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Pressurized Shape Memory Thin Films

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Dedicated to Roger Fosdick on the occasion of his 60th birthday.

Abstract. We study the behavior of a martensitic thin film with a hydrostatic pressure applied underneath the film. The problem is formulated in 3-D for a single crystal film of thickness h, and a Cosserat membrane theory is derived by Γ-convergence techniques in the limit $h \to 0$. The membrane theory is further simplified using a second Γ-convergence argument based on hard moduli. The resulting theory supports energy minimizing "tunnels": structures having the shape of part of a cylinder cut by a plane parallel to its axis, obtained by releasing the film from the substrate along a strip with a certain orientation. As the temperature is raised (at fixed pressure) the energy minimizing shape collapses gradually to the substrate, accompanied by a martensite-to-austenite phase transformation. During this process the tunnel supports a microstructure consisting of fine bands of austenite parallel to the axis of the tunnel, alternating with bands of a single variant of martensite. Formulas for the associated volume–temperature–pressure relation are given: in these the latent heat of transformation plays an important role.

1. Introduction

In recent years it has become possible to derive rigorously from 3-D nonlinear elasticity special theories for thin structures, without adopting an *ansatz* for the deformation (As an incomplete selection of these derivations, we list Acerbi et al. [1], Bhattacharya and James [11] and Le Dret and Raoult [20, 21]). If the starting theory is general 3-D finite elasticity, then it can be stated that these derivations give the definitive plate–shell–thin film theories. They therefore settle the long-standing question of which, among the many such theories available in the literature, is the appropriate theory for a thin body. There remains a lot to be done: currently, except for one formal argument, the only results are for the membrane theory. Bending theories evidently are obtained from a higher-order Γ-convergence argument.

These developments also allow one to approach with a certain measure of confidence the analysis of thin bodies of nonclassical materials, such as thin films of martensitic materials (cf. [11, 25]), which undergo a diffusionless phase transformation and have free energies with energy wells. Martensitic films can change shape when their temperature is changed, and some materials that undergo a reversible martensitic transformation display the shape memory effect. This effect is: the specimen is deformed by loads at low temperature (causing rearrangements of the different variants of martensite), the loads are removed but the specimen remains deformed (because all the variants have the same free energy density), the specimen is heated, causing a phase transformation from martensite to austenite, and it returns to its original shape. It is then cooled through the transformation with no macroscopic change of shape (because the martensite variants are able to form microstructures with this property), and the cycle can begin again. By alternately heating and loading a specimen, it can be used as an actuator. As discussed by Krulevitch [19], the NiTi shape memory material is among the actuator materials with the largest value of the work output per cycle per volume. Here, “volume” refers to the volume of the actuator. The “per volume” part of this formula is one reason that this material might perform well at small scales. Another reason is that the slow response of bulk shape memory actuators, caused by the necessity to cool them, is greatly improved by the rapid rate of heat transfer possible at small scales, especially in a thin film heated and cooled on its faces.

There are two simple overall design principles for these actuators that are sharply delineated by the Γ-convergence arguments. First, the membrane theory emerges at order h (unlike bending, which emerges at h^3). Therefore, to take advantage of the large work output *per volume* of the shape–memory materials, one should design the actuator to work in membrane mode. Second, the film should be released from the substrate, assuming it is an ordinary elastic material, so that the highly constraining effect of the substrate is eliminated.

There is another potentially attractive feature of the use of martensitic films for microactuators in MEMS (= micro-electro-mechanical systems). The techniques of microelectronics (e.g., molecular beam epitaxy) open up the possibility of making single crystal films. These could avoid the “fighting between the grains” that is associated with transformation in bulk polycrystals, which reduces the effective transformation strain much below its maximum value in single crystals. Oriented single crystal films could be released from the substrate on certain well-defined regions and undergo a large deformation, large work output, relatively high frequency shape memory effect. These ideas have motivated theoretical (Bhattacharya and James [11]) and experimental (Dong et al. [15]) studies. From the former have emerged some structures – “tents” and “tunnels” – that are energy minimizing under zero stress and exhibit large deformations.

As explained above, to produce the shape memory effect, loads have to be applied. Ideally, these are the actual loads that the actuator must do work against, but, in any case, the so-called bias stresses must be applied to produce the change of shape in the martensitic state in the first place. The idea explored in this paper is that the loads are produced by a pressure on the film. This could be done in the following way. The film could be released from the substrate on a well-defined region by back-etching the substrate (the judicious use of an etch-stop might be relevant here). Then a pressure could be applied from either above (advantageous, to prevent peeling of the film) or below. This scheme is suited to potential applications like pumps and valves in which the pressure is produced by the working fluid. The key questions are: what structure? what volume–temperature–pressure relation is predicted?

The membrane theory of a stiff, single phase material is the theory of isometric mappings of a plane. The deformed shapes are developable surfaces. A spherical bubble is not one, but a piece of a cylinder is. Moreover, according to the isoperimetric inequality, a circular cylinder holds the greatest volume for its area, among prismatic shapes. Thus, it is natural to consider a film released on a strip, so that under pressure, it bulges into a cylindrical shape. By raising the temperature, the film is made to undergo the shape memory effect and the film collapses to a flat shape. This is the situation studied in this paper. We speculate that a complex system of such tunnels could be patterned onto a chip and, by selective heating, bubbles of different fluids could be pumped around, mixed, reacted, etc.

The plan of this paper is as follows. We formulate the 3-D problem of a thin film of martensitic material of certain shape, acted upon by a pressure underneath, in Section 2. We include bulk and interfacial energies. The presence of the pressure necessitates that we impose slightly stronger growth conditions than is usual to prevent the film from blowing up. In Section 3 we rescale the pressure $p^h = Ph$, $P =$ const, and do the Γ-convergence argument, which is a modification of that of Bhattacharya and James [11]. The energy that emerges is a nonlinear membrane theory with an additive contribution pV where V is the volume enclosed between the membrane and a suitable plane (Section 3). To further simplify this energy, we do a second Γ-convergence argument in Section 4 based on the presence of hard moduli, a situation that is expected to be relevant for martensitic materials with large transformation strain. For the resulting *constrained theory* microstructures are replaced by Young measures having support on the energy wells, and the macroscopic deformation gradient is recovered as the center of mass of the Young measure. This theory is very easy to use and it supports tunnels with circular cross-section. They are proved to be energy minimizing among cylindrical deformations (Section 5) for suitable materials, and probably they are energy minimizing in general under their own boundary conditions. The dependence of the volume enclosed by the tunnel on pressure and temperature is found in Section 5 (Figures 5 and 6). An unexpected behavior is observed: instead of the tunnel collapsing suddenly to the flat shape as the temperature is raised, the collapse is more gradual (Figure 7), and complete collapse within the constrained theory is only possible with infinitely large temperature. During collapse, the tunnel exhibits a microstructure of axial bands of austenite and one variant of martensite (Figure 8). These results need to be explored for special materials, which we postpone to later work.

This research has been conducted in parallel with a related computational study [9].

2. The 3-dimensional problem

We assume that an origin $\mathbf{O}$ and an orthonormal basis, $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, have been chosen in the 3-dimensional Euclidean space. We identify the point $\mathbf{X}$ with the vector $\mathbf{X}-\mathbf{O}$, whose components will be indicated by $X_1,\ X_2,\ X_3$. In the reference configuration, the film is assumed to occupy the cylindrical region

$$\Omega^h := \big\{(X_1, X_2, X_3) \in \mathbb{R}^3\colon (X_1, X_2) \in S,\ \ X_3 \in (0, h)\big\}, \tag{2.1}$$

where S is an open bounded subset of $\mathbb{R}^2$ with Lipschitz boundary, and h is the thickness of the film. A deformation of the film is a mapping $\mathbf{y}\colon \Omega^h \mapsto \mathbb{R}^3$. Recalling that the film is attached to a substrate except for the released region S, we model the presence of the substrate by imposing the linear boundary condition

$$\mathbf{y} = \mathbf{y}_A(\mathbf{X}) := \mathbf{A}\mathbf{X}, \quad \mathbf{X} \in \partial S \times (0, h), \tag{2.2}$$

where $\mathbf{A}$ is a constant 3×3 matrix. The material of the film is supposed to be homogeneous and thermoelastic. At the temperature $\theta \in (0, +\infty)$, the strain energy of the film corresponding to the deformation $\mathbf{y}$ is given by

$$\int_{\Omega^h} \phi(\nabla\mathbf{y}; \theta)\, \mathrm{d}\mathbf{X}, \tag{2.3}$$

where $\phi\colon M^{3\times3} \times (0, +\infty) \mapsto [0, +\infty)$ is the strain energy density, and $M^{3\times3}$ is the set of all 3×3 matrices. The function ϕ is assumed to be smooth in both its arguments and to satisfy the following growth hypothesis: there exist positive constants c_1, c_2, c_3, c_4, and $3 < q < 6$ such that

$$c_1\, |\mathbf{F}|^3 - c_2 \leqslant \phi(\mathbf{F}; \theta) \leqslant c_3\, |\mathbf{F}|^q - c_4, \tag{2.4}$$

for all matrices $\mathbf{F}$ in $M^{3\times3}$, and for θ in $(0, +\infty)$. Furthermore, ϕ is assumed to be frame indifferent, i. e., it is assumed to satisfy the condition

$$\phi(\mathbf{QF}; \theta) = \phi(\mathbf{F}; \theta), \tag{2.5}$$

for all $\mathbf{F} \in M^{3\times3}$ and $\theta \in [0, +\infty)$, and for all proper rotations

$$\mathbf{Q} \in SO(3) := \big\{\mathbf{Q} \in M^{3\times3} | \mathbf{Q}\mathbf{Q}^T = \mathbf{I}, \det\mathbf{Q} = 1\big\}. \tag{2.6}$$

Stable deformations of the film, subject to a hydrostatic pressure p^h acting on its lower surface, are assumed to correspond to the deformations $\mathbf{y} \in W^{2,2}(\Omega^h; \mathbb{R}^3)$ which minimize the total energy

$$\begin{aligned} E^h(\mathbf{y}; \theta) \ := \ & \int_{\Omega^h} \big(\phi(\nabla\mathbf{y}; \theta) + \kappa |\nabla^2\mathbf{y}|^2\big)\, \mathrm{d}\mathbf{X} \\ & - \frac{p^h}{3} \int_{S\times\{0\}} \mathbf{y} \cdot (\mathbf{y}_{,1} \wedge \mathbf{y}_{,2})\, \mathrm{d}X_1\, \mathrm{d}X_2, \end{aligned} \tag{2.7}$$

in the class of deformations

$$\mathcal{H} := \{\mathbf{y} \in W^{2,2}(\Omega^h; \mathbb{R}^3) | \mathbf{y} = \mathbf{A}\mathbf{X},\ \mathbf{X} \in \partial S \times (0, h)\}. \tag{2.8}$$

In the expression (2.7) of the total energy, the notation $\mathbf{y}_{,i}$ denotes the vector $\nabla \mathbf{y}\mathbf{e}_i$, for $i = 1, 2$. The term $|\nabla^2 \mathbf{y}|^2$, which denotes the full $3 \times 3 \times 3$ matrix of second derivatives, not just the Laplacean, penalizes the formation of interfaces, κ being a small positive constant; the norm here is $\sqrt{\mathbf{y}_{,ij} \cdot \mathbf{y}_{,ij}}$. As discussed below, the term

$$V(\mathbf{y}) := \frac{1}{3} \int_{S \times \{0\}} \mathbf{y} \cdot (\mathbf{y}_{,1} \wedge \mathbf{y}_{,2})\, \mathrm{d}X_1\, \mathrm{d}X_2 \tag{2.9}$$

represents the volume enclosed between the plane $\mathbf{y}_A(S \times \{0\})$ and the deformed lower surface of the film. Physically, the last term of (2.7) may be interpreted as the energy of a gas under the film whose pressure can be assumed constant. This situation is realized in practice by micromachining a hole in the substrate and by pressurizing the film using a reservoir of gas whose volume is much larger than typical changes of volume due to film deformation. We note that if the volume of this reservoir is on the order of volume changes produced by the deformations of the film, then the last term of (2.7) would have to be replaced by the general expression for the free energy of such a gas, accounting for its compressibility. In the latter case our predictions concerning stability could be substantially changed [18].

The volume functional (2.9) evaluated at a deformation $\mathbf{y}$ of the film gives a reasonable expression for the volume of the region $\mathcal{V}_{\mathbf{y}}$ enclosed between the plane $\mathbf{y}_A(S \times \{0\})$ and the deformed lower surface of the film $\mathbf{y}(S \times \{0\})$. This can be seen through the following heuristic calculation. Let $|\mathcal{V}_{\mathbf{y}}|$ denote the three-dimensional Lebesgue measure of $\mathcal{V}_{\mathbf{y}}$, and let $\mathbf{Z} = (Z_1, Z_2, Z_3)$ be a point belonging to $\mathcal{V}_{\mathbf{y}}$. We have

$$|\mathcal{V}_{\mathbf{y}}| = \int_{\mathcal{V}_{\mathbf{y}}} \mathrm{d}\mathbf{Z} = \frac{1}{3} \int_{\mathcal{V}_{\mathbf{y}}} \mathrm{div}\mathbf{Z}\, \mathrm{d}\mathbf{Z}. \tag{2.10}$$

Using the divergence theorem, we get

$$|\mathcal{V}_{\mathbf{y}}| = \frac{1}{3} \int_{\mathbf{y}(S \times \{0\})} \mathbf{Z} \cdot \mathbf{n}(\mathbf{Z})\, \mathrm{d}a + \frac{1}{3} \int_{\mathbf{y}_A(S \times \{0\})} \mathbf{Z} \cdot \mathbf{n}(\mathbf{Z})\, \mathrm{d}a, \tag{2.11}$$

where, in each integral, $\mathbf{n}(\mathbf{Z})$ is the unit outward normal at the point $\mathbf{Z}$ to the surface on which the integral is defined. Since $\mathbf{Z} \cdot \mathbf{n}(\mathbf{Z}) = 0$ at each point $\mathbf{Z}$ of $\mathbf{y}_A(S \times \{0\})$, the second integral is zero. Taking $\mathbf{y}^{\mathrm{o}}$, the trace of $\mathbf{y}$ on the surface $S \times \{0\}$, as a parametric representation of the surface $\mathbf{y}(S \times \{0\})$, equation (2.11) may be rewritten as

$$\begin{aligned} |\mathcal{V}_{\mathbf{y}}| &= \frac{1}{3} \int_S \mathbf{y}^{\mathrm{o}} \cdot (\mathbf{y}^{\mathrm{o}}{}_{,1} \wedge \mathbf{y}^{\mathrm{o}}{}_{,2})\, \mathrm{d}X_1\, \mathrm{d}X_2 \\ &= \frac{1}{3} \int_{S \times \{0\}} \mathbf{y} \cdot (\mathbf{y}_{,1} \wedge \mathbf{y}_{,2})\, \mathrm{d}X_1\, \mathrm{d}X_2 = V(\mathbf{y}). \end{aligned} \tag{2.12}$$

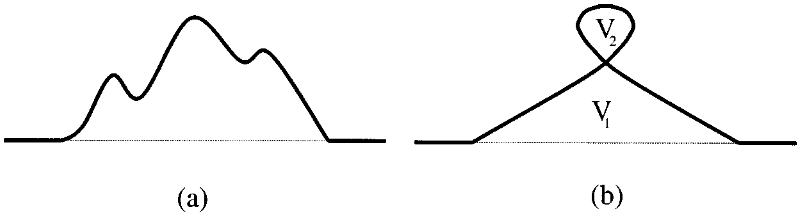

Figure 1. Globally invertible (a) and not globally invertible (b) deformation of the film.

We note that the expression (2.9) for the volume under the film has some pathologies related to global invertibility, illustrated in Figure 1 for the case $\mathbf{A} = \mathbf{I}$. In Figure 1(a), the volume under the film is expected to be reasonably modeled by (2.9). However, in Figure 1(b) the formula (2.9) gives the volume $V_1 - V_2$, which could not possibly be occupied by a fluid. Without Dirichlet boundary conditions on $\mathbf{y}$, we do not know how to impose global invertibility assumptions on the minimization of E^h, so we are not able to rule out situations as shown in Figure 1(b). However, we do not expect such configurations to arise from energy minimization.

