the energy integral is easily constructed and has the form

$$\bar{\bar{W}}(\tilde{\theta}'(s)) - \tilde{\theta}'(s)\bar{\bar{W}}'(\tilde{\theta}'(s)) = \text{const.}, \tag{4.17}$$

where $s \in [0, L]$ is any point of smoothness of $\tilde{\theta}'(\cdot)$. If $\tilde{\theta}'(\cdot)$ is smooth only on disconnected patches in $[0, L]$ the constant on the right hand side in (4.17) may differ from patch to patch. Clearly, when $\Delta/L$ is restricted according to Theorem 2 the same constant in (4.17) is valid throughout the interval $[0, L]$ since in this case $\tilde{\theta}'(s) = \Delta/L$ for all $s \in [0, L]$. What is not quite so obvious is that for *any* Eulerian state, even for those non-smooth ones which are characterized in Theorem 3, the constant in (4.17) is the same for all $s \in [0, L]$. To see this, it suffices to consider the situation when $(\Delta/L) \in (\lambda_1, \lambda_2)$ in Theorem 3 and to note that in this case (4.12), (4.14) and (4.16) yield

$$\begin{aligned}\bar{\bar{W}}(\lambda_2) - \lambda_2\bar{\bar{W}}'(\lambda_2) - \{\bar{\bar{W}}(\lambda_1) - \lambda_1\bar{\bar{W}}'(\lambda_1)\} \\ = \bar{\bar{W}}(\lambda_2) - \bar{\bar{W}}(\lambda_1) - (\lambda_2 - \lambda_1)\bar{\bar{W}}'(\lambda_1) = 0.\end{aligned} \tag{4.18}$$

When $(\Delta/L) \in (\lambda_1^-, \lambda_2^-)$, a similar argument holds.

One must be cautious, however, when considering weak Eulerian states in that there exists a large class of such states, none of which satisfy the energy integral (4.17) with the same constant for all $s \in [0, L]$.

While not so easily categorized as are the Eulerian states, the weak Eulerian states can dissipate energy in a sense we shall describe. Suppose the increment $\Delta$ is made to follow a cycle; $\Delta$ is given by an assigned continuously differentiable function of a parameter, $\Delta = \delta(t)$, $t \in [0, T]$, with $\delta(0) = \delta(T)$. Assume that for each $t \in [0, T]$, $\delta(t)/L$ belongs to $(a, b)$. According to Theorems 2 and 3, to each value of $\delta(\cdot)$ we can find a corresponding Eulerian state. $\tilde{M} = \tilde{m}(t)$ also becomes a continuous, single valued function of $t$, when evaluated for the chosen Eulerian states. Without providing details of the analysis, we assert that to each $t$, there corresponds also a weak Eulerian state, and if $\delta(t)/L$ belongs to the set $(\lambda_1^-, \lambda_2^-) \cup (\lambda_1, \lambda_2)$, this weak Eulerian state can be chosen to be different from any Eulerian state possible at that value of $t$. To each member of this one parameter family of weak Eulerian states, there will correspond a bending moment $\tilde{\tilde{M}} = \tilde{\tilde{m}}(t)$. No matter how we choose the one parameter family of Eulerian states, it must be true that

$$\int_0^T \tilde{m}(t)\delta'(t)\,dt = 0,$$

that is, the loading device can perform no net work on the elastica. However, there is a one parameter family of weak Eulerian states for certain functions $\delta(t)$, with the same initial and final states, such that

$$\int_0^T \tilde{\tilde{m}}(t)\delta'(t)\,dt > 0,$$

so that, in this sense, the elastica can dissipate the work done by the loading device. A detailed argument which we shall not pursue would also show that the analogue of Knowles' condition of dissipation [7] is satisfied in its strict sense across discontinuities of the curvature which occur in some of the weak Eulerian states, although it would be satisfied with equality across any discontinuity which appears in an Eulerian state.

We turn now to the *secondary Eulerian problem for the elastica* which is to determine, within the class C, those functions $\theta(\cdot):[0,L]\to\mathbb{R}$ which minimize the total potential energy

$$E_M[\theta] \equiv \int_0^L \bar{\bar{W}}(\theta'(s))\,ds - M[\theta(L)-\theta(0)], \tag{4.19}$$

where $M\in\mathbb{R}$ is the assigned applied bending moment. Given $M\in\mathbb{R}$, we shall call $\tilde{\theta}(\cdot)\in C$ a *minimizer of* $E_M$ if

$$E_M[\tilde{\theta}] \le E_M[\theta] \tag{4.20}$$

for all $\theta(\cdot)\in C$.

