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Micromagnetics of very thin films

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We determine by a scaling calculation the limiting form of the free energy governing a ferromagnetic film of vanishing thickness. Our theory generalizes Stoner and Wohlfarth's results for flat ellipsoids to arbitrary-shaped very thin films.

1. Introduction

In their well-known paper, Stoner & Wohlfarth (1948) made use of the idea that if a ferromagnetic particle is small, the energy-minimizing magnetization will be nearly constant. This becomes apparent by writing the micromagnetic energy per unit volume of a particle Ω_δ of typical diameter δ, and uniformly rescaling the energy onto a domain Ω of typical diameter 1. It is found that in the rescaled energy the exchange term is

$$\int_\Omega \frac{\alpha}{\delta^2}|\nabla m|^2 \,\mathrm{d}y, \tag{1.1}$$

where α is the exchange constant. Thus, as $\delta \to 0$, the exchange term contributes an unacceptably large energy unless $|\nabla m| \sim 0$.

In recent years an improved understanding of weak convergence methods has made possible various scaling calculations. The typical format of these calculations for a theory based on a variational principle is the following. Suppose $\mathbb{E}^{(\delta)}(m)$ is the total energy associated with a magnetization function m, and that this energy depends on a small scaling parameter δ. A minimizer $m^{(\delta)}$ of $\mathbb{E}^{(\delta)}$ will also depend upon δ. As $\delta \to 0$, $m^{(\delta)}$ may be expected to converge to a function $\tilde{m}$ which minimizes an energy functional $\mathbb{E}^0(m)$. Usually this limiting energy functional $\mathbb{E}^0(m)$ does not follow simply from setting $\delta = 0$ in $\mathbb{E}^{(\delta)}(m)$. The goal of the scaling calculation is then to (i) determine in what sense $m^{(\delta)}$ converges to $\tilde{m}$, and (ii) find the expression for $\mathbb{E}^0(m)$. The minimizer $\tilde{m}$ of $\mathbb{E}^0$ can then be used as an approximation

$$\tilde{m} \doteq m^{(\delta_{\mathrm{actual}})}, \tag{1.2}$$

where $\delta_{\mathrm{actual}} \ll 1$ is the 'measured' value of δ. It transpires from (1.2) that the type of convergence is important—the stronger the convergence the more $\tilde{m}$ resembles $m^{(\delta)}$. In some instances (e.g. in the limit $\delta \to \infty$, which corresponds to phase theory (see DeSimone 1993)), only certain averages of $m^{(\delta)}$ can be compared with those of $\tilde{m}$.

Sufficient conditions for the convergence of $\mathbb{E}^{(\delta)}$ to $\mathbb{E}^0$ are often stated using an abstract scheme termed Γ-convergence (De Giorgi 1975). We shall not use this framework, however, because the elementary viewpoint described above seems adequate for our purposes.

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A rigorous asymptotic analysis based on these ideas was used by DeSimone (1995) to study the problem of small ferromagnetic particles of arbitrary shape. His results explain why in such particles the energy is close to that associated with a constant magnetization, even though complex domain structures are expected near corners. Thus, the results of Stoner & Wohlfarth, albeit strictly valid for small ellipsoidal particles only, can be applied to small particles of arbitrary shape.

In this paper we consider a thin film Ω_h of thickness h and cross-section S. We associate Ω_h with the standard micromagnetic energy $\mathbb{E}^{(h)}(m)$ (Brown 1963). The relevant scaling parameter for this problem is the thickness h. Proceeding in the spirit of the scaling calculation sketched above, we find in § 4 that a minimizer $m^{(h)}$ of $\mathbb{E}^{(h)}$ converges in some sense to a function $\tilde{m}$ as $h \to 0$, where $\tilde{m}$ minimizes a limiting energy $\mathbb{E}^0(m)$, for which we obtain an expression (equation (4.17)). Remarkably, the limiting energy is completely local, that is to say, the magnetostatic equation which constrains the magnetization m in the original problem disappears from the limiting one. The minimizer $\tilde{m}$ is shown to be independent of the direction normal to the thin film; thus, the limiting problem is two dimensional. Furthermore, the form of $\mathbb{E}^0$ reveals that the thinness of the film imports an artificial anisotropy which disfavours out-of-plane magnetization. This feature is well-known in the special case of flat ellipsoids.

Metastability and hysteresis are important in ferromagnetism. Kohn & Sternberg (1989) have extended the scaling procedure to relative minimizers of $\mathbb{E}^{(\delta)}$ which converge to relative minimizers of $\mathbb{E}^0$; their results have in turn been applied to small ferromagnetic particles by DeSimone (1995). Thus, our theory (extended to relative minimizers) can be used to predict the square hysteresis loops observed in very thin films having in-plane easy axes.