The volume functional (2.9) is defined for the deformations $\mathbf{y}$ of $\mathcal{H}$. This can be seen as follows. By Sobolev's embedding theorem [2, Theorem 5.4], each element of $W^{2,2}(\Omega^h;\mathbb{R}^3)$ has trace on $S \times \{0\}$ belonging to $C^0(\overline{S};\mathbb{R}^3) \cap W^{1,2}(S;\mathbb{R}^3)$, with $\overline{S}$ denoting the closure of S. By the continuity of the trace, there exists a positive constant C such that

$$|V(\mathbf{y})| \leqslant C \int_S \left|\mathbf{y}^o{}_{,1} \wedge \mathbf{y}^o{}_{,2}\right| \mathrm{d}X_1\,\mathrm{d}X_2. \tag{2.13}$$

The functional

$$L(\mathbf{y}^o) := \int_S \left|\mathbf{y}^o{}_{,1} \wedge \mathbf{y}^o{}_{,2}\right| \mathrm{d}X_1\,\mathrm{d}X_2 \tag{2.14}$$

gives the Lebesgue area of the surface parametrized by $\mathbf{y}^o$ [12]. It is known that

$$\int_S \left|\mathbf{y}^o{}_{,1} \wedge \mathbf{y}^o{}_{,2}\right| \mathrm{d}X_1\,\mathrm{d}X_2 \leqslant \frac{1}{2}\int_S \left(|\mathbf{y}^o{}_{,1}|^2 + |\mathbf{y}^o{}_{,2}|^2\right)\mathrm{d}X_1\,\mathrm{d}X_2, \tag{2.15}$$

and the inequality holds only when $\mathbf{y}^o$ satisfies the conditions $|\mathbf{y}^o{}_{,1}| = |\mathbf{y}^o{}_{,2}|$, $(\mathbf{y}^o{}_{,1} \cdot \mathbf{y}^o{}_{,2}) = 0$ [16]. From (2.13), (2.15), and the regularity of the trace, it follows that $V(\mathbf{y})$ is defined in $\mathcal{H}$. Moreover, the volume functional (2.9) is weakly continuous, up to a subsequence, in $W^{2,2}(\Omega^h;\mathbb{R}^3)$. Indeed, consider a sequence $\mathbf{y}_n$ weakly converging to $\mathbf{y}$ in $W^{2,2}(\Omega^h;\mathbb{R}^3)$, by Sobolev's Theorem there exists a subsequence, not relabeled, such that

$$\mathbf{y}^o{}_n \rightharpoonup \mathbf{y}^o \quad \text{in } W^{1,q}\big(S;\mathbb{R}^3\big), \tag{2.16}$$

for $3 < q \leqslant 4$. This together with the weak continuity of minors [3], implies

$$(\mathbf{y}^o{}_{n,1} \wedge \mathbf{y}^o{}_{n,2}) \cdot \mathbf{e}_j \rightharpoonup (\mathbf{y}^o{}_{,1} \wedge \mathbf{y}^o{}_{,2}) \cdot \mathbf{e}_j \quad \text{in } L^{q/2}(S;\mathbb{R}), \tag{2.17}$$

for $j = 1, 2, 3$. Furthermore, by the Rellich–Kondrachov Theorem [2, Theorem 6.2], there exists a further subsequence, not relabeled, such that

$$\mathbf{y}^{\mathrm{o}}{}_n \to \mathbf{y}^{\mathrm{o}} \quad \text{in } C^0\big(\overline{S}; \mathbb{R}^3\big), \tag{2.18}$$

which, together with (2.17), implies $V(\mathbf{y}_n) \to V(\mathbf{y})$.

The fundamental estimate for dealing with the volume functional is the following isoperimetric inequality, whose proof can be found in [24, 26]. With an abuse of notation, with the same symbol V we shall denote the volume functional defined for deformations of $\mathcal{H}$, as in (2.9), and the volume functional defined for parametric surfaces in $C^0(\overline{S}; \mathbb{R}^3) \cap W^{1,2}(S; \mathbb{R}^3)$.

PROPOSITION 2.1. *Let* $\mathbf{f}$, $\mathbf{g} \in W^{1,2}(S; \mathbb{R}^3) \cap C^0(\overline{S}; \mathbb{R}^3)$ *be two parametrizations of two 3-dimensional surfaces such that* $\mathbf{f} = \mathbf{g}$ *on* ∂S. *Then,*

$$\big|V(\mathbf{f}) - V(\mathbf{g})\big|^2 \leqslant \frac{1}{36\pi}\big|L(\mathbf{f}) + L(\mathbf{g})\big|^3. \tag{2.19}$$

For the proof of the existence of minimizers of the total energy E^h, we shall need the following lemmas, which are applications of the isoperimetric inequality.

LEMMA 2.2. *Let* $t > 0$ *and let* $\mathbf{f} \in W^{1,3}(S; \mathbb{R}^3) \cap C^0(\overline{S}; \mathbb{R}^3)$ *be such that*

$$\mathbf{f} = \mathbf{A}\mathbf{e}_1 X_1 + \mathbf{A}\mathbf{e}_2 X_2 + \mathbf{A}\mathbf{e}_3 t, \quad (X_1, X_2) \in \partial S, \tag{2.20}$$

with $\mathbf{A}$ *a given* 3×3 *matrix. Then, denoting by* $|S|$ *the area of* S, *we have*

$$|V(\mathbf{f})| \leqslant \frac{|S|^{1/2}}{12\sqrt{\pi}}\|\nabla\mathbf{f}\|^3_{L^3(S)} + c, \tag{2.21}$$

with

$$c = \frac{\sqrt{2}}{6\sqrt{\pi}}|S|^{3/2}|\mathbf{A}\mathbf{e}_1 \wedge \mathbf{A}\mathbf{e}_2|^{3/2} + \frac{t}{3}|S||\det\mathbf{A}|.$$

Proof. By the triangle inequality, the isoperimetric inequality (2.19), Jensen's inequality [14, Theorem 2.2] and the inequality (2.15), we obtain

$$\begin{aligned} \big|V(\mathbf{f})\big| &\leqslant \big|V(\mathbf{f}) - V(\mathbf{g})\big| + \big|V(\mathbf{g})\big| \leqslant \frac{1}{6\sqrt{\pi}}\big|L(\mathbf{f}) + L(\mathbf{g})\big|^{3/2} + \big|V(\mathbf{g})\big| \\ &\leqslant \frac{\sqrt{2}}{6\sqrt{\pi}}\big(L(\mathbf{f})^{3/2} + L(\mathbf{g})^{3/2}\big) + \big|V(\mathbf{g})\big| \\ &\leqslant \frac{1}{12\sqrt{\pi}}\|\nabla\mathbf{f}\|^3_{L^2(S)} + \frac{\sqrt{2}}{6\sqrt{\pi}}L(\mathbf{g})^{3/2} + \big|V(\mathbf{g})\big|. \end{aligned} \tag{2.22}$$

Hence, in view of the imbedding of L^3 into L^2 [2, Theorem 2.8], we have

$$|V(\mathbf{f})| \leqslant \frac{|S|^{1/2}}{12\sqrt{\pi}}\|\nabla\mathbf{f}\|^3_{L^3(S)} + \frac{\sqrt{2}}{6\sqrt{\pi}}L(\mathbf{g})^{3/2} + \big|V(\mathbf{g})\big|. \tag{2.23}$$

We now choose

$$\mathbf{g} = \mathbf{A}\mathbf{e}_1 X_1 + \mathbf{A}\mathbf{e}_2 X_2 + \mathbf{A}\mathbf{e}_3 t. \tag{2.24}$$

Simple calculations give (2.21). □

We now extend the bound given in Lemma 2.2 to the interior of the film.

LEMMA 2.3. *For any* $\mathbf{y} \in \mathcal{H}$ *we have*

$$\begin{aligned}\left|V(\mathbf{y})\right| \leqslant\ & \left(\frac{|S|^{1/2}}{12\sqrt{\pi}h} + \frac{1}{3\sqrt{3}}\right)\|\nabla\mathbf{y}\|^3_{L^3(\Omega^h)} + \frac{\sqrt{2}}{6\sqrt{\pi}}|S|^{3/2}|\mathbf{A}\mathbf{e}_1 \wedge \mathbf{A}\mathbf{e}_2|^{3/2} \\ & + \frac{h}{2}|S||\det\mathbf{A}|.\end{aligned} \tag{2.25}$$

Proof. Fix t in $(0, h)$ and let $\mathbf{y}^t$ be the trace of $\mathbf{y}$ on the surface $S \times \{t\}$. Then $\mathbf{y}^t$ satisfies the hypothesis of Lemma 2.2 and thus it satisfies the inequality (2.21). In Appendix A, we prove that

$$V(\mathbf{y}) = V\left(\mathbf{y}^t\right) + \frac{2}{3}t|S|\det\mathbf{A} - \int_{S\times(0,t)} \det\nabla\mathbf{y}\,\mathrm{d}\mathbf{X}, \tag{2.26}$$

which, in view of (2.21), gives

$$\begin{aligned}\left|V(\mathbf{y})\right| \leqslant\ & \frac{|S|^{1/2}}{12\sqrt{\pi}}\|\nabla\mathbf{y}^t\|^3_{L^3(S)} + \frac{\sqrt{2}}{6\sqrt{\pi}}|S|^{3/2}|\mathbf{A}\mathbf{e}_1 \wedge \mathbf{A}\mathbf{e}_2|^{3/2} \\ & + t|S||\det\mathbf{A}| + \int_{S\times(0,t)} |\det\nabla\mathbf{y}|\,\mathrm{d}\mathbf{X}.\end{aligned} \tag{2.27}$$

Adding $|\mathbf{y}_{,3}(X_1, X_2, t)|^2$ to the argument of the square root in the first term of the right-hand side, and using the positivity of the integrand of the last term, we get

$$\begin{aligned}\left|V(\mathbf{y})\right| \leqslant\ & \frac{|S|^{1/2}}{12\sqrt{\pi}}\int_{S\times\{t\}}\left(\sum_{i=1}^{3}\left|\mathbf{y}_{,i}(X_1, X_2, t)\right|^2\right)^{3/2}\mathrm{d}X_1\,\mathrm{d}X_2 \\ & + \frac{\sqrt{2}}{6\sqrt{\pi}}|S|^{3/2}|\mathbf{A}\mathbf{e}_1 \wedge \mathbf{A}\mathbf{e}_2|^{3/2} + t|S||\det\mathbf{A}| \\ & + \int_{\Omega^h} |\det\nabla\mathbf{y}|\,\mathrm{d}\mathbf{X}.\end{aligned} \tag{2.28}$$

Integrating over $t \in (0, h)$ and applying Fubini's theorem, we find

$$\begin{aligned}h\left|V(\mathbf{y})\right| \leqslant\ & \frac{|S|^{1/2}}{12\sqrt{\pi}}\|\nabla\mathbf{y}\|^3_{L^3(\Omega^h)} + \frac{\sqrt{2}}{6\sqrt{\pi}}h|S|^{3/2}|\mathbf{A}\mathbf{e}_1 \wedge \mathbf{A}\mathbf{e}_2|^{3/2} \\ & + \frac{h^2}{2}|S||\det\mathbf{A}| + h\int_{\Omega^h} |\det\nabla\mathbf{y}|\,\mathrm{d}\mathbf{X}.\end{aligned} \tag{2.29}$$

We now divide by h and use the inequality

$$|\det \nabla \mathbf{y}| \leqslant \frac{1}{3\sqrt{3}} |\nabla \mathbf{y}|^3, \tag{2.30}$$

[23, Section 2.3], to obtain the result. □

The following proposition establishes, for suitable values of the pressure and for a fixed value of the thickness h, the existence of minimizers of the total energy functional in the class of deformations $\mathcal{H}$. In the argument, the temperature θ shall be held fixed, so we suppress it from the notation.

PROPOSITION 2.4. *Assume that* $\inf_{\mathbf{y}\in\mathcal{H}} E^h(\mathbf{y}) < \infty$. *If*

$$p^h \leqslant c_1 \Big/ \frac{|S|^{1/2}}{12\sqrt{\pi} h} + \frac{1}{3\sqrt{3}} =: p^h_{cr}, \tag{2.31}$$

there exists at least a minimizer of E^h *in* $\mathcal{H}$.

Proof. From the growth assumptions (2.4) and from the bound on the volume (2.25), we obtain

$$\begin{aligned} E^h(\mathbf{y}) \ \geqslant\ & \kappa \|\nabla^2 \mathbf{y}\|^2_{L^2(\Omega^h)} + \left(c_1 - p^h \left(\frac{|S|^{1/2}}{12\sqrt{\pi} h} + \frac{1}{3\sqrt{3}} \right)\right) \|\nabla \mathbf{y}\|^3_{L^3(\Omega^h)} \\ & - c_2 |\Omega^h| - \frac{\sqrt{2}}{6\sqrt{\pi}} |S|^{3/2} |\mathbf{A}\mathbf{e}_1 \wedge \mathbf{A}\mathbf{e}_2|^{3/2} - \frac{h}{2} |S| |\det \mathbf{A}|. \end{aligned} \tag{2.32}$$

In view of the assumption (2.31), the second term on the right-hand side is non-negative and thus we can write

$$\begin{aligned} E^h(\mathbf{y}) \ \geqslant\ & \kappa \|\nabla^2 \mathbf{y}\|^2_{L^2(\Omega^h)} - c_2 |\Omega^h| - \frac{\sqrt{2}}{6\sqrt{\pi}} |S|^{3/2} |\mathbf{A}\mathbf{e}_1 \wedge \mathbf{A}\mathbf{e}_2|^{3/2} \\ & - \frac{h}{2} |S| |\det \mathbf{A}|. \end{aligned} \tag{2.33}$$

According to the generalized Poincaré inequality, [22, Theorem 3.6.4], there exist two positive constants C_1 and C_2 such that

$$\|\mathbf{y}\|^p_{L^p(\Omega^h)} + \|\nabla \mathbf{y}\|^p_{L^p(\Omega^h)} \leqslant C_1 \|\nabla^2 \mathbf{y}\|^p_{L^p(\Omega^h)} + C_2 \tag{2.34}$$

for each p in $[1, \infty)$ and for each $\mathbf{y}$ in $\mathcal{H}$. Combining (2.33) and (2.34) written for $p = 2$, we obtain

$$\begin{aligned} E^h(\mathbf{y}) \ \geqslant\ & \min\left\{ \frac{\kappa}{2}; \frac{\kappa}{2C_1} \right\} \|\mathbf{y}\|^2_{W^{2,2}(\Omega^h)} - C_2 - c_2 \big|\Omega^h\big| \\ & - \frac{\sqrt{2}}{6\sqrt{\pi}} |S|^{3/2} |\mathbf{A}\mathbf{e}_1 \wedge \mathbf{A}\mathbf{e}_2|^{3/2} - \frac{h}{2} |S| |\det \mathbf{A}|. \end{aligned} \tag{2.35}$$

Let now $\mathbf{y}_n$ be a minimizing sequence for E^h. Because the infimum of E^h is bounded by hypothesis, from the bound (2.35) it follows that the sequence $\mathbf{y}_n$ is uniformly bounded in $W^{2,2}(\Omega^h;\mathbb{R}^3)$. We can then extract a subsequence, not relabeled, such that

$$\mathbf{y}_n \rightharpoonup \bar{\mathbf{y}} \quad \text{in } W^{2,2}\big(\Omega^h;\mathbb{R}^3\big). \tag{2.36}$$

The limit $\bar{\mathbf{y}}$ belongs to $\mathcal{H}$. Indeed, by the compact embedding of $W^{2,2}(\Omega^h;\mathbb{R}^3)$ into $C^0(\bar{\Omega}^h;\mathbb{R}^3)$, [2, Theorem 6.2], we can extract a further subsequence uniformly converging to $\bar{\mathbf{y}}$ in the closure of Ω^h, $\bar{\Omega}^h$. Therefore, $\bar{\mathbf{y}}$ satisfies the boundary condition (2.2). Using the convexity of the second gradient term and the continuity of ϕ, together with the bound (2.4), we also have

$$\liminf_{n\to\infty}\int_{\Omega^h}\big(\phi(\nabla\mathbf{y}_n)+\kappa\big|\nabla^2\mathbf{y}_n\big|^2\big)\,\mathrm{d}\mathbf{X} \geqslant \int_{\Omega^h}\big(\phi(\nabla\bar{\mathbf{y}})+\kappa\big|\nabla^2\bar{\mathbf{y}}\big|^2\big)\,\mathrm{d}\mathbf{X}, \tag{2.37}$$

[14, Theorem 3.4]. From (2.37) and from the weak continuity of the volume functional in $W^{2,2}(\Omega^h;\mathbb{R}^3)$, it follows that the energy functional E^h is weakly lower semicontinuous in $\mathcal{H}$. Therefore, $\bar{\mathbf{y}}$ is a minimizer. □

If $p^h > p^h_{cr}$ and the lower bound of the strain energy density (2.4) holds with equality, then the infimum of the energy in $W^{2,2}(\Omega^h;\mathbb{R}^3)$ is expected to be $-\infty$, corresponding to rupture of the film. We also note that, since the constant which multiplies $\|\nabla\mathbf{y}\|_{L^3}$ in the bound on the volume (2.25) is not optimal, p^h_{cr} provides only a lower bound for the pressure at rupture. The presence of the thickness h and of the area $|S|$ of the film in the expression of p^h_{cr} introduces a scale effect. Indeed, if we keep $|S|$ fixed and let h increase, i.e., if we consider a thicker and thicker film, then, as the physical intuition suggests, p^h_{cr} increases and thus the film can sustain larger and larger value of the pressure. On the other hand, if we keep h fixed and increase $|S|$, i.e., if we consider a film of fixed thickness which becomes wider and wider, then p^h_{cr} decreases and thus the film sustains smaller and smaller values of the pressure.