While the primary and secondary Eulerian problems are distinct, they do have a common physical root and their solutions are intimately related as we see in the following

THEOREM 4. *Let $\tilde{\theta}(\cdot)\in C(\Delta)$ be an Eulerian state for some $\Delta\in\mathbb{R}$, and let $\tilde{M}$ denote the associated bending moment. Then $\tilde{\theta}(\cdot)\in C$ is a minimizer of $E_{\tilde{M}}$. Conversely, suppose $\tilde{\theta}(\cdot)\in C$ is a minimizer of $E_M$ for some $M\in\mathbb{R}$ and define*

$$\tilde{\Delta} \equiv \int_0^L \tilde{\theta}'(s)\,ds. \tag{4.21}$$

*Then, $\tilde{\theta}(\cdot)\in C(\tilde{\Delta})$ is an Eulerian state whose bending moment is M.*

*Proof.* If $\tilde{\theta}(\cdot)\in C(\Delta)$ is an Eulerian state for some $\Delta\in\mathbb{R}$ and if $\tilde{M}$ is the associated bending moment, then Theorem 1 shows that almost everywhere

$$\bar{\bar{W}}(\tilde{\theta}'(s)) - \tilde{M}\tilde{\theta}'(s) \le \bar{\bar{W}}(\lambda) - \tilde{M}\lambda \tag{4.22}$$

for all $\lambda\in\mathcal{I}$. Then, if $\theta(\cdot)\in C$ is arbitrary it follows, by replacing $\lambda$ in (4.16) with $\theta'(s)$ and by integrating, that $E_{\tilde{M}}[\tilde{\theta}]\le E_{\tilde{M}}[\theta]$ for all $\theta(\cdot)\in C$, so that $\tilde{\theta}(\cdot)\in C$ is a minimizer of $E_{\tilde{M}}$.

On the other hand, suppose $\tilde{\theta}(\cdot)\in C$ is a minimizer of $E_M$ for some $M\in\mathbb{R}$ and define $\tilde{\Delta}\in\mathbb{R}$ as in (4.21). Then, (4.19) and (4.20) yield

$$\int_0^L \bar{\bar{W}}(\tilde{\theta}'(s))\,ds - M\tilde{\Delta} \le \int_0^L \bar{\bar{W}}(\theta'(s))\,ds - M[\theta(L)-\theta(0)]$$

for all $\theta(\cdot)\in C$. *A fortiori*, for all $\theta(\cdot)\in C(\tilde{\Delta})$ we have

$$\int_0^L \bar{\bar{W}}(\tilde{\theta}'(s))\,ds \leq \int_0^L \bar{\bar{W}}(\theta'(s))\,ds,$$

so that $\tilde{\theta}(\cdot)\in C(\tilde{\Delta})$ is an Eulerian state. To see that the bending moment of this Eulerian state is $M$, we need only recall our definition following Theorem 1 and note that since $\tilde{\theta}(\cdot)$ is a minimizer of $E_M$, the integrated form of the Euler equation, $\bar{\bar{W}}'(\tilde{\theta}'(s))=M$, must hold almost everywhere.

This theorem, together with our results concerning Eulerian states, shows that if $M$ is in the open set,

$$M\in(\bar{\bar{W}}'(a),\ \bar{\bar{W}}'(\lambda_1^-))\cup(\bar{\bar{W}}'(\lambda_2^-),\ \bar{\bar{W}}'(\lambda_1))\cup(\bar{\bar{W}}'(\lambda_2),\ \bar{\bar{W}}'(b)), \tag{4.23}$$

where

$$\bar{\bar{W}}'(a)\equiv\lim_{\varepsilon\to0}\bar{\bar{W}}'(a+\varepsilon),\qquad \bar{\bar{W}}'(b)\equiv\lim_{\varepsilon\to0}\bar{\bar{W}}'(b-\varepsilon), \tag{4.24}$$

then a minimizer of $E_M$, $\tilde{\theta}(\cdot)\in C$, exists, and its curvature $\tilde{\theta}'(\cdot)$ is unique and is given by

$$\tilde{\theta}'(s)=\Delta/L \tag{4.25}$$

for all $s\in[0,L]$, where the constant $\Delta$ is such that $\Delta/L$ is the unique inverse of $\bar{\bar{W}}'(\Delta/L)=M$ in the domain $(a,\lambda_1^-)\cup(\lambda_2^-,\lambda_1)\cup(\lambda_2,b)$. Note from (4.25) that $\Delta=\tilde{\theta}(L)-\tilde{\theta}(0)$.