The scaling approach described above does not lead to a criterion indicating how thin is a 'thin' film. In other words, the method does not give precise limits of h for which (1.2) is satisfied within some prescribed tolerance. However, it is easy to obtain an upper bound h_c for the acceptable values of h. Even though this is a rather imprecise way of assessing the applicability of the theory to any given case, it is nonetheless sufficient for many practical purposes. In § 5 we give estimates of h_c for a few materials of interest, and discuss engineering applications where our formulation may prove useful.

2. Micromagnetics

Consider a ferromagnetic thin film defined by $\Omega_h \equiv \{(x_1, x_2) \in S,\ 0 < x_3 < h\}$, where $h \ll 1$ is a measure of the film thickness, which we assume constant, see figure 1. Here, S is a suitable domain in $\mathbb{R}^2$, with area A, representing the shape of the film in plan. In the classical theory of micromagnetics, the free energy per unit volume of such a ferromagnetic film is given by the following expression (Brown 1963),

$$\bar{\mathbb{E}}^{(h)}(\bar{m}) = \frac{1}{Ah} \int_{\Omega_h} \alpha |\nabla \bar{m}|^2 + \varphi(\bar{m}) + \tfrac{1}{2} \nabla \bar{\zeta} \cdot \bar{m} \, \mathrm{d}x, \tag{2.1}$$

subject to

$$\nabla \cdot (-\nabla \bar{\zeta} + \bar{m}) = 0 \quad \text{on} \quad \mathbb{R}^3, \tag{2.2}$$

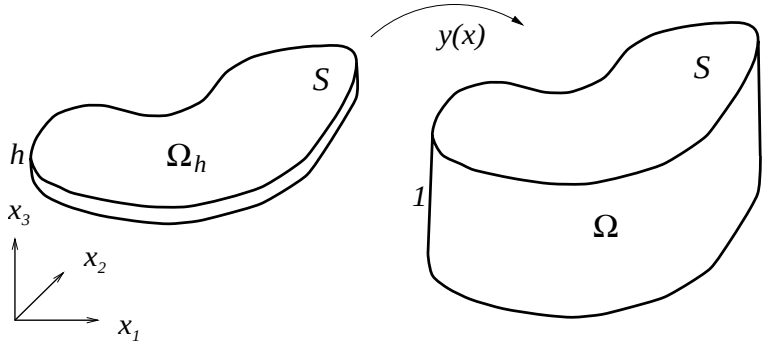

Figure 1. Thin film Ω_h, reference body Ω, and mapping $y(x)$.

and

$$|\bar{m}| = m_s, \tag{2.3}$$

where $\bar{m} : \Omega_h \mapsto \mathbb{R}^3$ is the magnetization, and $\bar{\zeta}$ is a scalar potential for the magnetic field $H \equiv -\nabla\bar{\zeta}$ (whereby Maxwell's equation $\nabla \times H = 0$ is automatically fulfilled). The different terms in (2.1) represent, from left to right, the exchange energy ($\alpha > 0$ is a material constant), the anisotropy energy, and the magnetostatic energy. The exchange energy penalizes spatial variations of the magnetization, thus embodying the tendency of the magnetization vectors associated with neighboring atoms of the underlying atomic structure to remain parallel to one another. The anisotropy energy models the existence of preferred directions of magnetization (the *easy axes*); φ is a continuous, non-negative, even function, exhibiting, whenever pertinent, crystallographic symmetry. Finally, the magnetostatic energy is the energy of the magnetic field which stems from the magnetization $\bar{m}$. The constraint (2.2) is simply the magnetostatic equation $\nabla \cdot B = 0$ (with the choice of units adopted here $H \equiv B - \bar{m}$). For the purpose of solving the magnetostatic equation (2.2) it is understood that $\bar{m}(x) = 0$ for x in $\mathbb{R}^3 - \Omega_h$. Finally, (2.3) embodies a fundamental constraint of micromagnetics, whereby a ferromagnetic body is always locally magnetized to a *saturation magnetization* $m_s(T)$, where T is the local temperature. Denoting the Curie temperature by T_C, $m_s > 0$ unless $T \geqslant T_C$; in the latter case $m_s = 0$ and the material ceases to behave ferromagnetically. As a consequence of (2.3), a specimen at $T < T_C$ can achieve a demagnetized state only in an average sense. Throughout this work we suppose constant temperature in the ferromagnetic regime.