Another interesting point is that, to get the lower bound on the energy (2.35) and thus to guarantee the existence of a continuous minimizer, not only the condition on the pressure is needed, but it also necessary that the exponent which appears in the growth condition (2.4) be not less than three. This issue arises from the competition between the loading potential, which, as it can be seen from (2.25), is dominated by the cube of $\|\nabla\mathbf{y}\|_{L^3(\Omega^h)}$, and the strain energy. In this respect, an analogy can be established with the phenomenon of cavitation [4].

3. Thin Film Theory

In the previous section, for each value of thickness h, we proved the existence of at least one minimizer, say $\mathbf{y}^h$, of the total energy functional E^h in the class of admissible deformations $\mathcal{H}$. We now consider the behavior of the minimizers $\mathbf{y}^h$

as $h \to 0^+$. The approach which we follow was introduced by Bhattacharya and James in [11] for a film whose total energy is given by (2.7) with $p^h = 0$. In the following, we adopt the notation ∇_p for the gradient in the plane of the film:

$$\nabla_p \mathbf{y}^h := \mathbf{y}^h_{,1} \otimes \mathbf{e}_1 + \mathbf{y}^h_{,2} \otimes \mathbf{e}_2, \tag{3.1}$$

and the notation $\mathbf{y}^h_{,1}\ |\mathbf{y}^h_{,2}|\ \mathbf{y}^h_{,3}$ for $\mathbf{y}^h_{,1} \otimes \mathbf{e}_1 + \mathbf{y}^h_{,2} \otimes \mathbf{e}_2 + \mathbf{y}^h_{,3} \otimes \mathbf{e}_3$. We begin by considering an equivalent minimization problem set on a cylindrical domain of fixed height. This can be obtained through the change of variables

$$Z_1 = X_1, \quad Z_2 = X_2, \quad Z_3 = \frac{X_3}{h}, \tag{3.2}$$

which maps the reference configuration of the film, Ω^h, into the configuration

$$\Omega_1 := \big\{(Z_1, Z_2, Z_3) \in \mathbb{R}^3\colon\ (Z_1, Z_2) \in S,\ Z_3 \in (0, 1)\big\}. \tag{3.3}$$

Then, if $\mathbf{y} \in \mathcal{H}$ is an admissible deformation of the film, the rescaled deformation

$$\mathbf{y} = \mathbf{y}\big(\mathbf{X}(\mathbf{Z})\big) =: \tilde{\mathbf{y}}(\mathbf{Z}), \tag{3.4}$$

belongs to the set

$$\mathcal{H}_1 := \big\{\tilde{\mathbf{y}} \in W^{2,2}\big(\Omega_1; \mathbb{R}^3\big) | \tilde{\mathbf{y}} = (\mathbf{A}\mathbf{e}_1 | \mathbf{A}\mathbf{e}_1 | h\mathbf{A}\mathbf{e}_3)\mathbf{Z},\ \mathbf{Z} \in \partial S \times (0, 1)\big\}. \tag{3.5}$$

We accordingly rescale the total energy by setting

$$\begin{aligned} E_1^h(\tilde{\mathbf{y}}) \ :=\ & \frac{1}{h} E^h(\tilde{\mathbf{y}}) = \int_{\Omega_1} \kappa\Big(|\nabla_p^2 \tilde{\mathbf{y}}|^2 + \frac{2}{h^2}|\nabla_p \tilde{\mathbf{y}}_{,3}|^2 + \frac{1}{h^4}|\tilde{\mathbf{y}}_{,33}|^2\Big)\, \mathrm{d}\mathbf{Z} \\ & + \int_{\Omega_1} \phi\Big(\tilde{\mathbf{y}}_{,1} | \tilde{\mathbf{y}}_{,2} | \frac{1}{h}\tilde{\mathbf{y}}_{,3}\Big)\, \mathrm{d}\mathbf{Z} - \frac{p^h}{3h} \int_{S\times\{0\}} \tilde{\mathbf{y}} \cdot (\tilde{\mathbf{y}}_{,1} \wedge \tilde{\mathbf{y}}_{,2})\, \mathrm{d}Z_1\, \mathrm{d}Z_2. \end{aligned} \tag{3.6}$$

The existence of minimizers of E_1^h in $\mathcal{H}_1$ follows from the existence of minimizers of E^h in $\mathcal{H}$ (Proposition 2.4), and from the fact that if $\mathbf{y}^h$ is a minimizer of E^h in $\mathcal{H}$, then $\tilde{\mathbf{y}}^h$ is a minimizer of E_1^h in $\mathcal{H}_1$. Note also that because

$$\|\nabla \mathbf{y}\|^3_{L^3(\Omega^h)} = h \Big\| (\tilde{\mathbf{y}}_{,1} | \tilde{\mathbf{y}}_{,2} | \frac{1}{h}\tilde{\mathbf{y}}_{,3}) \Big\|^3_{L^3(\Omega_1)}, \tag{3.7}$$

then, by (2.25) any $\tilde{\mathbf{y}} \in \mathcal{H}_1$ satisfies the inequality

$$\big|V(\tilde{\mathbf{y}})\big| \leqslant \bigg(\frac{|S|^{1/2}}{12\sqrt{\pi}} + \frac{h}{3\sqrt{3}}\bigg) \Big\| (\tilde{\mathbf{y}}_{,1} | \tilde{\mathbf{y}}_{,2} | \frac{1}{h}\tilde{\mathbf{y}}_{,3}) \Big\|^3_{L^3(\Omega_1^h)} + \widetilde{D}, \tag{3.8}$$

with

$$\widetilde{D} := \frac{\sqrt{2}}{6\sqrt{\pi}} |S|^{3/2} |\mathbf{A}\mathbf{e}_1 \wedge \mathbf{A}\mathbf{e}_2|^{3/2} + \frac{h}{2}|S||\det \mathbf{A}|.$$

For the asymptotic approach, it is also essential that the magnitude of the applied pressure be scaled appropriately in the thickness. From (3.6) (see also (2.31)), it turns out that the right order of magnitude is $p^h = Ph$, where P is a constant independent of h.

THEOREM 3.1. *Assume that*

$$P < \frac{12\sqrt{\pi}c_1}{|S|^{1/2}}. \tag{3.9}$$

Then, the family of minimizers $\tilde{\mathbf{y}}^h \in \mathcal{H}_1$ *has a subsequence, not relabeled, such that*

$$\left.\begin{aligned} &\nabla_p^2 \tilde{\mathbf{y}}^h \to \nabla^2 \hat{\mathbf{y}},\\ &\frac{1}{h}\nabla_p \tilde{\mathbf{y}}^h_{,3} \to \nabla \hat{\mathbf{b}},\\ &\frac{1}{h^2}\tilde{\mathbf{y}}^h_{,33} \to 0 \end{aligned}\right\} \quad \text{in } L^2, \tag{3.10}$$

where $\hat{\mathbf{y}} \in W^{2,2}(S;\mathbb{R}^3)$ *and* $\hat{\mathbf{b}} \in W^{1,2}(S;\mathbb{R}^3)$ *are vector fields independent of* Z_3. *The couple* $(\hat{\mathbf{y}}, \hat{\mathbf{b}})$ *minimizes the limit energy*

$$\begin{aligned} E(\mathbf{y},\mathbf{b}) := \int_S \Big\{ &\kappa\big\{|\nabla^2 \mathbf{y}|^2 + 2|\nabla \mathbf{b}|^2\big\} + \phi(\mathbf{y}_{,1}|\mathbf{y}_{,2}|\mathbf{b})\\ &- \frac{P}{3}\mathbf{y}\cdot(\mathbf{y}_{,1}\wedge \mathbf{y}_{,2})\Big\}\mathrm{d}Z_1\,\mathrm{d}Z_2, \end{aligned} \tag{3.11}$$

among all couples $(\mathbf{y},\mathbf{b}) \in W^{2,2}(S;\mathbb{R}^3) \times W^{1,2}(S;\mathbb{R}^3)$ *satisfying the boundary conditions*

$$\left.\begin{aligned} &\mathbf{y} = \mathbf{A}\mathbf{e}_1 Z_1 + \mathbf{A}\mathbf{e}_2 Z_2,\\ &\mathbf{b} = \mathbf{A}\mathbf{e}_3 \end{aligned}\right\} \quad (Z_1, Z_2) \in \partial S. \tag{3.12}$$

Proof. The lower bound in (2.4) and the inequality (3.8) give for the rescaled energy the following lower bound :

$$\begin{aligned} E_1^h(\tilde{\mathbf{y}}^h) \geqslant\ &\kappa\left\{ \|\nabla_p^2 \tilde{\mathbf{y}}^h\|^2_{L^2(\Omega_1)} + 2\left\|\frac{1}{h}\nabla_p \tilde{\mathbf{y}}^h_{,3}\right\|^2_{L^2(\Omega_1)} + \left\|\frac{1}{h^2}\tilde{\mathbf{y}}^h_{,33}\right\|^2_{L^2(\Omega_1)}\right\}\\ &+ \left\{c_1 - P\left(\frac{|S|^{1/2}}{12\sqrt{\pi}} + \frac{h}{3\sqrt{3}}\right)\right\}\left\|\left(\tilde{\mathbf{y}}_{,1}|\tilde{\mathbf{y}}_{,2}|\frac{1}{h}\tilde{\mathbf{y}}_{,3}\right)\right\|^3_{L^3(\Omega_1^h)}\\ &- \widetilde{D} + c_2|\Omega_1|, \end{aligned} \tag{3.13}$$

from which, in view of the assumption (3.9), we get

$$\begin{aligned} E_1^h(\tilde{\mathbf{y}}^h) \geqslant\ &\kappa\left\{ \left\|\nabla_p^2 \tilde{\mathbf{y}}^h\right\|^2_{L^2(\Omega_1)} + 2\left\|\frac{1}{h}\nabla_p \tilde{\mathbf{y}}^h_{,3}\right\|^2_{L^2(\Omega_1)} + \left\|\frac{1}{h^2}\tilde{\mathbf{y}}^h_{,33}\right\|^2_{L^2(\Omega_1)}\right\}\\ &- D + c_2|\Omega_1|, \end{aligned} \tag{3.14}$$

for h sufficiently small, with

$$D = \frac{\sqrt{2}}{6\sqrt{\pi}}|S|^{3/2}|\mathbf{Ae}_1 \wedge \mathbf{Ae}_2|^{3/2}.$$

Because $\tilde{\mathbf{y}}^h$ is a minimizer, we can test it against the affine deformation $\mathbf{y} = (\mathbf{Ae}_1|\mathbf{Ae}_2|h\mathbf{Ae}_3)\mathbf{Z}$ of $\mathcal{H}_1$ to obtain the upper bound

$$E_1^h(\tilde{\mathbf{y}}^h) \leqslant |\Omega_1|\phi\big(\mathbf{Ae}_1|\mathbf{Ae}_2|\mathbf{Ae}_3\big). \tag{3.15}$$

Combining (3.14) with (3.15) and setting $c := |\Omega_1|\phi(\mathbf{Ae}_1|\mathbf{Ae}_2|\mathbf{Ae}_3)/\kappa + D$, we get

$$\big\|\nabla_p^2\tilde{\mathbf{y}}^h\big\|^2_{L^2(\Omega_1)} \leqslant c, \tag{3.16}$$

$$2\left\|\frac{1}{h}\nabla_p\tilde{\mathbf{y}}^h_{,3}\right\|^2_{L^2(\Omega_1)} \leqslant c, \tag{3.17}$$

$$\left\|\frac{1}{h^2}\tilde{\mathbf{y}}^h_{,33}\right\|^2_{L^2(\Omega_1)} \leqslant c. \tag{3.18}$$

Because we assume $h \leqslant 1$, we also have

$$2\big\|\nabla_p\tilde{\mathbf{y}}^h_{,3}\big\|^2_{L^2(\Omega_1)} \leqslant c, \tag{3.19}$$

$$\big\|\tilde{\mathbf{y}}^h_{,33}\big\|^2_{L^2(\Omega_1)} \leqslant c. \tag{3.20}$$

By (3.16), (3.19), and (3.20), we see that

$$\big\|\nabla^2\tilde{\mathbf{y}}^h\big\|^2_{L^2(\Omega_1)} \leqslant c, \tag{3.21}$$

and this bound, together with the Poincaré inequality (2.34) written for $\tilde{\mathbf{y}}^h$ and $p = 2$, gives in turn

$$\big\|\tilde{\mathbf{y}}^h\big\|^2_{L^2(\Omega_1)} + \big\|\nabla\tilde{\mathbf{y}}^h\big\|^2_{L^2(\Omega_1)} \leqslant C_1c + C_2. \tag{3.22}$$

Therefore, the family of minimizers $\tilde{\mathbf{y}}^h$ is uniformly bounded in $W^{2,2}$, and thus it contains a subsequence, not relabed, such that

$$\tilde{\mathbf{y}}^h \rightharpoonup \hat{\mathbf{y}} \quad \text{in } W^{2,2}\big(\Omega_1;\mathbb{R}^3\big). \tag{3.23}$$

In view of (3.17) and (3.18), we also have

$$\left\|\nabla\left(\frac{1}{h}\tilde{\mathbf{y}}^h_{,3}\right)\right\|^2_{L^2(\Omega_1)} = \left\|\frac{1}{h}\nabla_p\tilde{\mathbf{y}}^h_{,3}\right\|^2_{L^2(\Omega_1)} + \left\|\frac{1}{h^2}\tilde{\mathbf{y}}^h_{,33}\right\|^2_{L^2(\Omega_1)} \leqslant \frac{3}{2}c, \tag{3.24}$$

which, together with the Poincaré inequality written in the form

$$\int_{\Omega_1}\left|\frac{1}{h}\tilde{\mathbf{y}}^h_{,3}\right|^2 \mathrm{d}\mathbf{Z} \leqslant \overline{C}_1\left\{\int_{\Omega_1}\left|\frac{1}{h}\nabla\tilde{\mathbf{y}}^h_{,3}\right|^2 \mathrm{d}\mathbf{Z} + \left|\int_{\partial S\times(0,1)}\frac{1}{h}\tilde{\mathbf{y}}^h_{,3}\,\mathrm{d}a\right|^2\right\} \tag{3.25}$$