It can also be argued, using Theorem 4, that if either

$$\text{(i)}\ \ M=\bar{\bar{W}}'(\lambda_1^-)=\bar{\bar{W}}'(\lambda_2^-), \tag*{(4.26)$_1$}$$

or

$$\text{(ii)}\ \ M=\bar{\bar{W}}'(\lambda_1)=\bar{\bar{W}}'(\lambda_2), \tag*{(4.26)$_2$}$$

then while a minimizer $\tilde{\theta}(\cdot)\in C$ of $E_M$ exists, its curvature $\tilde{\theta}'(\cdot)$ is neither unique nor even unique up to a rearrangement. In each of these cases, however, the corresponding curvature must be of the respective form

$$\text{(i)}\quad \tilde{\theta}'(s)=\begin{cases}\lambda_1^- & \textit{for}\quad s\in\mathcal{P},\\ \lambda_2^- & \textit{for}\quad s\in[0,L]-\mathcal{P},\end{cases} \tag*{(4.27)$_1$}$$

or

$$\text{(ii)}\quad \tilde{\theta}'(s)=\begin{cases}\lambda_1 & \textit{for}\quad s\in\mathcal{P},\\ \lambda_2 & \textit{for}\quad s\in[0,L]-\mathcal{P},\end{cases} \tag*{(4.27)$_2$}$$

where in either case $\mathcal{P}\subseteq[0,L]$ is the union of any finite number of intervals whose total length may be arbitrarily specified so that $0\leq l(\mathcal{P})\leq L$. For each specified length $l(\mathcal{P})$ it follows that the respective curvatures are unique up to a rearrangement and that, correspondingly, by integration of (4.27) we have

$$\text{(i)}\ \ \Delta=\lambda_1^-l(\mathcal{P})+\lambda_2^-(L-l(\mathcal{P})), \tag*{(4.28)$_1$}$$

or

(ii) $\Delta = \lambda_1 l(\mathscr{P}) + \lambda_2(L - l(\mathscr{P}))$, $(4.28)_2$

where $\Delta \equiv \tilde{\theta}(L) - \tilde{\theta}(0)$ represents the unique total angle change that is common to the complete class of minimizers of $E_M$.

Finally, our earlier work together with Theorem 4 shows that if $M$ is prescribed to be any other number besides those covered in (4.23) and (4.26) then $E_M$ does not possess a minimizer. Thus, from Figure 2a if $M$ satisfies either $M \leq \bar{\bar{W}}'(a)$ or $M \geq \bar{\bar{W}}'(b)$, a minimizer of $E_M$ does not exist.

Imagine, now, that an increasing sequence of equilibrated bending moments, starting at zero, are applied to the ends of a slender and initially straight homogeneous rod. For $M = 0$, (4.25) applies with $\Delta = 0$ so that we must have $\tilde{\theta}'(s) = 0$ for all $s \in [0, L]$ and the rod is straight as pictured in Figure 3a. As $M$ is increased the solution (4.25) continues to apply and the rod must take on a circular form with a