3. Thin-film scaling

In this section, we endeavour to ascertain how the peculiar geometry of the thin film Ω_h affects the relative importance of the different terms in (2.1). To that end, we rescale the thin film Ω_h into a reference body Ω wherein all characteristic dimensions are of order 1, see figure 1. Without loss of generality we choose the reference domain to be a cylinder of unit height and cross-section S of area $A = 1$, $\Omega \equiv \{(x_1, x_2) \in S, 0 < x_3 < 1\}$. The method consists in expressing the integral in (2.1) on the reference domain, whereupon the dependence of the different energy terms on the thickness h becomes explicit, and their relative weight can be assessed in the thin-film limit of interest, $h \to 0$.

We adopt the following one-to-one mapping $y : \Omega_h \mapsto \Omega$ of the thin-film domain

onto the reference domain:

$$y_1 = x_1, \qquad y_2 = x_2, \qquad y_3 = \frac{1}{h}x_3. \tag{3.1}$$

Furthermore, to any magnetization $\bar{m} : \Omega_h \mapsto \mathbb{R}^3$ we associate a magnetization $m : \Omega \mapsto \mathbb{R}^3$ via

$$m(y(x)) = \bar{m}(x), \qquad x \in \Omega_h. \tag{3.2}$$

Equivalently,

$$m(y) = \bar{m}(x(y)), \qquad y \in \Omega. \tag{3.3}$$

This is a convenient scaling hypothesis because, along with the constraint (2.3), it automatically leads to $|m| = m_s$. It follows from (3.2) that

$$\nabla\bar{m}(x) = m_{,\alpha}(y(x)) \otimes e^\alpha + \frac{1}{h}\, m_{,3}(y(x)) \otimes e^3, \tag{3.4}$$

where $\alpha = 1, 2$, and e^1, e^2, and e^3 are unit vectors in the directions of the corresponding coordinates. Finally, corresponding to the magnetostatic potential $\bar{\zeta}$ we associate a potential ζ given by

$$\zeta(y(x)) = \bar{\zeta}(x), \qquad x \in \mathbb{R}^3. \tag{3.5}$$

It bears emphasis that even though in principle we could have adopted other one-to-one scalings in lieu of (3.3) and (3.5), our choices allow for a straightforward analysis.

We now express the free energy per unit volume of thin film associated with a magnetization $\bar{m}$ (with $\bar{m}$ defined on Ω_h), as a free energy per unit volume of thin film associated with a magnetization m (with m defined on Ω), with the following result,

$$\begin{aligned}\bar{\mathbb{E}}^{(h)}(\bar{m}) &= \mathbb{E}^{(h)}(m) \\ &= \int_\Omega \alpha\left(|\nabla_p m|^2 + \frac{1}{h^2}|m_{,3}|^2\right) + \varphi(m) + \frac{1}{2}\left(\nabla_p\zeta \cdot m_p + \frac{1}{h}\,\zeta_{,3}m_3\right) \mathrm{d}y,\end{aligned} \tag{3.6}$$

subjected to the constraints

$$\nabla_p \cdot (-\nabla_p\zeta + m_p) + \frac{1}{h}\left(-\frac{1}{h}\,\zeta_{,3} + m_3\right)_{,3} = 0 \qquad \text{on } \mathbb{R}^3 \tag{3.7}$$

(the magnetostatic equation), and

$$|m| = m_s, \tag{3.8}$$

where we have used the notation

$$m_p \equiv m_\alpha e^\alpha, \qquad \nabla_p\zeta \equiv \zeta_{,\alpha}e^\alpha, \qquad \nabla_p m \equiv m_{,\alpha} \otimes e^\alpha, \qquad \alpha = 1, 2. \tag{3.9}$$

Because the mapping $y \mapsto x$ is one to one, and $\bar{\mathbb{E}}^{(h)}(\bar{m}) = \mathbb{E}^{(h)}(m)$, it follows that minimizing $\bar{\mathbb{E}}^{(h)}(\bar{m})$ over magnetizations $\bar{m} : \Omega_h \mapsto \mathbb{R}^3$ is equivalent to minimizing $\mathbb{E}^{(h)}(m)$ over magnetizations $m : \Omega \mapsto \mathbb{R}^3$.

We now proceed to conveniently rephrase the magnetostatic equation (3.7). To that end, let $m|_\Omega \in L^2(\Omega)$, and consider the following variational principle,

$$\min_{\zeta \in V} \frac{1}{2}\int_{\mathbb{R}^3} |\nabla_p\zeta - m_p|^2 + \left|\frac{1}{h}\zeta_{,3} - m_3\right|^2 \mathrm{d}y, \tag{3.10}$$

where

$$V = \left\{ v : \mathbb{R}^3 \mapsto \mathbb{R},\ \nabla v \in L^2(\mathbb{R}^3),\ \int_B v \,\mathrm{d}y = 0 \right\}, \tag{3.11}$$