[13, Theorem 6.1-8], implies

$$\left\| \frac{1}{h}\tilde{\mathbf{y}}^h_{,3} \right\|^2_{L^2(\Omega_1)} \leqslant \overline{C}_1 \left[\frac{3}{2}c + |\partial S|^2 |A\mathbf{e}_3|^2 \right], \tag{3.26}$$

where $|\partial S|$ denotes the length of the boundary curve ∂S of S. In view of (3.24) and (3.26), we conclude that $(1/h)\tilde{\mathbf{y}}^h_{,3}$ is uniformly bounded in $W^{1,2}(\Omega_1; \mathbb{R}^3)$. Therefore, there exists a sequence, not relabeled, such that

$$\frac{1}{h}\tilde{\mathbf{y}}^h_{,3} \rightharpoonup \hat{\mathbf{b}} \quad \text{in} W^{1,2}\big(\Omega_1; \mathbb{R}^3\big), \tag{3.27}$$

and thus, up to a further subsequence, $\tilde{\mathbf{y}}^h_{,3}$ converges to zero almost everywhere in Ω_1. Because Ω_1 is convex in the Z_3 direction, the limit $\hat{\mathbf{y}}$ is independent of Z_3. Note also that, by (3.23) and by Rellich's Theorem, there exists a subsequence of $\tilde{\mathbf{y}}^h$, not relabeled, uniformly converging to $\hat{\mathbf{y}}$ in the closure of Ω_1. Therefore, $\hat{\mathbf{y}}$ satisfies the boundary condition $(3.12)_1$. From (3.18) we also get that $\tilde{\mathbf{y}}^h_{,33}$ converges to zero almost everywhere in Ω_1, and again from the convexity of Ω_1 in the Z_3 direction, the limit $\hat{\mathbf{b}}$ turns out to be independent of Z_3. Finally, by the trace theorem [17, 4.3], $\hat{\mathbf{b}}$ satisfies the boundary condition $(3.12)_2$. We now write

$$\nabla_p^2 \tilde{\mathbf{y}}^h = \nabla_p^2 \hat{\mathbf{y}} + e_p^h, \qquad e_p^h \rightharpoonup 0 \quad \text{in } L^2, \tag{3.28}$$

$$\frac{1}{h}\nabla_p \tilde{\mathbf{y}}^h_{,3} = \nabla \hat{\mathbf{b}} + e_3^h, \quad e_3^h \rightharpoonup 0 \quad \text{in } L^2. \tag{3.29}$$

Let $n \mapsto \hat{\mathbf{b}}_n \in C^\infty(\overline{S})$ be an approximating smooth sequence strongly converging to $\hat{\mathbf{b}}$ in $W^{1,2}(\Omega_1; \mathbb{R}^3)$ such that $\hat{\mathbf{b}}_n(Z_1, Z_2) = \mathbf{Ae}_3$ for $(Z_1, Z_2) \in \partial S$ and $\hat{\mathbf{b}}_{n,3}(\mathbf{Z}) = 0$ for each $\mathbf{Z}$ in Ω_1. Because $\tilde{\mathbf{y}}^h$ is a minimizer of E_1^h, we can test it against the deformation $\tilde{\mathbf{y}}^h_n := \hat{\mathbf{y}} + h\hat{\mathbf{b}}_n Z_3$, which, in virtue of the properties enjoyed by $\hat{\mathbf{b}}_n$, belongs to the set $\mathcal{H}_1$. Using (3.28) and (3.29), we get

$$\begin{aligned}
&\int_{\Omega_1} \kappa \bigg\{ \big|\nabla_p^2 \hat{\mathbf{y}}\big|^2 + \big|e_p^h\big|^2 + 2\nabla_p^2 \hat{\mathbf{y}} \cdot e_p^h + 2\big(\big|\nabla \hat{\mathbf{b}}\big|^2 + \big|e_3^h\big|^2 + 2\nabla \hat{\mathbf{b}} \cdot e_3^h\big) \\
&\qquad + \left| \frac{1}{h^2}\tilde{\mathbf{y}}^h_{,33} \right|^2 \bigg\} + \phi\left(\tilde{\mathbf{y}}^h_{,1} \big| \tilde{\mathbf{y}}^h_{,2} \big| \frac{1}{h}\tilde{\mathbf{y}}^h_{,3} \right) \mathrm{d}\mathbf{Z} \\
&\qquad - \frac{P}{3} \int_{S\times\{0\}} \tilde{\mathbf{y}}^h \cdot \big(\tilde{\mathbf{y}}^h_{,1} \wedge \tilde{\mathbf{y}}^h_{,2}\big) \mathrm{d}Z_1\, \mathrm{d}Z_2 \\
&\quad \leqslant \int_{\Omega_1} \kappa \big\{ \big|\nabla_p^2 \hat{\mathbf{y}}\big|^2 + h^2 Z_3^2 \big|\nabla_p^2 \hat{\mathbf{b}}_n\big|^2 + 2h^2 Z_3^2 \nabla_p^2 \hat{\mathbf{y}} \cdot \nabla^2 \hat{\mathbf{b}}_n + 2\big|\nabla \hat{\mathbf{b}}_n\big|^2 \big\} \\
&\qquad + \phi\big(\hat{\mathbf{y}}_{,1} + hZ_3 \hat{\mathbf{b}}_{n,1} \big| \hat{\mathbf{y}}_{,2} + hZ_3 \hat{\mathbf{b}}_{n,2} \big| \hat{\mathbf{b}}_n\big)\, \mathrm{d}\mathbf{Z} \\
&\qquad - \frac{P}{3} \int_S \hat{\mathbf{y}} \cdot \big(\hat{\mathbf{y}}_{,1} \wedge \hat{\mathbf{y}}_{,2}\big)\, \mathrm{d}Z_1\, \mathrm{d}Z_2.
\end{aligned} \tag{3.30}$$

After simplifying the first term on both sides, fix n and take the lim sup as $h \to 0^+$. Because $\hat{\mathbf{b}}_n$ is smooth, we can simplify the second and the third term on the right-hand side. Moreover, in view of (3.28) and (3.29), the third and sixth term on the left-hand side converge to zero. Using the upper bound in (2.4), the Lebesgue theorem and the weak continuity of the volume functional in $W^{2,2}$, we reduce to

$$\limsup_{h\to 0^+} \int_{\Omega_1} \kappa \left\{ |e_p^h|^2 + 2|e_3^h|^2 + \left| \frac{1}{h^2} \tilde{\mathbf{y}}^h_{,33} \right|^2 \right\} \mathrm{d}\mathbf{Z} \leqslant \int_{\Omega_1} 2\kappa \left\{ |\nabla \hat{\mathbf{b}}_n|^2 - |\nabla \hat{\mathbf{b}}|^2 \right\} \mathrm{d}\mathbf{Z} + \int_{\Omega_1} \left(\phi(\hat{\mathbf{y}}_{,1}|\hat{\mathbf{y}}_{,2}|\hat{\mathbf{b}}_n) - \phi(\hat{\mathbf{y}}_{,1}|\hat{\mathbf{y}}_{,2}|\hat{\mathbf{b}}) \right) \mathrm{d}\mathbf{Z}. \tag{3.31}$$

Let now $n \to \infty$ and use again the upper bound in (2.4). Then the "sup" can be dropped in (3.31) and we have improved the convergence in (3.28) and (3.29) to strong. This also shows that the limit energy is given by (3.11) evaluated at $(\hat{\mathbf{y}}, \hat{\mathbf{b}})$.

To establish the minimum principle, we choose the deformation $\hat{\mathbf{y}}^h := \hat{\mathbf{y}}(Z_1, Z_2) + h\hat{\mathbf{b}}(Z_1, Z_2)Z_3$ as a test function, with $\hat{\mathbf{y}}$ and $\hat{\mathbf{b}} \in C^\infty(\overline{S}; \mathbb{R}^3)$ satisfying the boundary conditions (3.12). Repeating the argument from (3.30) to (2.30) gives the minimum principle for smooth competitors. The minimum principle for competitors in $W^{2,2}(S; \mathbb{R}^3) \times W^{1,2}(S; \mathbb{R}^3)$ follows by approximation. □

The limit energy (3.11) turns out to depend upon two independent vector fields, $\hat{\mathbf{y}}$ and $\hat{\mathbf{b}}$, which describe the deformation of the middle surface of the film S, and the deformation in the direction perpendicular to the film, respectively. Therefore, Theorem 3.1 provides a 2-dimensional Cosserat theory, with $\hat{\mathbf{b}}$ the Cosserat director. The energy (3.11) is a membrane energy supplemented by an interfacial energy term (the term multiplying κ). The latter has a similar form as a bending energy, but its physical origins are the same as the analogous term in the 3-dimensional energy (2.7). That is, it is intended to model energy associated with lattice curvature, arising from lattice radii of curvature that are comparable to atomic spacing, which occur in the present case when there are interfaces between variants or phases. Thus κ should not be considered as a classical bending modulus.

Because of this interpretation, κ is expected to be much smaller than a typical modulus that describes the growth of ϕ away from its energy wells. It is therefore known from many studies that the presence of κ merely smooths interfaces, and this could also be verified by an elementary Γ-convergence argument. Hence, in the following section we drop the term multiplying κ and study the pressurized membrane energy alone.

4. Martensitic Thin Films

Martensitic crystals display a diffusionless solid-to-solid phase transformation between a symmetric high temperature phase (austenite) and different symmetry-related variants of a low temperature phase (martensite). For temperatures above the transformation temperature θ_{cr}, the austenite phase is the stable phase, while for temperatures below θ_{cr}, the martensite is stable. At the transformation temperature θ_{cr}, both phases are stable. To model the change of phase, we follow Ball and James [7, 8] in introducing a nonconvex strain energy density ϕ with energy wells at the matrices

$$\mathcal{A} := SO(3), \tag{4.1}$$

for the austenite phase, and

$$\mathcal{M} := \left\{\mathbf{F} \in M^{3\times 3} \mid \exists \mathbf{R} \in SO(3), \exists \mathbf{U} \in \{\mathbf{U}_1, \mathbf{U}_2, \ldots \mathbf{U}_n\}\colon \mathbf{F} = \mathbf{RU}\right\}, \tag{4.2}$$

for the martensite variants. $SO(3)$ is the set of all proper rotations, and $\mathbf{U}_i$, $i = 1, 2, \ldots, N$, are distinct positive definite symmetric matrices representing the transformation strains of the variants of martensite from the austenite, taken as reference configuration; they can be determined by measurements of the lattice parameters of the material. For the austenite and the martensite phases, we introduce nonconvex strain energy densities ϕ_a, ϕ_m, which are continuous non negative scalar functions defined over $M^{3\times 3}$ and such that ϕ_a is minimized on $\mathcal{A}$ and the minimum value is zero, while ϕ_m is minimized on $\mathcal{M}$ and the minimum value is zero. The energies ϕ_a, ϕ_m, are also supposed to satisfy the following growth hypotheses: there exist positive constants c_{a1}, c_{a2}, c_{a3}, c_{a4}, c_{m1}, c_{m2}, c_{m3}, c_{m4} and $3 < q < 6$ such that

$$c_{a1}|\mathbf{F}|^3 - c_{a2} \leqslant \phi_a(\mathbf{F}) \leqslant c_{a3}|\mathbf{F}|^q - c_{a4}, \tag{4.3}$$

$$c_{m1}|\mathbf{F}|^3 - c_{m2} \leqslant \phi_m(\mathbf{F}) \leqslant c_{m3}|\mathbf{F}|^q - c_{m4}, \tag{4.4}$$

for each $\mathbf{F} \in M^{3\times 3}$. These hypotheses are consistent with the growth assumptions (2.4).

Introducing a $\chi \in C^0(M^{3\times 3}; [0, 1])$ such that

$$\chi(\mathbf{F}) = 0 \quad \Leftrightarrow \quad \mathbf{F} \in \mathcal{A}, \tag{4.5}$$

$$\chi(\mathbf{F}) = 1 \quad \Leftrightarrow \quad \mathbf{F} \in \mathcal{M}, \tag{4.6}$$

we assume for the strain energy density ϕ the simple form

$$\phi(\mathbf{F}; \theta) = \chi(\mathbf{F})\big(\phi_m(\mathbf{F}) + l_m(\theta)\big) + (1 - \chi(\mathbf{F}))\big(\phi_a(\mathbf{F}) + l_a(\theta)\big). \tag{4.7}$$

The expression (4.7) is a simple but realistic way of modeling the exchange of stability between austenite and martensite. The terms $l_m(\theta)$ and $l_a(\theta)$, which are related to the latent heat of transformation, are positive material constants depending continuously upon the temperature θ and such that

$$\left.\begin{array}{ll} l_a(\theta) > l_m(\theta) & \text{if } \theta < \theta_{cr}, \\ l_a(\theta) < l_m(\theta) & \text{if } \theta > \theta_{cr}, \\ l_a(\theta_{cr}) = l_m(\theta_{cr}). & \end{array}\right\} \tag{4.8}$$

In view of the definition (4.7) and of the assumptions (4.1), (4.2),(4.5), (4.6), and (4.8), ϕ has a multi-well structure, the terms $l_m(\theta)$, $l_a(\theta)$ corresponding to the heights of the martensite and of the austenite wells, respectively. In particular, if $\theta < \theta_{cr}$, $\phi(\cdot;\theta)$ attains the absolute minimum at all matrices belonging to the set of martensite wells $\mathcal{M}$. Indeed, because χ has values in [0, 1] and because ϕ_a and ϕ_m are positive, we have

$$\inf_{\mathbf{F}\in M^{3\times 3}} \phi(\mathbf{F};\theta) \geqslant \min\bigl\{l_a(\theta); l_m(\theta)\bigr\}. \tag{4.9}$$

If $\theta < \theta_{cr}$, the right-hand side coincides with $l_m(\theta)$, and since $\phi(\mathbf{F};\theta) = l_m(\theta)$ for all $\mathbf{F} \in \mathcal{M}$, then ϕ is minimized at $\mathcal{M}$. Analogously, it can be shown that if $\theta > \theta_{cr}$, then $\phi(\cdot;\theta)$ is minimized at $\mathcal{A}$ and $\phi(\cdot;\theta_{cr})$ is minimized at $\mathcal{A} \cup \mathcal{M}$.