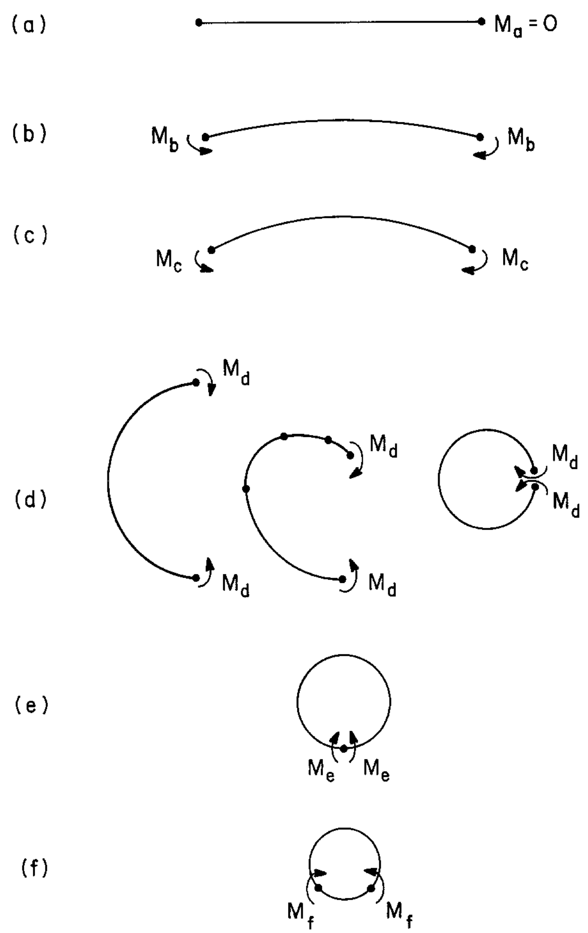


Figure 3. Minimizing placements which correspond to an increasing sequence of bending moments: $0 = M_a < M_b < M_c < M_d = \bar{\bar{W}}'(\lambda_1) = \bar{\bar{W}}'(\lambda_2) < M_e < M_f$.

unique but diminishing radius, as pictured in Figures 3b and 3c, until $M$ reaches the value of $\bar{\bar{W}}'(\lambda_1) = \bar{\bar{W}}'(\lambda_2)$. At this value the curvature of the rod is governed by $(4.27)_2$ and we may only conclude that the rod is either of a circular form with radius $1/\lambda_1$, or of a smooth form which combines radii $1/\lambda_1$ and $1/\lambda_2$ in patches whose individual lengths are indeterminate and restricted only to the extent that they sum to the total length $L$ of the given inextensible elastica. Some of the possible shapes are pictured in Figure 3d. Finally, as $M$ is further increased the solution (4.25) again applies and the rod will take on circular forms of smaller radii as pictured in Figures 3e and 3f.

The behavior displayed by this sequence of solutions is reminiscent of certain phenomena commonly observed in practice. The shapes in Figure 3 look remarkably like configurations that are obtained from a casual experiment on the bending of a steel pocket measuring tape. In fact, at a certain applied bending moment it is easy to produce a large class of configurations which contain one, two, or more severely bent patches surrounded by patches that are not so severely bent. It is possible that this bending moment corresponds to the value $\bar{\bar{W}}'(\lambda_1) = \bar{\bar{W}}'(\lambda_2)$ in Figure 2b.

In general, the connection between Eulerian states and the observed behavior of thin metal or polymer rods may be less obvious. When certain metal or polymer rods are loaded by sufficiently large terminal bending moments, they show patches of severe "yield" which are separated from the rest of the rod by fairly sharp boundaries. In some cases, when the end moments are relaxed the "yielded" patches remain curved and the remaining sections return to their straight form. Such a final configuration could not correspond to an Eulerian state within the present work since we have shown that the straight configuration is the only moment free Eulerian state. However, if the constitutive relation pictured in Figure 2a was allowed to dip below the abscissa for $\lambda > 0$, or above the abscissa for $\lambda < 0$, then it would be possible to produce a *weak Eulerian state*, corresponding to zero terminal bending moments, which has this partially bent and partially straight form. The bent patches for this weak Eulerian state would all be circular and of the same radius.

**Acknowledgement**

We gratefully acknowledge the U.S. National Science Foundation, Grant No. ENG 7726616 for its support of this work.

REFERENCES

[1] Euler, Leonhard, Methodus inveniendi lineas curvas maximi minimive proprietate gaudentes, additamentum I de curvis elastics. Lausanne and Geneve. *Opera Omnia I* 24 (1744) 231–297.
[2] Crandall, Stephen H., Norman C. Dahl and Thomas J. Lardner, *An Introduction to the Mechanics of Solids*, second ed. with SI units. McGraw-Hill 1978.
[3] Ericksen, J. L., Equilibrium of bars. *Journal of Elasticity* 5 (1975) 191–201.

[4] James, Richard D., Co-existent phases in the one dimensional static theory of elastic bars. *Arch. Rational Mech. Anal.* 72 (1979) 99–140.
[5] Dunn, J. E., R. L. Fosdick and R. Stone, On minimizing states and their stability in elastic bars. In preparation.
[6] Dunn, J. Ernest, and Roger L. Fosdick, The morphology and stability of material phases. *Arch. Rational Mech. Anal.* 74 (1980) 1–99.
[7] Knowles, James K., On the dissipation associated with equilibrium shocks in finite elasticity. *Journal of Elasticity* 9 (1979) 131–158.