and B is a ball containing Ω. The condition $\int_B v \,\mathrm{d}y = 0$ prevents trivial translations $v \to v + c$. V can be turned into a Hilbert space by introducing the natural inner product,

$$(v, w)_V \equiv \int_{\mathbb{R}^3} \nabla_p v \cdot \nabla_p w + \frac{1}{h^2} v_{,3} w_{,3} \,\mathrm{d}y, \tag{3.12}$$

which readily leads to the definition of the norm

$$\|v\|_V \equiv (v, v)^{1/2}. \tag{3.13}$$

The direct method of the calculus of variations yields a unique minimizer of (3.10) in V (James & Kinderlehrer 1990), and this minimizer satisfies the Euler–Lagrange equations,

$$\int_{\mathbb{R}^3} (\nabla_p \zeta - m_p) \cdot \nabla_p \xi + \frac{1}{h}\left(\frac{1}{h}\,\zeta_{,3} - m_3\right)\xi_{,3} \,\mathrm{d}y = 0 \qquad \forall \xi \in V, \tag{3.14}$$

i.e. the weak form of (3.7). Setting $\xi = \zeta$ in (3.14), and taking into account that m vanishes outside Ω, we obtain

$$\int_\Omega \nabla_p \zeta \cdot m_p + \frac{1}{h}\zeta_{,3} m_3 \,\mathrm{d}y = \int_{\mathbb{R}^3} |\nabla_p \zeta|^2 + \frac{1}{h^2}|\zeta_{,3}|^2 \,\mathrm{d}y. \tag{3.15}$$

The left-hand side of this expression is twice the magnetostatic energy $\mathbb{E}^{(h)}_{\mathrm{mag}}$ (i.e. the last two terms of (3.6)), whereas the right-hand side is, by definition, the second power of the norm of the potential ζ in V. We can therefore rewrite (3.15) in the form $\mathbb{E}^{(h)}_{\mathrm{mag}} = \frac{1}{2}\,\|\zeta\|^2_V$.

4. Derivation of the limiting variational principle

Proposition 4.1. *Suppose $m^{(h)} \to \tilde{m}$ in $L^2(\mathbb{R}^3)$, $m^{(h)} = 0$ on $\mathbb{R}^3 - \Omega$, and let $\zeta^{(h)}$ be the magnetostatic potential corresponding to $m^{(h)}$ (i.e. the minimizer of (3.10) with $m = m^{(h)}$). Then,*

$$\nabla \zeta^{(h)} \to 0, \qquad \frac{1}{h}\zeta^{(h)}_{,3} \to \tilde{m}_3 \qquad \text{in } L^2(\mathbb{R}^3) \tag{4.1}$$

and

$$\mathbb{E}^{(h)}_{\mathrm{mag}}(m^{(h)}) \to \mathbb{E}^0_{\mathrm{mag}}(\tilde{m}) \equiv \frac{1}{2}\int_\Omega \tilde{m}_3^2 \,\mathrm{d}y. \tag{4.2}$$

Proof. For a given $m^{(h)}$, $\zeta^{(h)}$ minimizes the potential (3.10) among all possible competitors in V; in particular, that potential takes a larger value for $\zeta = 0$ than for $\zeta^{(h)}$, that is to say,

$$\int_{\mathbb{R}^3} |\nabla_p \zeta^{(h)} - m^{(h)}_p|^2 + \left|\frac{1}{h}\zeta^{(h)}_{,3} - m^{(h)}_3\right|^2 \mathrm{d}y \leqslant m_s^2 \Omega. \tag{4.3}$$

By applying the triangle inequality and recalling that $|m^{(h)}| = m_s$, it follows that

$$\|\nabla_p \zeta^{(h)}\|_{L^2(\mathbb{R}^3)} \leqslant C_1 \tag{4.4}$$

and

$$\left\| \frac{1}{h}\zeta^{(h)}_{,3} \right\|_{L^2(\mathbb{R}^3)} \leqslant C_2, \tag{4.5}$$

C_1 and C_2 being two positive constants. Finally, from (4.4) and (4.5) we conclude that

$$\|\nabla\zeta^{(h)}\|_{L^2(\mathbb{R}^3)} \leqslant C_3, \tag{4.6}$$

where $\nabla\zeta^{(h)} = \nabla_p\zeta^{(h)} + \zeta^{(h)}_{,3}e^3$, and C_3 is a positive constant.