Experiments indicate that, at equilibrium, the deformation gradient stays very close to the wells, even though it could not be precisely at the minima, because of the presence of the term $-PV(\mathbf{y})$. For many martensitic materials with “hard moduli”, this suggests that the equilibrium microstructures can be approximately described by deformations whose gradients satisfy the constraint of lying on the wells. The idea of this approach, called the *constrained theory of martensite* and first proposed in [6], is to study the asymptotic behavior of the sequence of total energy functionals

$$E^n(\mathbf{y},\mathbf{b}) := \int_S \Bigl(\phi_n\bigl(\mathbf{y}_{,1}|\mathbf{y}_{,2}|\mathbf{b}\bigr) - \frac{P}{3}\mathbf{y}\cdot(\mathbf{y}_{,1}\wedge\mathbf{y}_{,2})\Bigr)\,\mathrm{d}Z_1\,\mathrm{d}Z_2, \tag{4.10}$$

as $n \to \infty$, with

$$\phi_n(\mathbf{F};\theta) := \chi(\mathbf{F})\bigl(n\phi_m(\mathbf{F}) + l_m(\theta)\bigr) + \bigl(1-\chi(\mathbf{F})\bigr)\bigl(n\phi_a(\mathbf{F}) + l_a(\theta)\bigr). \tag{4.11}$$

The interfacial energy term $\kappa\{|\nabla_p^2\mathbf{y}|^2 + 2|\nabla_p\mathbf{b}|^2\}$ has been neglected in (4.10). Indeed, if the film is large enough, the elastic energy is much larger than the interfacial energy. In this respect, each element of the sequence (4.10) provides a reasonable approximation of the expression (3.11) of the energy of a very thin film with strain energy density ϕ_n growing more and more steeply away from the wells. Each element of the sequence (4.10) is assumed to be defined on the set of functions

$$\begin{aligned}\mathcal{K} := \bigl\{&(\mathbf{y},\mathbf{b}) \in W^{1,q}(S;\mathbb{R}^3)\times L^q(S;\mathbb{R}^3)|\mathbf{y} = \mathbf{A}\mathbf{e}_1 Z_1 + \mathbf{A}\mathbf{e}_2 Z_2,\\ &\mathbf{b} = \mathbf{A}\mathbf{e}_3,\ (Z_1,Z_2)\in\partial S\bigr\},\end{aligned} \tag{4.12}$$

where q is the exponent which appears in (4.3), (4.4). This set turns out to be the “natural” domain of the energy functionals in (4.10). Indeed, in view of the upper bounds in (4.3), (4.4) and of the bound on the volume functional (2.21), each couple $(\mathbf{y},\mathbf{b})$ in $\mathcal{K}$ has energy $E^n(\mathbf{y},\mathbf{b})$ finite. As shown by the following theorem, the result of the constrained theory is a new simplified variational problem whose solutions are searched among fine mixtures; these are mathematically described by families of Young measures with supports contained in the set $\mathcal{A} \cup \mathcal{M}$. We recall the essential property of the Young measure [5]. Let $C_0(M^{3\times 3};\mathbb{R})$ denote the continuous functions on $M^{3\times 3}$ with compact support. Given a sequence $\mathbf{F}^n \in L^1(S;M^{3\times 3})$, we may find a family of probability measures $(\nu_{\mathbf{Z}}), \mathbf{Z} \in S$ and a subsequence of $\mathbf{F}^n$, not relabeled, such that, for any $\psi \in C_0(M^{3\times 3};\mathbb{R})$,

$$\psi\big(\mathbf{F}^n\big) \overset{*}{\rightharpoonup} \int_{M^{3\times 3}} \psi(\mathbf{F})\,\mathrm{d}\nu_{\mathbf{Z}}(\mathbf{F}) \quad \text{in } L^\infty(S;\mathbb{R}). \tag{4.13}$$

Young measures are useful tools for the analysis of the microstructure [7, 8]. The family of measures $(\nu_{\mathbf{Z}}), \mathbf{Z} \in S$ characterizes the local limit distribution of the values $\mathbf{F}^n$ as $n \to \infty$. If the sequence $\mathbf{F}^n$ is thought of as representing a sequence $(\mathbf{y}^n_{,1}|\mathbf{y}^n_2|\mathbf{b}^n)$ and if the measures $\nu_{\mathbf{Z}}$ are supported on the set $\mathcal{M} \cup \mathcal{A}$, then the $\nu_{\mathbf{Z}}$ turn out to describe the local proportions of phases and the microstructure of the material.

THEOREM 4.1. *Assume that there exists a constant C, independent of n, such that*

$$\inf_{(\mathbf{y},\mathbf{b})\in\mathcal{K}} E^n(\mathbf{y},\mathbf{b}) \leqslant C < +\infty, \tag{4.14}$$

and a sequence $(\mathbf{y}^n,\mathbf{b}^n)$ in $\mathcal{K}$ such that

$$E^n\big(\mathbf{y}^n,\mathbf{b}^n\big) \leqslant \inf_{(\mathbf{y},\mathbf{b})\in\mathcal{K}} E^n(\mathbf{y},\mathbf{b}) + \frac{1}{n}. \tag{4.15}$$

Then, there exists a subsequence, not relabeled, such that

$$\mathbf{y}^n \rightharpoonup \hat{\mathbf{y}} \quad \text{in } W^{1,3}\big(S;\mathbb{R}^3\big), \tag{4.16}$$

$$\mathbf{b}^n \rightharpoonup \hat{\mathbf{b}} \quad \text{in } L^3\big(S;\mathbb{R}^3\big), \tag{4.17}$$

and the family of Young measures, $(\hat{\nu}_{\mathbf{Z}}), \mathbf{Z} \in S$, generated by the sequence $(\mathbf{y}^n_{,1}|\mathbf{y}^n_{,2}|\mathbf{b}^n)$ is such that the couple $(\hat{\nu}_{\mathbf{Z}},\hat{\mathbf{y}})$ is a minimizer of the limit energy

$$\begin{aligned} e(\nu_{\mathbf{Z}};\mathbf{y}) := &\int_S \left\{ \big(l_m(\theta) - l_a(\theta)\big) \int_{\mathcal{M}} \mathrm{d}\nu_{\mathbf{Z}}(\mathbf{F}) - \frac{P}{3}\mathbf{y}\cdot(\mathbf{y}_{,1}\wedge\mathbf{y}_{,2}) \right\} \mathrm{d}Z_1\,\mathrm{d}Z_2 \\ &+ l_a(\theta)|S|, \end{aligned} \tag{4.18}$$

among all couples $(\nu_{\mathbf{Z}},\mathbf{y})$ such that $(\nu_{\mathbf{Z}}), \mathbf{Z} \in S$, is a family of Young measures with supports in the set $\mathcal{M} \cup \mathcal{A}$ and $\mathbf{y} \in W^{1,3}(S;\mathbb{R}^3)$ satisfies the constraint

$$\mathbf{y}_{,\alpha}(\mathbf{Z}) = \int_{\mathcal{M}\cup\mathcal{A}} \mathbf{F}\mathbf{e}_\alpha\,\mathrm{d}\nu_{\mathbf{Z}}(\mathbf{F}), \quad \alpha = 1, 2, \tag{4.19}$$

at almost every $\mathbf{Z} \in S$, and the boundary condition

$$\mathbf{y} = \mathbf{A}\mathbf{e}_1 Z_1 + \mathbf{A}\mathbf{e}_2 Z_2, \quad \mathbf{Z} \in \partial S. \tag{4.20}$$

REMARKS. We note that (4.14) places a restriction on the boundary conditions in (4.12). This could be quantified, but we don't do it here. In the next section, we give an example (the tunnel) in which (4.14) holds with the boundary conditions (5.1). In Theorem 4.1, condition (4.15) is to allow for the possibility of nonattainment of the minimum. Also, Young measure refers in this paper to Young measures arising from a sequence that is bounded in $\mathcal{K}$.

Proof. In view of the hypotheses (4.14) and (4.15) and recalling that $n \geqslant 1$, we have

$$E^n\big(\mathbf{y}^n, \mathbf{b}^n\big) \leqslant C + 1. \tag{4.21}$$

Substituting to E^n its expression (4.10) with ϕ_n given by (4.11) and dividing by n, we obtain

$$\begin{aligned}&\int_S \Big\{ \chi\big(\mathbf{y}^n_{,1}|\mathbf{y}^n_{,2}|\mathbf{b}^n\big)\phi_m(\mathbf{y}^n_{,1}|\mathbf{y}^n_{,2}|\mathbf{b}^n) + \big(1-\chi(\mathbf{y}^n_{,1}|\mathbf{y}^n_{,2}|\mathbf{b}^n)\big)\phi_a\big(\mathbf{y}^n_{,1}|\mathbf{y}^n_{,2}|\mathbf{b}^n\big)\\ &\quad + \frac{1}{n}\big[\chi\big(\mathbf{y}^n_{,1}|\mathbf{y}^n_{,2}|\mathbf{b}^n\big)l_m(\theta) + \big(1-\chi\big(\mathbf{y}^n_{,1}|\mathbf{y}^n_{,2}|\mathbf{b}^n\big)\big)l_a(\theta)\big]\Big\}\mathrm{d}Z_1\,\mathrm{d}Z_2\\ &\quad -\frac{P}{n}V\big(\mathbf{y}^n\big) \leqslant \frac{C+1}{n}.\end{aligned} \tag{4.22}$$

Now we use the bound on the volume functional (2.21), written for $t = 0$, and the lower bounds in (4.3) and (4.4) to get

$$\begin{aligned}&\int_S \Big\{ \big[c_{m1}\chi\big(\mathbf{y}^n_{,1}|\mathbf{y}^n_{,2}|\mathbf{b}^n\big) + c_{a1}\big(1-\chi\big(\mathbf{y}^n_{,1}|\mathbf{y}^n_{,2}|\mathbf{b}^n\big)\big)\big]\big|\big(\mathbf{y}^n_{,1}|\mathbf{y}^n_{,2}|\mathbf{b}^n\big)\big|^3\\ &\quad -\big[c_{m2}\chi\big(\mathbf{y}^n_{,1}|\mathbf{y}^n_{,2}|\mathbf{b}^n\big) + c_{a2}\big(1-\chi\big(\mathbf{y}^n_{,1}|\mathbf{y}^n_{,2}|\mathbf{b}^n\big)\big)\big]\\ &\quad +\frac{1}{n}\big[\chi\big(\mathbf{y}^n_{,1}|\mathbf{y}^n_{,2}|\mathbf{b}^n\big)l_m(\theta) + \big(1-\chi\big(\mathbf{y}^n_{,1}|\mathbf{y}^n_{,2}|\mathbf{b}^n\big)\big)l_a(\theta)\big]\Big\}\mathrm{d}Z_1\,\mathrm{d}Z_2\\ &\quad -\frac{P}{n}\left[\frac{|S|^{1/2}}{12\sqrt{\pi}}\big\|\nabla\mathbf{y}^n\big\|^3_{L^3(S)} + c\right] \leqslant \frac{C+1}{n}.\end{aligned} \tag{4.23}$$

Using the inequality $|(\mathbf{y}^n_{,1}|\mathbf{y}^n_{,2}|\mathbf{b}^n)|^3 \geqslant |\nabla\mathbf{y}^n|^3 + |\mathbf{b}^n|^3$ and recalling that $0 \leqslant \chi \leqslant 1$, we also have

$$\begin{aligned}&\left(\min\{c_{a1}; c_{m1}\} - \frac{P|S|^{1/2}}{12n\sqrt{\pi}}\right)\big\|\nabla\mathbf{y}^n\big\|^3_{L^3(S)} + \min\{c_{a1}; c_{m1}\}\big\|\nabla\mathbf{b}^n\big\|^3_{L^3(S)}\\ &\quad +|S|\left(\frac{1}{n}\min\big\{l_m(\theta); l_a(\theta)\big\} - \max\big\{c_{m2}; c_{a2}\big\}\right) - \frac{P}{n}c \leqslant \frac{C+1}{n}.\end{aligned} \tag{4.24}$$

Therefore, for n sufficiently large, there exists a constant $\overline{C}$, independent of n, such that

$$\big\|\nabla\mathbf{y}^n\big\|^3_{L^3(S)} + \big\|\nabla\mathbf{b}^n\big\|^3_{L^3(S)} \leqslant \overline{C}. \tag{4.25}$$

By the weak compactness in Sobolev's spaces, it then follows that

$$\nabla \mathbf{y}^n \rightharpoonup \nabla \hat{\mathbf{y}} \quad \text{in } L^3\big(S; \mathbb{R}^3\big), \tag{4.26}$$

$$\mathbf{b}^n \rightharpoonup \hat{\mathbf{b}} \quad \text{in } L^3\big(S; \mathbb{R}^3\big), \tag{4.27}$$

up to a subsequence. In view of (4.26) and of the Poincaré inequality (2.34) written for $\mathbf{y}^n$ and $p = 3$, $\mathbf{y}^n$ is uniformly bounded in $W^{1,3}(S; \mathbb{R}^3)$, and thus (4.16) holds up to a subsequence. Besides, by Rellich's theorem, there exists a further subsequence, not relabeled, such that

$$\mathbf{y}^n \to \hat{\mathbf{y}} \quad \text{in } C^0\big(\overline{S}; \mathbb{R}^3\big). \tag{4.28}$$

Therefore, the limit $\hat{\mathbf{y}}$ satisfies the boundary condition (4.20). From (4.26) and from the bound on the volume (2.21), we also get

$$\frac{1}{n} V\big(\mathbf{y}^n\big) \to 0, \tag{4.29}$$

which, in view of (4.21) and of the positivity of ϕ_m, ϕ_a, χ and $1-\chi$, in turn implies

$$\int_S \chi\big(\mathbf{y}^n_{,1}|\mathbf{y}^n_{,2}|\mathbf{b}^n\big)\phi_m\big(\mathbf{y}^n_{,1}|\mathbf{y}^n_{,2}|\mathbf{b}^n\big)\mathrm{d}Z_1\,\mathrm{d}Z_2 \to 0, \tag{4.30}$$

$$\int_S \big(1-\chi(\mathbf{y}^n_{,1}|\mathbf{y}^n_{,2}|\mathbf{b}^n)\big)\phi_a\big(\mathbf{y}^n_{,1}|\mathbf{y}^n_{,2}|\mathbf{b}^n\big)\mathrm{d}Z_1\,\mathrm{d}Z_2 \to 0. \tag{4.31}$$

Therefore, by the fundamental property of the Young measures (4.13), there exists another subsequence of $(\mathbf{y}^n_{,1}|\mathbf{y}^n_{,2}|\mathbf{b}^n)$, generating a family of Young measures $(\hat{\nu}_{\mathbf{Z}})$, $\mathbf{Z} \in S$, such that

$$\int_S \int_{M^{3\times 3}} \chi(\mathbf{F})\phi_m(\mathbf{F})d\hat{\nu}_{\mathbf{Z}}(\mathbf{F})\,\mathrm{d}Z_1\,\mathrm{d}Z_2 = 0, \tag{4.32}$$

$$\int_S \int_{M^{3\times 3}} (1-\chi(\mathbf{F}))\phi_a(\mathbf{F})d\hat{\nu}_{\mathbf{Z}}(\mathbf{F})\,\mathrm{d}Z_1\,\mathrm{d}Z_2 = 0. \tag{4.33}$$

These imply that the support of $\hat{\nu}_{\mathbf{Z}}$ is contained in $\mathcal{A} \cup \mathcal{M}$ for almost every $\mathbf{Z} \in S$ [8, Lemma 3.3]. Using (4.30), (4.31), we now construct a further "rare" subsequence, not relabeled, such that

$$\frac{C+1}{n^2} \geqslant \int_S \big\{\chi\big(\mathbf{y}^n_{,1}|\mathbf{y}^n_{,2}|\mathbf{b}^n\big)\phi_m\big(\mathbf{y}^n_{,1}|\mathbf{y}^n_{,2}|\mathbf{b}^n\big) + \big(1-\chi\big(\mathbf{y}^n_{,1}|\mathbf{y}^n_{,2}|\mathbf{b}^n\big)\big)\phi_a\big(\mathbf{y}^n_{,1}|\mathbf{y}^n_{,2}|\mathbf{b}^n\big)\big\}\,\mathrm{d}Z_1\,\mathrm{d}Z_2. \tag{4.34}$$

This subsequence has the same Young measure, $(\hat{\nu}_{\mathbf{Z}})$, $\mathbf{Z} \in S$, but it is such that

$$n \int_S \big\{\chi\big(\mathbf{y}^n_{,1}|\mathbf{y}^n_{,2}|\mathbf{b}^n\big)\phi_m\big(\mathbf{y}^n_{,1}|\mathbf{y}^n_{,2}|\mathbf{b}^n\big) + \big(1-\chi\big(\mathbf{y}^n_{,1}|\mathbf{y}^n_{,2}|\mathbf{b}^n\big)\big)\phi_a\big(\mathbf{y}^n_{,1}|\mathbf{y}^n_{,2}|\mathbf{b}^n\big)\big\}\mathrm{d}Z_1\,\mathrm{d}Z_2 \to 0. \tag{4.35}$$