The satisfaction of (4.6) and of the condition $\int_B \zeta^{(h)}\,\mathrm{d}y = 0$ guarantees the existence of a function $\zeta \in W^{1,2}(\mathbb{R}^3)$ such that on extraction of a suitable subsequence (not relabelled)

$$\nabla\zeta^{(h)} \rightharpoonup \nabla\zeta \qquad \text{in } L^2(\mathbb{R}^3). \tag{4.7}$$

Similarly, (4.5) implies the existence of a function g such that for a suitable subsequence (not relabelled)

$$\frac{1}{h}\,\zeta^{(h)}_{,3} \rightharpoonup g \qquad \text{in } L^2(\mathbb{R}^3). \tag{4.8}$$

From (4.8), $\zeta_{,3} = 0$ a.e. in $\mathbb{R}^3$, and it is possible to write $\zeta(y) = \tilde{\zeta}(y_1, y_2)$ for $y \in \mathbb{R}^3$. In view of this latter conclusion and Fubini's theorem, (4.6) implies that

$$C_3 \geqslant \int_{\mathbb{R}^3} |\nabla\zeta|^2\,\mathrm{d}y \geqslant \int_a^b \left(\int_{\mathbb{R}^2} |\nabla_p\tilde{\zeta}|^2\,\mathrm{d}y_1\mathrm{d}y_2 \right) \mathrm{d}y_3 = \|\nabla_p\tilde{\zeta}\|^2_{L^2(\mathbb{R}^2)}\,(b-a) \tag{4.9}$$

for all real numbers a and b such that $a < b$. Since $(b-a)$ can be any arbitrarily large positive number, it follows from (4.9) that $\|\nabla_p\tilde{\zeta}\|^2_{L^2(\mathbb{R}^2)} = 0$, or, equivalently, that $\nabla_p\tilde{\zeta} = 0$ a.e. in $\mathbb{R}^2$. Hence $\nabla\zeta = 0$ a.e. in $\mathbb{R}^3$.

From (4.7) and (4.8) we can write

$$\nabla_p\zeta^{(h)} = \nabla_p\zeta + a^{(h)}_p, \qquad \frac{1}{h}\zeta^{(h)}_{,3} = g + a^{(h)}_3, \tag{4.10}$$

where $a^{(h)}_p \rightharpoonup 0$ and $a^{(h)}_3 \rightharpoonup 0$ in $L^2(\mathbb{R}^3)$. To prove that $a^{(h)}_p$ and $a^{(h)}_3$ converge strongly to zero, we would like to construct a test function based on $\tilde{m}_3$; however, $\tilde{m}_3$ is not sufficiently smooth for differentiation. To bypass this problem, we let $\tilde{m}^\epsilon \in C^\infty_0(\mathbb{R}^3)$ be a family of smooth functions supported on Ω and converging strongly to $\tilde{m}$ in $L^2(\mathbb{R}^3)$. We now compare the value of the potential in (3.10) computed for $\zeta^{(h)}$ (the minimizer) with that computed for the test function

$$\zeta^{\epsilon,\lambda} = h\int_0^{y_3} \tilde{m}^\epsilon_3(y_1, y_2, s)\,\mathrm{d}s - \frac{h}{\lambda}\int_1^{y_3} \chi_{[1,1+\lambda]}(r)\,\mathrm{d}r \int_0^1 \tilde{m}^\epsilon_3(y_1, y_2, s)\,\mathrm{d}s + c^\epsilon, \tag{4.11}$$

where $\chi_{[1,1+\lambda]}$ is the characteristic function of $[1, 1+\lambda]$. The second term in (4.11) guarantees that $\nabla\zeta^{\epsilon,\lambda} \in L^2(\mathbb{R}^3)$, and the constant c^ϵ is chosen in such a way as to have $\zeta^{\epsilon,\lambda} \in V$. Upon expanding the square on the left-hand side of the inequality we get

$$\begin{aligned}
&\int_{\mathbb{R}^3} |a^{(h)}_p|^2 - 2\,a^{(h)}_p \cdot m^{(h)}_p + |m^{(h)}_p|^2 + |g - m^{(h)}_3|^2 + 2\,a^{(h)}_3 \cdot (g - m^{(h)}_3) + |a^{(h)}_3|^2\,\mathrm{d}y \\
&\quad\leqslant \int_{\mathbb{R}^3} \left| h\nabla_p \int_0^{y_3} \tilde{m}^\epsilon_3\,\mathrm{d}s - \frac{h}{\lambda}\int_1^{y_3} \chi_{[1,1+\lambda]}(r)\,\mathrm{d}r\nabla_p \int_0^1 \tilde{m}^\epsilon_3(y_1, y_2, s)\,\mathrm{d}s - m^{(h)}_p \right|^2 \\
&\qquad + \left| \tilde{m}^\epsilon_3 - \frac{1}{\lambda}\chi_{[1,1+\lambda]}(y_3) \int_0^1 \tilde{m}^\epsilon_3(y_1, y_2, s)\,\mathrm{d}s - m^{(h)}_3 \right|^2 \mathrm{d}y.
\end{aligned} \tag{4.12}$$

Fixing ϵ and λ in (4.12) and passing to the limit $h\to 0$, we note that the second and fifth terms in the left-hand side vanish (weak×strong), whereas the third term in the left-hand side and the first term in the right-hand side cancel each other. Thus,

$$\limsup_{h\to 0}\int_{\mathbb{R}^3}|a_p^{(h)}|^2+|g-m_3^{(h)}|^2+|a_3^{(h)}|^2\,\mathrm{d}y \leqslant \int_{\mathbb{R}^3}\left|\tilde{m}_3^{\epsilon}-\frac{1}{\lambda}\chi_{[1,1+\lambda]}(y_3)\int_0^1\tilde{m}_3^{\epsilon}(y_1,y_2,s)\,\mathrm{d}s-\tilde{m}_3\right|^2\mathrm{d}y. \tag{4.13}$$

By applying the triangle inequality to the right-hand side of (4.13), and taking into account that

$$\int_{-\infty}^{\infty}\chi^2_{[1,1+\lambda]}(y_3)\,\mathrm{d}y_3=\lambda,$$

we get

$$\limsup_{h\to 0}\int_{\mathbb{R}^3}|a_p^{(h)}|^2+|g-m_3^{(h)}|^2+|a_3^{(h)}|^2\,\mathrm{d}y \leqslant \int_{\mathbb{R}^3}|m_3^{\epsilon}-\tilde{m}_3|^2\,\mathrm{d}y+\frac{1}{\lambda}\int_{\mathbb{R}^2}\left[\int_0^1\tilde{m}_3^{\epsilon}(y_1,y_2,s)\,\mathrm{d}s\right]^2\mathrm{d}y_1\mathrm{d}y_2. \tag{4.14}$$

Finally, we pass to the limit $\epsilon\to 0$ and $\lambda\to\infty$ in (4.14); it follows that $a_p^{(h)}\to 0$ and $a_3^{(h)}\to 0$ in $L^2(\mathbb{R}^3)$, and $g=\tilde{m}_3$. Therefore, we have proved the convergence in (4.7) and (4.8) to be strong, implying (4.1) and (4.2). Even though in the treatment of both terms of the magnetostatic energy we have worked with subsequences, it is seen that in each case any convergent subsequence would have given the same limit. Therefore, proposition 4.1 holds for the whole sequence. ■

We now turn to the micromagnetic energy,

$$\mathbb{E}^{(h)}(m)=\int_{\Omega}\alpha\left(|\nabla_p m|^2+\frac{1}{h^2}|m_{,3}|^2\right)+\varphi(m)\,\mathrm{d}y+\mathbb{E}^{(h)}_{\mathrm{mag}}(m). \tag{4.15}$$

It is well known that a minimizer of $\mathbb{E}^{(h)}(m)$ exists in $H^1(\Omega)\cap\{|m|=m_s\}$ for each $h>0$.

Theorem 4.1. *Let $m^{(h)}$ be the minimizer of $\mathbb{E}^{(h)}(m)$ on $H^1(\Omega)\cap\{|m|=m_s\}$. Then, for a suitable subsequence (not relabelled) $m^{(h)}\to\tilde{m}$ in $H^1(\Omega)$,*

$$\tilde{m}_{,3}=0 \qquad \text{on } \Omega, \tag{4.16}$$

and $\tilde{m}$ is a minimizer of the limiting micromagnetic energy

$$\mathbb{E}^0(m)=\int_S\alpha|\nabla_p m|^2+\varphi(m)+\frac{1}{2}m_3^2\,\mathrm{d}y_1\mathrm{d}y_2 \tag{4.17}$$

on $H^1(S)\cap\{|m|=m_s\}$.

Proof. Since $m^{(h)}$ is the energy minimizer, we have that $\mathbb{E}^{(h)}(m^{(h)})\leqslant\mathbb{E}^{(h)}(m_s bbe_1)$ for a suitable constant $e_1\in\mathbb{R}^3$, whereupon

$$\alpha\|\nabla_p m^{(h)}\|^2_{L^2(\mathbb{R}^3)}\leqslant D_1, \qquad \frac{\alpha}{h^2}\|m^{(h)}_{,3}\|^2_{L^2(\mathbb{R}^3)}\leqslant D_2, \tag{4.18}$$

where D_1 and D_2 are two fixed constants. It follows from (4.18) that for a suitable

subsequence $m^{(h)}$,

$$\nabla m^{(h)} \rightharpoonup \nabla \tilde{m}, \qquad m^{(h)}_{,3} \to 0 \tag{4.19}$$