Note that by (4.26), (4.28) and the weak continuity of minors [3], we have

$$\int_S \mathbf{y}^n \cdot \left(\mathbf{y}^n_{,1} \wedge \mathbf{y}^n_{,2}\right) \mathrm{d}Z_1\, \mathrm{d}Z_2 \to \int_S \hat{\mathbf{y}} \cdot \left(\hat{\mathbf{y}}_{,1} \wedge \hat{\mathbf{y}}_{,2}\right) \mathrm{d}Z_1\, \mathrm{d}Z_2, \tag{4.36}$$

and this, together with (4.30), (4.31) and the fact that the support of $(\hat{\nu}_{\mathbf{Z}}), \mathbf{Z} \in S$ is contained in $\mathcal{A} \cup \mathcal{M}$, implies that the limit of $E^n(\mathbf{y}^n, \mathbf{b}^n)$ is given by (4.18) evaluated at $(\hat{\mathbf{y}}, \hat{\nu}_{\mathbf{Z}})$. To establish the minimum principle, we consider a family of Young measures $(\nu_{\mathbf{Z}}), \mathbf{Z} \in S$, supported on $\mathcal{A} \cup \mathcal{M}$, arising from a sequence $(\bar{\mathbf{y}}^n_{,1} | \bar{\mathbf{y}}^n_{,2} | \bar{\mathbf{b}}^n)$, with $(\bar{\mathbf{y}}^n, \bar{\mathbf{b}}^n) \in \mathcal{K}$, such that (without loss of generality)

$$\nabla \bar{\mathbf{y}}^n \rightharpoonup \mathbf{y} \quad \text{in } L^3(S), \tag{4.37}$$

$$\bar{\mathbf{b}}^n \rightharpoonup \mathbf{b} \quad \text{in } L^3(S), \tag{4.38}$$

$$\begin{aligned} n \int_S &\left\{ \chi\left(\bar{\mathbf{y}}^n_{,1} | \bar{\mathbf{y}}^n_{,2} | \bar{\mathbf{b}}^n\right) \phi_m\left(\bar{\mathbf{y}}^n_{,1} | \bar{\mathbf{y}}^n_{,2} | \bar{\mathbf{b}}^n\right) \right. \\ &\left. + \left(1 - \chi\left(\bar{\mathbf{y}}^n_{,1} | \bar{\mathbf{y}}^n_{,2} | \bar{\mathbf{b}}^n\right)\right) \phi_a\left(\bar{\mathbf{y}}^n_{,1} | \bar{\mathbf{y}}^n_{,2} | \bar{\mathbf{b}}^n\right) \right\} \mathrm{d}Z_1\, \mathrm{d}Z_2 \to 0. \end{aligned} \tag{4.39}$$

From (4.15) we find

$$\begin{aligned} E^n(\mathbf{y}^n, \mathbf{b}^n) &= \int_S \left\{ \chi\left(\mathbf{y}^n_{,1} | \mathbf{y}^n_{,2} | \mathbf{b}^n\right) \left[n\phi_m\left(\mathbf{y}^n_{,1} | \mathbf{y}^n_{,2} | \mathbf{b}^n\right) + l_m(\theta) \right] \right. \\ &\quad + \left(1 - \chi\left(\mathbf{y}^n_{,1} | \mathbf{y}^n_{,2} | \mathbf{b}^n\right)\right) \left[n\phi_a\left(\mathbf{y}^n_{,1} | \mathbf{y}^n_{,2} | \mathbf{b}^n\right) + l_a(\theta) \right] \\ &\quad \left. - \frac{P}{3} \mathbf{y}^n \cdot (\mathbf{y}^n_{,1} \wedge \mathbf{y}^n_{,2}) \right\} \mathrm{d}Z_1\, \mathrm{d}Z_2 \\ &\leqslant \inf_{(\mathbf{y}, \mathbf{b}) \in \mathcal{K}} E^n(\mathbf{y}, \mathbf{b}) + \frac{1}{n} \leqslant E^n(\bar{\mathbf{y}}^n, \bar{\mathbf{b}}^n) + \frac{1}{n} \\ &= \int_S \left\{ \chi\left(\bar{\mathbf{y}}^n_{,1} | \bar{\mathbf{y}}^n_{,2} | \bar{\mathbf{b}}^n\right) \left[n\phi_m\left(\bar{\mathbf{y}}^n_{,1} | \bar{\mathbf{y}}^n_{,2} | \bar{\mathbf{b}}^n\right) + l_m(\theta) \right] \right. \\ &\quad + \left(1 - \chi\left(\bar{\mathbf{y}}^n_{,1} | \bar{\mathbf{y}}^n_{,2} | \bar{\mathbf{b}}^n\right)\right) \left[n\phi_a\left(\bar{\mathbf{y}}^n_{,1} | \bar{\mathbf{y}}^n_{,2} | \bar{\mathbf{b}}^n\right) + l_a(\theta) \right] \\ &\quad \left. - \frac{P}{3} \bar{\mathbf{y}}^n \cdot \left(\bar{\mathbf{y}}^n_{,1} \wedge \bar{\mathbf{y}}^n_{,2}\right) \right\} \mathrm{d}Z_1\, \mathrm{d}Z_2 + \frac{1}{n}. \end{aligned} \tag{4.40}$$

Taking the limit as $n \to \infty$ and using (4.35), (4.36), (4.37), (4.38) and the weak continuity of minors, we obtain the minimum principle. □

According to Theorem 4.1, the behavior of the film is governed by the limit energy (4.18), defined for couples $(\nu_{\mathbf{Z}}; \mathbf{y})$, where $(\nu_{\mathbf{Z}}), \mathbf{Z} \in S$, is a family of Young measures supported on $\mathcal{M} \cup \mathcal{A}$, and $\mathbf{y}$ is a vector field satisfying the boundary conditions (4.20). The vector field $\mathbf{y}$ and the measures $\nu_{\mathbf{Z}}$ are related through the constraint (4.19). The vector field $\hat{\mathbf{b}}$ does not appear explicitly in the expression of the limit energy (4.18). However, because the family of Young measures $(\hat{\nu}_{\mathbf{Z}})$, $\mathbf{Z} \in S$, is generated by the sequence $(\mathbf{y}^n_{,1}|\mathbf{y}^n_{,2}|\mathbf{b}^n)$, we have

$$\hat{\mathbf{b}} = \int_{\mathcal{A}\cup\mathcal{M}} \mathbf{F}\mathbf{e}_3 \,\mathrm{d}\hat{\nu}_{\mathbf{Z}}(\mathbf{F}) \tag{4.41}$$

at almost every $\mathbf{Z} \in S$. The energy (4.18) is much easier to study than the original energy.

5. Tunnels

Thin film deformations involving gradients only from the martensite and the austenite wells are studied in [10, 11]. A particularly interesting deformation, especially in connection with the possible applications in the design of microactuators, is the *tunnel deformation*, sketched in Figure 2. To illustrate this deformation, we consider a martensitic film released on the rectangular region $S = (0, l_1) \times (0, l_2)$, for which we adopt boundary conditions more general than (4.20). In particular, we assume

$$\begin{cases} \mathbf{y}(Z_1, Z_2) = Z_1\mathbf{e}_1 + Z_2\mathbf{e}_2 & \text{for } Z_1 \in [0, l_1],\ Z_2 = 0, l_2, \\ \mathbf{y}(Z_1, Z_2)\cdot\mathbf{e}_1 = Z_1 & \text{for } Z_1 = 0, l_1,\ Z_2 \in [0, l_2]. \end{cases} \tag{5.1}$$

These conditions model the situation of a rectangular film attached to the substrate only along the edges parallel to the direction of $\mathbf{e}_1$. The edges parallel to the direction of $\mathbf{e}_2$ are restricted to move on planes perpendicular to the plane of the film.

To ensure the existence of the tunnel deformation, it is necessary to make suitable assumptions on the set of the martensite wells $\mathcal{M}$ [11]. In particular, we assume that the conditions

$$\mathbf{e}_3\cdot \mathrm{Adj}\big(\mathbf{U}^2 - \mathbf{I}\big)\mathbf{e}_3 = 0, \tag{5.2}$$

$$\mathrm{tr}\mathbf{U}^2 - \mathbf{e}_3\cdot\mathbf{U}^2\mathbf{e}_3 - 2 > 0 \tag{5.3}$$

hold for some symmetric matrix $\mathbf{U} \in \mathcal{M}$. These, in turn, are satisfied if and only if there exist a rotation $\mathbf{Q} \in SO(3)$ and a vector $\mathbf{e}$ such that

$$(\mathbf{QU} - \mathbf{I})\mathbf{e} = 0, \quad \mathbf{e}\cdot\mathbf{e}_3 = 0, \quad |\mathbf{e}| = 1, \tag{5.4}$$

$$\mathbf{n}\cdot\mathbf{U}^2\mathbf{e} = 0, \qquad \text{where } \mathbf{n} = \mathbf{e}\wedge\mathbf{e}_3, \tag{5.5}$$

$$|\mathbf{Un}| > 1, \tag{5.6}$$

[11, Proposition 5.2]. Condition (5.4) says that an interface between the austenite and a variant of martensite described by $\mathbf{U}$ can be formed in the direction of $\mathbf{e}$. Equation (5.5) is a condition of vanishing shear, while the inequality (5.6) says that the film is stretched in the direction perpendicular to the interface. If we orient the film so that the two directions of $\mathbf{e}$ and of $\mathbf{n}$ coincide with the directions of $\mathbf{e}_1$ and of $\mathbf{e}_2$, respectively, then the conditions from (5.4) to (5.6) become

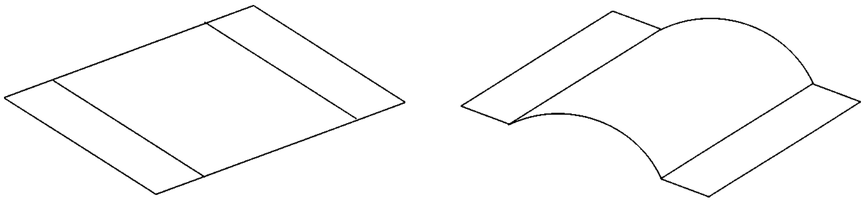

Figure 2. The flat and the tunnel configurations.

$$\mathbf{QUe}_1 = \mathbf{e}_1, \tag{5.7}$$

$$\mathbf{e}_1 \cdot \mathbf{U}^2\mathbf{e}_2 = 0. \tag{5.8}$$

$$|\mathbf{Ue_2}| > 1. \tag{5.9}$$

In the the constrained theory, the tunnel deformation is described by the sequence $(\mathbf{y}^n, \mathbf{b}^n)$ with $\mathbf{y}^n$ the sequence of cylindrical deformations

$$\mathbf{y}^n(Z_1, Z_2) = Z_1\mathbf{e}_1 + \hat{u}(Z_2)\mathbf{e}_2 + \hat{v}(Z_2)\mathbf{e}_3, \tag{5.10}$$

with

$$\hat{u}(Z_2) := |\mathbf{Ue}_2| \int_0^{Z_2} \cos\left(\alpha - \frac{2\alpha}{l_2}t\right)\mathrm{d}t, \tag{5.11}$$

$$\hat{v}(Z_2) := |\mathbf{Ue}_2| \int_0^{Z_2} \sin\left(\alpha - \frac{2\alpha}{l_2}t\right)\mathrm{d}t, \tag{5.12}$$

and with α the solution in $(0, 2\pi)$ to the equation

$$\sin\alpha = \frac{\alpha}{|\mathbf{Ue}_2|}. \tag{5.13}$$

The sequence of Cosserat directors $\mathbf{b}^n$ is given by $\mathbf{R}(Z_2)\overline{\mathbf{R}}\mathbf{QUe}_3, \mathbf{R}(Z_2)$ and $\overline{\mathbf{R}}$ being the rotation matrices

$$\begin{aligned}\mathbf{R}(Z_2) &:= \mathbf{e}_1 \otimes \mathbf{e}_1 + \cos\left(\alpha - \frac{2\alpha}{l_2}Z_2\right)[\mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3] \\ &\quad + \sin\left(\alpha - \frac{2\alpha}{l_2}Z_2\right)[-\mathbf{e}_2 \otimes \mathbf{e}_3 + \mathbf{e}_3 \otimes \mathbf{e}_2],\end{aligned} \tag{5.14}$$

$$\begin{aligned}\overline{\mathbf{R}} &:= \mathbf{e}_1 \otimes \mathbf{e}_1 + \frac{\mathbf{QUe}_2 \cdot \mathbf{e}_2}{|\mathbf{Ue}_2|}[\mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3] \\ &\quad - \frac{\mathbf{QUe}_2 \cdot \mathbf{e}_3}{|\mathbf{Ue}_2|}[-\mathbf{e}_2 \otimes \mathbf{e}_3 + \mathbf{e}_3 \otimes \mathbf{e}_2].\end{aligned} \tag{5.15}$$

In the plane $(\mathbf{e}_2, \mathbf{e}_3)$, the couple $(\hat{u}, \hat{v})$ describes the circular arch of length $l_2|\mathbf{Ue}_2|$, starting at the point $(0, 0)$, ending at $(0, l_2)$, and lying in the positive half-plane. The family of Young measures arising from the sequence $(\mathbf{y}^n_{,1}|\mathbf{y}^n_{,2}|\mathbf{b}^n)$ is simply a Dirac mass centered at $\mathbf{R}(Z_2)\overline{\mathbf{R}}\mathbf{QU}$:

$$\nu_{\mathbf{Z}} = \delta_{\mathbf{R}(Z_2)\bar{\mathbf{R}}\mathbf{QU}}. \tag{5.16}$$

It is easy to see that the tunnel deformation is energy minimizing in the constrained theory when $\theta \leqslant \theta_{cr}$ and no pressure is applied under the film. In this case, the total energy (4.18) reduces to

$$e(\nu_{\mathbf{Z}}; \mathbf{y}) = \int_S \left\{ \big(l_m(\theta) - l_a(\theta)\big) \int_{\mathcal{M}} \mathrm{d}\nu_{\mathbf{Z}}(\mathbf{F}) \right\} \mathrm{d}Z_1\, \mathrm{d}Z_2 + l_a(\theta)|S|, \tag{5.17}$$

to be minimized among the Young measures $\nu_{\mathbf{Z}}$ such that

$$\text{supp } \nu_{\mathbf{Z}} \subset \mathcal{A} \quad \text{if } \theta > \theta_{cr}, \tag{5.18}$$

$$\text{supp } \nu_{\mathbf{Z}} \subset \mathcal{M} \quad \text{if } \theta < \theta_{cr}, \tag{5.19}$$

$$\text{supp } \nu_{\mathbf{Z}} \subset \mathcal{A} \cup \mathcal{M} \quad \text{if } \theta = \theta_{cr}, \tag{5.20}$$

and whose center of mass satisfies the constraint (4.19) and the boundary conditions (5.1). If $\theta \leqslant \theta_{cr}$, then, in view of (5.19), the couple $(\nu_{\mathbf{Z}}; \mathbf{y})$ given by (5.16) and (5.10) is minimizing. If $\theta > \theta_{cr}$, then, from (5.18), minimizers involve only the austenite phase, and therefore the couple $(\delta_{\mathbf{I}}, \mathbf{Z})$, with $\delta_{\mathbf{I}}$ the Dirac mass centered at the identity $\mathbf{I}$ and arising from a sequence of "flat" deformations, is minimizing. The reversible, temperature activated change of stability between the tunnel and the flat configurations, both sketched in Figure 2, makes it possible to employ the film as an actuator [10, 11].

Let us turn to the case $P \neq 0$. Now the total energy (4.18) is the sum of the bulk energy (5.17) and the free energy of the gas $-PV(\mathbf{y})$, and thus minimizing deformations $\mathbf{y}$ involve gradients from the wells which maximize the volume $V(\mathbf{y})$. In this respect, the tunnel deformation is a good candidate to be a minimizer. In the next subsection, we prove that the tunnel deformation is a minimizer in certain ranges of pressure and temperature. In the proof, we restrict ourselves to consider only cylindrical deformations $\mathbf{y}$, but we believe that our results also hold under weaker hypotheses on $\mathbf{y}$.