in $L^2(\Omega)$. Consequently, $\tilde{m}_{,3} = 0$, $m^{(h)} \to \tilde{m}$ in $L^2(\Omega)$, and $|\tilde{m}| = m_s$. To show that $\nabla_p m^{(h)}$ converges strongly to $\nabla_p \tilde{m}$ in $L^2(\Omega)$, we now compare $\mathbb{E}^{(h)}(m^{(h)})$ to $\mathbb{E}^{(h)}(\tilde{m})$:

$$\begin{aligned} &\int_\Omega \alpha \left(|\nabla_p m^{(h)}|^2 + \frac{1}{h^2} |m^{(h)}_{,3}|^2 \right) + \varphi(m^{(h)}) \, \mathrm{d}y + \mathbb{E}^{(h)}_{\mathrm{mag}}(m^{(h)}) \\ &\leqslant \int_\Omega \alpha |\nabla_p \tilde{m}|^2 + \varphi(\tilde{m}) \, \mathrm{d}y + \mathbb{E}^{(h)}_{\mathrm{mag}}(\tilde{m}). \end{aligned} \tag{4.20}$$

By proposition 4.1 and the strong convergence of $m^{(h)}$, both the anisotropy and magnetostatic energy converge, and (4.20) simplifies to

$$\int_\Omega |\nabla_p m^{(h)}|^2 + \frac{1}{h^2} |m^{(h)}_{,3}|^2 \, \mathrm{d}y \leqslant \int_\Omega |\nabla_p \tilde{m}|^2 \, \mathrm{d}y + O(h). \tag{4.21}$$

From (4.19a), we can write $\nabla_p m^{(h)} = \nabla_p \tilde{m} + b^{(h)}$ with $b^{(h)} \rightharpoonup 0$ in $L^2(\Omega)$. Substituting this expression into the left-hand side of (4.21), it follows that

$$\int_\Omega |\nabla_p \tilde{m}|^2 + 2 \nabla_p \tilde{m} \cdot b^{(h)} + |b^{(h)}|^2 + \frac{1}{h^2} |m^{(h)}_{,3}|^2 \, \mathrm{d}y \leqslant \int_\Omega |\nabla_p \tilde{m}|^2 \, \mathrm{d}y + O(h). \tag{4.22}$$

Since $b^{(h)} \rightharpoonup 0$, the second term in (4.22) tends to zero and, therefore,

$$\int_\Omega |b^{(h)}|^2 \, \mathrm{d}y \to 0, \qquad \frac{1}{h^2} \int_\Omega |m^{(h)}_{,3}|^2 \, \mathrm{d}y \to 0, \tag{4.23}$$

whereupon

$$\nabla_p m^{(h)} \to \nabla_p \tilde{m} \qquad \text{in } L^2(\Omega). \tag{4.24}$$

Hence, the limiting energy associated with $m^{(h)}$ is

$$\lim_{h \to 0} \mathbb{E}^{(h)}(m^{(h)}) = \mathbb{E}^0(\tilde{m}) = \int_S \alpha |\nabla_p \tilde{m}|^2 + \varphi(\tilde{m}) + \tfrac{1}{2} \tilde{m}_3^2 \, \mathrm{d}y_1 \mathrm{d}y_2, \tag{4.25}$$

where the domain of integration reduces to S because the integrand is independent of y_3. By comparing $\mathbb{E}^{(h)}(m^{(h)})$ with $\mathbb{E}^{(h)}(m)$ for any arbitrary $m \in H^1(\Omega)$ with $m_{,3} = 0$, we can easily verify that indeed $\tilde{m}$ renders $\mathbb{E}^0$ minimum in $H^1(S) \cap \{|m| = m_s\}$. ■

5. Discussion

We have shown that the magnetization $\tilde{m}$ associated with a film of vanishing thickness can be derived from the minimization of the limiting free energy (4.17). This functional is substantially simpler than the original expression (4.15). Thus, theorem 4.1 proves $\tilde{m}$ to be independent of the space coordinate normal to the film: if magnetic domains are present, the attendant walls are necessarily perpendicular to the film. Furthermore, by proposition 4.1 the magnetostatic potential converges to a constant at precisely the right rate so as to make the limiting magnetostatic energy entirely local. There is no remnant magnetostatic equation for the limiting problem.

Inspection of (4.17) readily reveals that the limiting energy may be thought of as corresponding to an effective anisotropy function $\varphi(m) + \frac{1}{2} m_3^2$, where the artificial

anisotropy $\frac{1}{2}m_3^2$ is actually the vestigial magnetostatic energy. As a result, out-of-plane magnetizations are disfavoured. It is also apparent from (4.17) that the reason for the presence of magnetic domains—the competition between the anisotropy and magnetostatic energies—is no longer present in the limiting problem: unless oriented in so special a way that the effective anisotropy function has multiple minima, the film is uniformly magnetized, just as in the classical solutions of Stoner & Wohlfarth. Indeed, the term $\frac{1}{2}m_3^2$ in the limiting energy (4.17) may be recognized from tables (Osborn 1945) as the demagnetization energy density for a uniformly magnetized flat ellipsoid. We conclude that our calculation extends the applicability of Stoner & Wohlfarth's results for flat ellipsoids to thin films of arbitrary shape in plan.