5.1. CYLINDRICAL DEFORMATIONS

Let us consider deformation $(\nu_{\mathbf{Z}}, \mathbf{y})$ of the constrained theory with $\nu_{\mathbf{Z}}$ independent of Z_1 and with $\mathbf{y}$ of the type (5.10). Again we assume $\mathcal{M}$ containing a matrix with positive strain so that

$$\gamma := \max\big\{|\mathbf{F}\mathbf{e}_2| : \mathbf{F} \in \mathcal{M}\big\} > 1. \tag{5.21}$$

We assume the film made by a good "tunnel material", so that the maximizer of (5.21) satisfies (5.4) and (5.5). Using the constraint (4.19) and the kinematic assumption (5.10), we find

$$u(Z_2) = \int_{\mathcal{M}} \mathbf{F}\mathbf{e}_2 \cdot \mathbf{e}_2\, \mathrm{d}\nu_{Z_2}(\mathbf{F}), \tag{5.22}$$

$$v(Z_2) = \int_{\mathcal{M}} \mathbf{F}\mathbf{e}_2 \cdot \mathbf{e}_3\, \mathrm{d}\nu_{Z_2}(\mathbf{F}), \tag{5.23}$$

at almost every $Z_2 \in (0, l_2)$. Besides, each deformation of the type (5.10) automatically satisfies the boundary condition $(5.1)_2$, while $(5.1)_1$ gives

$$u(0) = v(0) = 0, \tag{5.24}$$

$$u(l_2) = l_2, \qquad v(l_2) = 0. \tag{5.25}$$

In view of (5.10), (5.24), and (5.25), the volume of the gas under the film has the expression

$$V(\mathbf{y}) = \int_0^{l_2} u'(Z_2)v(Z_2)\,\mathrm{d}Z_2, \tag{5.26}$$

where the prime denotes the first derivative. Therefore, the energy (4.18) reduces to

$$\begin{aligned} e(\nu_{\mathbf{Z}}; \mathbf{y}) &= l_1\left\{\big(l_m(\theta) - l_a(\theta)\big)l_2\bar{\lambda} + l_a(\theta)l_2 - P\int_0^{l_2} u'(Z_2)v(Z_2)\,\mathrm{d}Z_2\right\} \\ &=: \mathcal{E}(\bar{\lambda}; u, v), \end{aligned} \tag{5.27}$$

where

$$\bar{\lambda} := \frac{1}{l_2}\int_0^{l_2}\int_{\mathcal{M}} \mathrm{d}\nu_{Z_2}(\mathbf{F})\,\mathrm{d}Z_2 \in [0, 1] \tag{5.28}$$

indicates the average volume fraction of martensite along the direction of $\mathbf{e}_2$. Using the constraints (5.22), (5.23), the triangle inequality, the definitions (5.21) and (5.28), and recalling that ν_{Z_2} is a probability measure, we get

$$\begin{aligned} \int_0^{l_2} \sqrt{(u')^2 + (v')^2}\,\mathrm{d}Z_2 &\leqslant \int_0^{l_2}\left|\int_{\mathcal{M}} \mathbf{F}\mathbf{e}_2\,\mathrm{d}\nu_{Z_2}\right|\mathrm{d}Z_2 \\ &\leqslant \int_0^{l_2}\int_{\mathcal{M}} |\mathbf{F}\mathbf{e}_2|\,\mathrm{d}\nu_{Z_2}\,\mathrm{d}Z_2 \leqslant l_2(\gamma\bar{\lambda} + 1 - \bar{\lambda}). \end{aligned} \tag{5.29}$$

This inequality provides an upper bound on the length of the curve describing the deformed configuration of the cross-section of the film. Ignoring other possible compatibility conditions between $\bar{\lambda}$ and (u, v) arising from the constraints (5.22), (5.23), we minimize the energy (5.27) with $(\bar{\lambda}; u, v)$ satisfying the constraint (5.29) and the boundary conditions (5.24), (5.25), thereby giving a lower bound for $\mathcal{E}(\bar{\lambda}; u, v)$. We begin by keeping $\bar{\lambda}$ fixed and by minimizing over (u, v). This means that we seek the curve lying in the $(\mathbf{e}_2, \mathbf{e}_3)$ plane, joining the origin with the point $(0, l_2)$, having length not greater than $l_2(\gamma\bar{\lambda} + 1 - \bar{\lambda})$ and enclosing the largest area. In Appendix 2, we prove that the circular arch of length $l_2(\gamma\bar{\lambda}+1-\bar{\lambda})$, parametrized by the couple (U, V) with

$$U(Z_2) := (\gamma\bar{\lambda} + 1 - \bar{\lambda})\int_0^{Z_2} \cos\left(\alpha - \frac{2\alpha}{l_2}t\right)\mathrm{d}t, \tag{5.30}$$

$$V(Z_2) := (\gamma\bar{\lambda} + 1 - \bar{\lambda})\int_0^{Z_2} \sin\left(\alpha - \frac{2\alpha}{l_2}t\right)\mathrm{d}t, \tag{5.31}$$

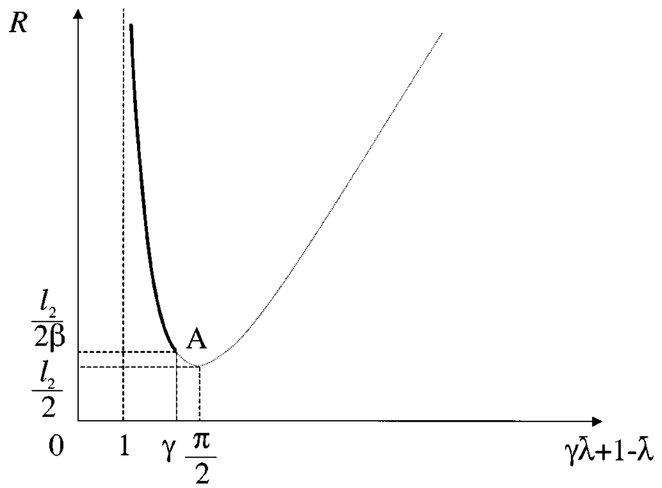

Figure 3. Radius R of the cross-section of the tilm versus $\gamma\bar{\lambda}+1-\bar{\lambda}$.

and α the solution in $(0, 2\pi)$ to the equation

$$\sin\alpha = \frac{\alpha}{(\gamma\bar{\lambda}+1-\bar{\lambda})}, \tag{5.32}$$

encloses the largest area, given by

$$\int_0^{l_2} U'(Z_2)V(Z_2)\,\mathrm{d}Z_2 = \begin{cases} \frac{1}{2}l_2R(\gamma\bar{\lambda}+1-\bar{\lambda}) - \frac{l_2}{2}\sqrt{R^2-\frac{l_2^2}{4}}, & \text{for } 1 < (\gamma\bar{\lambda}+1-\bar{\lambda}) \leqslant \frac{\pi}{2}, \\ \frac{1}{2}l_2R(\gamma\bar{\lambda}+1-\bar{\lambda}) + \frac{l_2}{2}\sqrt{R^2-\frac{l_2^2}{4}}, & \text{for } (\gamma\bar{\lambda}+1-\bar{\lambda}) > \frac{\pi}{2}. \end{cases} \tag{5.33}$$

In the last equation, R, which denotes the radius of the arch parametrized by (U, V), satisfies the implicit relation

$$\sin\left(\left(\gamma\bar{\lambda}+1-\bar{\lambda}\right)\frac{l_2}{2R}\right) = \frac{l_2}{2R}, \tag{5.34}$$

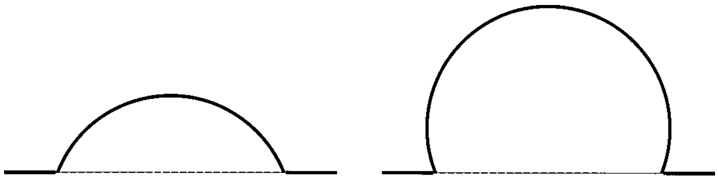

Figure 4. Two possible tunnel configurations.

plotted in Figure 3. The presence of the two expression in (5.33) refers to the two cases shown in Figure 4. Substituting (5.33) into the expression of the total energy (5.27), we get the lower bound

$$\mathcal{E}(\bar{\lambda}; u, v) \geqslant \begin{cases} l_1 l_2 \Big\{ \big(l_m(\theta) - l_a(\theta)\big)\bar{\lambda} + l_a(\theta) - \dfrac{P}{2}(\gamma\bar{\lambda} + 1 - \bar{\lambda})R \\ \quad + \dfrac{P}{2}\sqrt{R^2 - \dfrac{l_2^2}{4}} \Big\}, \quad \text{for } 1 < (\gamma\bar{\lambda} + 1 - \bar{\lambda}) \leqslant \dfrac{\pi}{2}, \\ l_1 l_2 \Big\{ \big(l_m(\theta) - l_a(\theta)\big)\bar{\lambda} + l_a(\theta) - \dfrac{P}{2}(\gamma\bar{\lambda} + 1 - \bar{\lambda})R \\ \quad - \dfrac{P}{2}\sqrt{R^2 - \dfrac{l_2^2}{4}} \Big\}, \quad \text{for } (\gamma\bar{\lambda} + 1 - \bar{\lambda}) > \dfrac{\pi}{2}. \end{cases} \tag{5.35}$$

Because the right-hand side turns out to depend only upon $\bar{\lambda}$, the bound can be further improved by minimizing with respect to $\bar{\lambda}$. Let $\bar{\lambda}_{\min} \in [0, 1]$ denote the minimizer. Now we show that there exists a family of Young measures which achieves the lower bound (5.35). This family arises from the sequence $(\mathbf{y}^n_{,1}|\mathbf{y}^n_{,2}|\mathbf{b}^n)$ with

$$\mathbf{y}^n(Z_1, Z_2) = Z_1\mathbf{e}_1 + u_n(Z_2)\mathbf{e}_2 + v_n(Z_2)\mathbf{e}_3, \tag{5.36}$$

$$u_n(Z_2) := \int_0^{Z_2} r_n(t)\cos(\beta_n t + \alpha_n)\,\mathrm{d}t, \tag{5.37}$$

$$v_n(Z_2) := \int_0^{Z_2} r_n(t)\sin(\beta_n t + \alpha_n)\,\mathrm{d}t, \tag{5.38}$$

and with r_n the piecewise constant periodic function with period l_2/n such that

$$r_n(t) := \begin{cases} \gamma & \text{for } p\frac{l_2}{n} \leqslant t \leqslant p\frac{l_2}{n} + \bar{\lambda}_{\min}\frac{l_2}{n}, \\ 1 & \text{for } p\frac{l_2}{n} + \bar{\lambda}_{\min}\frac{l_2}{n} < t \leqslant (p+1)\frac{l_2}{n}, \end{cases} \tag{5.39}$$

p being an integer between 0 and $n - 1$. The sequence of Cosserat directors $\mathbf{b}^n$ is defined by

$$\mathbf{b}^n(Z_2) = \mathbf{R}(Z_2)\overline{\mathbf{R}}\mathbf{QUe}_3 \quad \text{wherever } r_n(Z_2) = \gamma, \tag{5.40}$$

$$\mathbf{b}^n(Z_2) = \mathbf{R}(Z_2)\mathbf{e}_3 \quad \text{wherever } r_n(Z_2) = 1. \tag{5.41}$$

Here, $\mathbf{R}(Z_2)$ and $\overline{\mathbf{R}}$ are the rotations (5.14), (5.15), $\mathbf{U}$ a solution to the maximum problem in (5.21), and $\mathbf{Q}$ a rotation such that $\mathbf{QU}$ satisfies the condition (5.7)–(5.9) for building up a tunnel.

By construction, $u_n(0) = v_n(0) = 0$. In Appendix 3, we prove the existence of $\alpha_n^*, \beta_n^* \in [0, 2\pi]$ for which the boundary conditions (5.25) are also satisfied. We also show that

$$\alpha_n^* \to \alpha, \tag{5.42}$$

$$\beta_n^* \to -\frac{2\alpha}{l_2}, \tag{5.43}$$

up to a subsequence, with α as in (5.32). Using (5.42), (5.43), the weak convergence of r_n and $\mathbf{b}^n$ to their averages [14, Theorem 1.5], and the definitions (5.14), (5.15), we find

$$\big(\mathbf{y}^n_{,1}|\mathbf{y}^n_{,2}|\mathbf{b}^n\big) \overset{*}{\rightharpoonup} \bar{\lambda}_{\min}\mathbf{R}(Z_2)\overline{\mathbf{R}}\mathbf{QU} + \big(1 - \bar{\lambda}_{\min}\big)\mathbf{R}(Z_2) \quad \text{in } L^\infty. \tag{5.44}$$

To compute the Young measure generated by $(\mathbf{y}^n_{,1}|\mathbf{y}^n_{,2}|\mathbf{b}^n)$, we note that for any function $\psi \in C_0(M^{3\times3}; \mathbb{R})$

$$\psi\big(\mathbf{y}^n_{,1}|\mathbf{y}^n_{,2}|\mathbf{b}^n\big) \overset{*}{\rightharpoonup} \bar{\lambda}_{\min}\psi(\mathbf{R}(Z_2)\overline{\mathbf{R}}\mathbf{QU}) + (1 - \bar{\lambda}_{\min})\psi\big(\mathbf{R}(Z_2)\big) \quad \text{in } L^\infty \tag{5.45}$$

[23, Corollary 3.3]. Thus, in view of (4.13), the family of Young measures arising from $(\mathbf{y}^n_{,1}|\mathbf{y}^n_{,2}|\mathbf{b}^n)$ is given by

$$\hat{\nu}_{Z_2} = \bar{\lambda}_{\min}\delta_{\mathbf{R}(Z_2)\overline{\mathbf{R}}\mathbf{QU}} + (1 - \bar{\lambda}_{\min})\delta_{\mathbf{R}(Z_2)}. \tag{5.46}$$

Because

$$\left|\int_{\mathcal{M}} \mathbf{F}\mathbf{e}_2 \, \mathrm{d}\hat{\nu}_{Z_2}\right| = \bar{\lambda}_{\min}\gamma + 1 - \bar{\lambda}_{\min}, \tag{5.47}$$

the family of Young measures $\hat{\nu}$ achieves the lower bound in (5.35).

Let us now turn to evaluating $\bar{\lambda}_{\min}$. We differentiate the right-hand side of (5.35) with respect to $\bar{\lambda}$ and use (5.34) to get

$$\frac{\mathrm{d}\mathcal{E}(\bar{\lambda}; U, V)}{\mathrm{d}\bar{\lambda}} = l_1 l_2\big[(l_m(\theta) - l_a(\theta)) - P(\gamma - 1)R\big], \tag{5.48}$$

which vanishes at the unique solution

$$R = \frac{(l_m(\theta) - l_a(\theta))}{P(\gamma - 1)}. \tag{5.49}$$

While there is a unique stationary point R, the corresponding value of $\bar{\lambda}$ may not be unique (i.e., see Figure 3 with $\gamma > \pi/2$). The minimizing values of $\bar{\lambda}$ are the following:

- Case 1.

$$\gamma \leqslant \pi/2, \quad \text{and} \quad P < 2\beta \frac{(l_m(\theta) - l_a(\theta))}{l_2(\gamma - 1)} =: P_1,$$

with β the solution in $(0, \pi/(2\gamma))$ to the equation

$$\sin(\gamma\beta) = \beta. \tag{5.50}$$

There is a unique minimizer $\bar{\lambda}_{\min}$ given by

$$\bar{\lambda}_{\min} = \frac{2(l_m(\theta) - l_a(\theta))}{l_2 P(\gamma - 1)^2} \arcsin\left[\frac{l_2 P(\gamma - 1)}{2(l_m(\theta) - l_a(\theta)}\right] - \frac{1}{(\gamma - 1)}. \tag{5.51}$$

- Case 2. $\gamma \leqslant \pi/2$, and $P \geqslant P_1$. The unique global minimizer is $\bar{\lambda}_{\min} = 1$.
- Case 3.

$$\gamma > \pi/2, \quad \text{and} \quad P < 2\xi \frac{(l_m(\theta) - l_a(\theta))}{l_2(\gamma - 1)} =: P_2,$$

with $\xi \in (0, 1)$ the solution to the equation

$$\frac{1}{\xi^2} \arcsin \xi + \frac{1}{\xi}\sqrt{1 - \xi^2} = \frac{2\gamma}{\xi} - \frac{1}{\beta}\left[\gamma + \sqrt{1 - \beta^2}\right], \tag{5.52}$$

where β is now the solution in $(\pi/(2\gamma); 1)$ to (5.50). The unique global minimizer is again given by (5.51).
- Case 4. $\gamma > \pi/2$, and $P = P_2$. There are two global minimizer, one at $\bar{\lambda}$ given by (5.51) and one at $\bar{\lambda}_{\min} = 1$.
- Case 5. $\gamma > \pi/2$, and $P > P_2$. The unique global minimizer is $\bar{\lambda}_{\min} = 1$.