Our calculation can be easily extended to the case of an applied field h by substituting $\varphi - h \cdot m$ for φ in the equations (with $h \in L^2(\Omega)$, for example). If the applied field is constant, then the energy minimizer is a single domain. If the material is also isotropic, the magnetization m is contained in both (i) the plane of the film and (ii) the plane defined by the applied field h and the normal to the film; furthermore, m is such that $h \cdot m > 0$. Whenever $h = 0$, m is contained in the plane of the film, but its orientation is undetermined. Therefore, an alternating h perpendicular to the film does not lead to magnetic hysteresis, whereas an alternating h in the plane of the film causes rectangular hysteresis loops. These predictions coincide with those of Stoner & Wohlfarth for flat ellipsoids, and are supported by many experimental studies, including some of the earliest published work on magnetic thin films (see, for example, Blois (1955), where the hysteresis loops obtained for different orientations are illustrated, and Conger (1955), where the absence of domain walls is compellingly demonstrated).

Stable multi-domain patterns are possible when h is non-constant and $\varphi + \frac{1}{2}m_3^2 - h \cdot m$ has multiple minima. Our theory can serve as a starting point towards the prediction of these patterns. The metastability of domain patterns and the resulting hysteresis can be studied by analysing the relative minimizers of our limiting energy. This leads to a simple problem which is analogous to elastic buckling.

The bounding argument upon which the derivation of the limiting energy is predicated in theorem 4.1 hinges on the assumption that α in (4.15) be a material parameter unrelated to the thickness h. Although a fundamental explanation of the exchange energy must rely on a quantum mechanical approach, sensible classical analogies have been proposed which suggest that, for crystalline solids, the only length scale involved in α is the lattice parameter (see Carey & Isaac 1966). In the case of amorphous materials, it also appears reasonable to assume that α involves a length scale strictly associated with the nanostructure of the material. It follows that α is independent of the film thickness, and the assumption introduced in theorem 4.1 is compatible with the underlying physics.

Theorem 4.1 acquires practical value from its application to thin films of finite thicknesses, where the limit $h \to 0$ is not completely effected. This practical use of theorem 4.1 necessitates a more detailed consideration of the step going from (4.18b) to (4.19b). It is apparent that $m_{,3}^{(h)}$ can be regarded close to zero (in other words, the partial fulfillment of the limit (4.19b) can be judged sufficient for $\tilde{m}$ to stand as a good practical approximation to $m^{(h)}$) if and only if h is small enough for the following inequality to hold,

$$h \ll h_c \equiv \sqrt{\alpha}. \tag{5.1}$$

By normalizing the energy (4.15) so that the anisotropy function φ is nondimensional,

Table 1. *Theoretical and experimental estimates of critical thicknesses (Bloch wall width)*
(Theoretical, after Lilley (1950); experimental, after Wade (1962).)

material	theor. h_c (Å)	exper. h_c (Å)
Fe	730–1410	1000±400
Ni	1160–2060	550±150
Co	160	—
50–50 Permalloy	—	1800±500

α is seen to be the ratio of an exchange-related numerator to an anisotropy-related denominator (which are customarily denoted A and K, respectively). Thus, h_c is relatively small for a material exhibiting strong anisotropy and weak or moderate exchange effects. Interestingly, such a trade off between anisotropy and exchange energy takes place within domain walls in classical domain theory, and the width w_c of Bloch walls scales with the square root of α, just as h_c does. Values of w_c are readily available both in the form of theoretical and experimental estimates. To get an impression of the order of magnitude of h_c (which we take to equal w_c), we have compiled table 1.

Continuous thin films with thicknesses of about 250 Å are customarily being used in a series of applications, and the current trend is towards ever thinner films which, for all practical purposes, fulfil condition (5.1). In the area of storage devices increasing recording densities demand the reduction of all the key size parameters—including not only film thickness, but film roughness and trackwidth as well (Yeak-Scranton 1993). For instance, magnetic tapes with continuous thin film media (as opposed to the traditional particulate media) and thicknesses of less than 100 Å have been intensively studied in recent years for very high density audio and video recording (Hirota 1991). Thin film discs in the same thickness range have been considered for computer data storage. Our results could prove useful in the development of these new technologies.

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