In Case 3, there is also a relative minimum at $\bar{\lambda} = 1$ if

$$2\beta \frac{(l_m(\theta) - l_a(\theta))}{l_2(\gamma - 1)} < P < P_2, \tag{5.53}$$

while in Case 5 there is a relative minimum at $\bar{\lambda}$ given by (5.51) if

$$P_2 < P < 2\frac{(l_m(\theta) - l_a(\theta))}{l_2(\gamma - 1)}. \tag{5.54}$$

To give a physical interpretation of these results, in all cases we fix the temperature θ above the transformation temperature and the material (so γ is fixed), and we increase the pressure starting from an appropriate value. In Case 1, a fine mixture of austenite and martensite is globally stable. The volume fraction of martensite $\bar{\lambda}_{\min}$ increases as P increases starting from zero. Correspondingly, the length of the cross-section, which has the form of a circular arch, increases and the film encloses

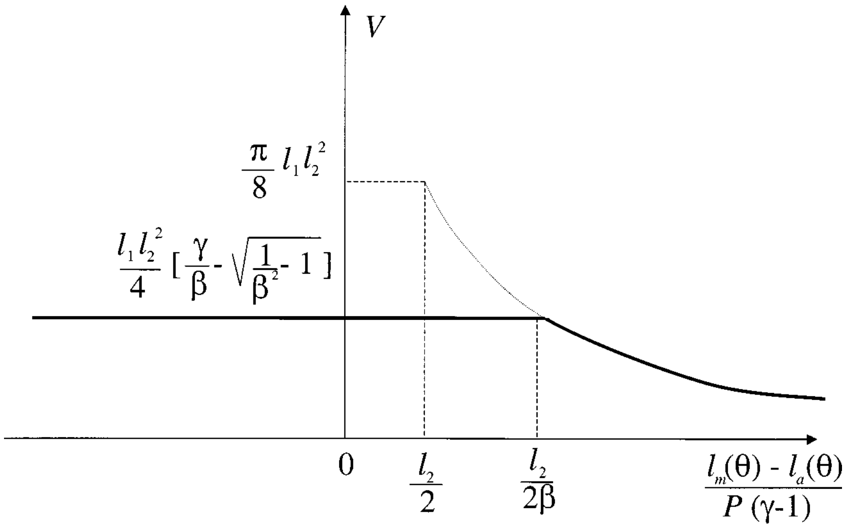

Figure 5. Volume V enclosed by the film versus the ratio $(l_m(\theta) - l_a(\theta))/(P(\gamma - 1))$ in the case $\gamma \in (1, \pi/2]$.

more and more volume. From (5.33), (5.49), and (5.51), we find that this volume is given by

$$V = \frac{l_1 l_2^2}{4}\left[\frac{4(l_m(\theta) - l_a(\theta))^2)}{P l_2^2(\gamma - 1)^2} \arcsin\left(\frac{(P l_2(\gamma - 1))}{2(l_m(\theta) - l_a(\theta))}\right) - \sqrt{\frac{4(l_m(\theta) - l_a(\theta))^2)}{P l_2^2(\gamma - 1)^2} - 1}\,\right], \tag{5.55}$$

which corresponds to the curve plotted in Figure 5. For $P = P_1$ (see Case 2), we have $\bar{\lambda}_{\min} = 1$ and thus the austenite has completely transformed. At this point the length of the cross-section reaches its maximum value $l_2\gamma$ and the film encloses the volume

$$V = \frac{l_1 l_2^2}{4}\left[\frac{\gamma}{\beta} - \frac{1}{\beta}\sqrt{1 - \beta^2}\,\right], \tag{5.56}$$

with β defined as in Case 1. If the pressure is further increased from P_1, the length of the cross-section and the volume enclosed remain constant.

In Case 3, as the pressure increases, a mixture of austenite and martensite with increasing volume fraction (5.51) is globally stable. The volume enclosed by the film is still given by the relation (5.55) plotted now in Figure 6. For P as in (5.53), the martensite becomes a metastable configuration and at $P = P_2$, (see Case 4), both the martensite and the mixture are globally stable. If P is further increased (Case 5), then the mixture becomes metastable while the martensite becomes globally stable. The presence of metastable states introduces the possibility of a hysteresis loop in Figure 6.

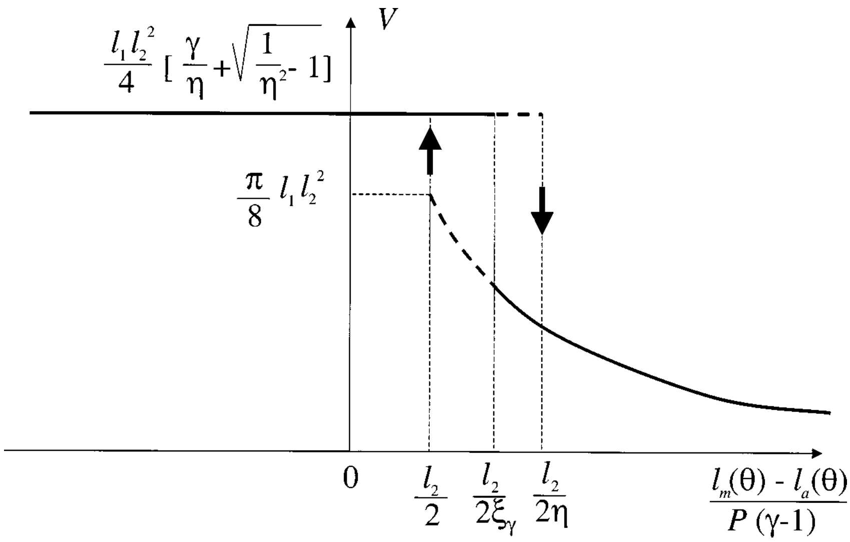

Figure 6. Volume V enclosed by the film versus the ratio $(l_m(\theta) - l_a(\theta))/(P(\gamma - 1))$ in the case $\gamma \in (\pi/2, +\infty)$. Dashed curve corresponds to relative minimizers.

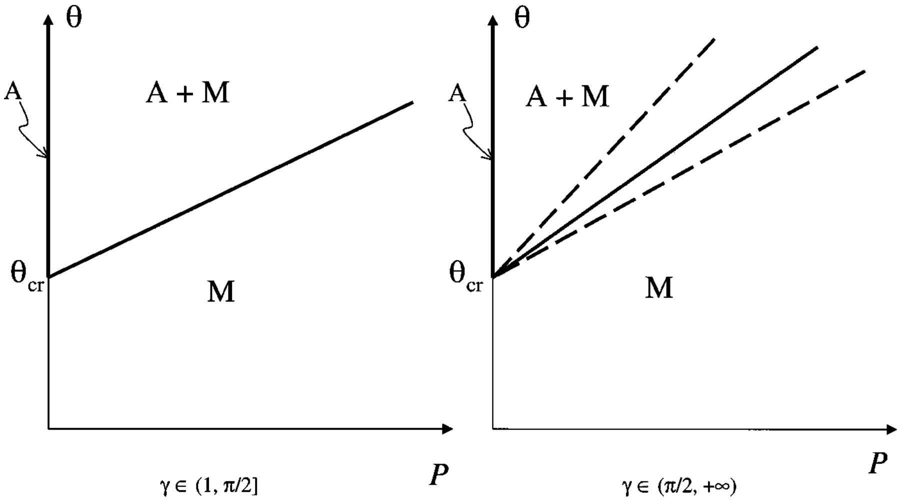

Figure 7. Pressure–temperature phase diagram for the material of the film. The cases drawn correspond to linear relations $l_m(\theta)$, $l_a(\theta)$. The sector enclosed by dashed lines contains metastable states. Note that some martensite is present even at high temperatures.

If we assume $l_m(\theta)$ and $l_a(\theta)$ to depend linearly upon the temperature, we can summarize these results in the pressure-temperature phase diagram depicted in Figure 7, in which the two cases $\gamma \leqslant \pi/2$, and $\gamma > \pi/2$ are drawn separately. For values of the pressure and of the temperature lying in the regions marked with M, the tunnel deformation with $\mathbf{y}$ given by (5.10), (5.30), (5.31) with $\bar{\lambda}_{\min} = 1$, and $\mathbf{b} = \mathbf{R}(Z_2)\overline{\mathbf{R}}\mathbf{QUe}_3$ is globally stable. At each point of the film, the material

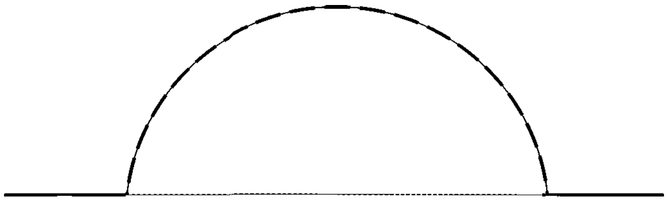

Figure 8. Microstructure of the film at equilibrium.

is in the variant of martensite **U**. For values of pressure and temperature lying in the regions marked with $A + M$, the macroscopic deformation of the film is now given by the tunnel deformation (5.10), (5.30), (5.31) evaluated at $\bar{\lambda} = \bar{\lambda}_{\min}$, and by $\mathbf{b} = (\bar{\lambda}_{\min}\mathbf{R}(Z_2)\overline{\mathbf{R}}\mathbf{QU} + (1 - \bar{\lambda}_{\min}))\mathbf{R}(Z_2))\mathbf{e}_3$; correspondingly, the material is a fine mixture of austenite and of the variant of martensite **U**. The family of Young measures (5.46) describes the microstructure of the material. Recalling the construction of the sequence generating the measures (5.46), the microstructure is found to consist of martensitic regions alternated with austenitic region, both regions having the shape of thin strips parallel to the axis of the tunnel, as depicted in Figure 8.

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Appendix A

We now give a proof of the identity (2.26).

LEMMA A.1. *Let* $\mathbf{y} \in \mathcal{H}$ *and* $t > 0$. *Then*

$$\frac{1}{3}\int_{S\times\{t\}} \mathbf{y}\cdot(\mathbf{y}_{,1}\wedge\mathbf{y}_{,2})\,\mathrm{d}X_1\,\mathrm{d}X_2 - \frac{1}{3}\int_{S\times\{0\}} \mathbf{y}\cdot(\mathbf{y}_{,1}\wedge\mathbf{y}_{,2})\,\mathrm{d}X_1\,\mathrm{d}X_2 + \frac{2}{3}t|S|\det\mathbf{A} = \int_{S\times(0,t)} \det\nabla\mathbf{y}\,\mathrm{d}\mathbf{X}. \tag{A.1}$$

Proof. Let $\mathbf{y}^n \in C^\infty(\Omega^h;\mathbb{R}^3)$ be a smooth sequence approximating $\mathbf{y}$ in $\mathcal{H}$. Denoting with ϵ_{ijk} the Ricci tensor, we have

$$\int_{S\times(0,t)} \det\nabla\mathbf{y}^n\,\mathrm{d}\mathbf{X} = \int_{S\times(0,t)} \frac{1}{6}\,\epsilon_{ijk}\epsilon_{pqr}y^n_{i,p}y^n_{j,q}y^n_{k,r}\,\mathrm{d}\mathbf{X}. \tag{A.2}$$

By integrating by parts and applying the divergence theorem, we obtain

$$\int_{S\times(0,t)} \det\nabla \mathbf{y}^n \, \mathrm{d}\mathbf{X} = -\int_{S\times(0,t)} \frac{1}{6}\epsilon_{ijk}\epsilon_{pqr}\left(\mathrm{y}^n_{i,p}\mathrm{y}^n_{j,q}\right)_{,r}\mathrm{y}^n_k \, \mathrm{d}\mathbf{X} + \int_{\partial(S\times(0,t))} \frac{1}{6}\epsilon_{ijk}\epsilon_{pqr} y^n_{i,p} y^n_{j,q} y^n_k N_r \, \mathrm{d}a, \tag{A.3}$$

where $\mathbf{N} = N_1\mathbf{e}_1 + N_2\mathbf{e}_2 + N_3\mathbf{e}_3$ is the outward unit normal to $\partial(S \times (0, t))$. The first term on the right hand side of (A.3) is zero because it is the inner product of symmetric and antisymmetric tensors. Hence,

$$\begin{aligned}\int_{S\times(0,t)} \det\nabla \mathbf{y}^n \, \mathrm{d}\mathbf{X} &= \int_{\partial(S\times(0,t))} \frac{1}{3}\left(\mathrm{Adj}\nabla\mathbf{y}^n\right)\mathbf{y}^n \cdot \mathbf{N}\,\mathrm{d}a,\\ &= -\int_{S\times\{0\}} \frac{1}{3}\left(\mathrm{Adj}\nabla\mathbf{y}^n\right)\mathbf{y}^n \cdot \mathbf{e}_3 \,\mathrm{d}X_1\,\mathrm{d}X_2\\ &\quad + \int_{S\times\{t\}} \frac{1}{3}\left(\mathrm{Adj}\nabla\mathbf{y}^n\right)\mathbf{y}^n \cdot \mathbf{e}_3 \,\mathrm{d}X_1\,\mathrm{d}X_2\\ &\quad + \int_{\partial S\times\{0,t\}} \frac{1}{3}\left(\mathrm{Adj}\nabla\mathbf{y}^n\right)\mathbf{y}^n \cdot \mathbf{N}\,\mathrm{d}a. \end{aligned} \tag{A.4}$$

Let $\mathbf{X}(s)$ be a parametrization of the boundary of S, with $\mathbf{X}(0) = \mathbf{X}(1)$. Then, $(\mathbf{X}(s), X_3)$ is a parametrization of the surface $\partial S \times \{0, t\}$, which is assumed to orient the surface $\partial S \times \{0, t\}$ so that the vector

$$\frac{\mathbf{X}_{,s} \wedge \mathbf{e}_3}{|\mathbf{X}_{,s} \wedge \mathbf{e}_3|} \tag{A.5}$$

is the outward pointing unit normal. Using the identities

$$(\mathrm{Adj}\mathbf{A})^{\mathrm{T}}(\mathbf{b} \wedge \mathbf{c}) = \mathbf{Ab} \wedge \mathbf{Ac},$$
$$(\mathrm{Adj}\mathbf{A})\mathbf{A} = (\det\mathbf{A})\mathbf{I},$$

and the boundary conditions (2.2), we obtain

$$\begin{aligned}&\int_{\partial S\times\{0,t\}} \left(\mathrm{Adj}\nabla\mathbf{y}^n(\mathbf{X})\right)\mathbf{y}^n(\mathbf{X}) \cdot \mathbf{N}(\mathbf{X})\,\mathrm{d}a\\ &= \int_0^t\int_0^1 \left(\mathrm{Adj}\nabla\mathbf{y}^n(\mathbf{X}(s), X_3)\right)\mathbf{y}^n(\mathbf{X}(s), X_3) \cdot (\mathbf{X}(s)_{,s} \wedge \mathbf{e}_3)\,\mathrm{d}s\,\mathrm{d}X_3\\ &= \int_0^t\int_0^1 \mathbf{y}^n\left(\mathbf{X}(s), X_3\right) \cdot \left(\mathrm{Adj}\nabla\mathbf{y}^n(\mathbf{X}(s), X_3)\right)^{\mathrm{T}}(\mathbf{X}(s)_{,s} \wedge \mathbf{e}_3)\,\mathrm{d}s\,\mathrm{d}X_3\\ &= \int_0^t\int_0^1 \mathbf{y}^n\left(\mathbf{X}(s), X_3\right) \cdot \left(\nabla\mathbf{y}^n(\mathbf{X}(s), X_3)\mathbf{X}(s)_{,s} \wedge \nabla\mathbf{y}^n\left(\mathbf{X}(s), X_3\right)\mathbf{e}_3\right)\mathrm{d}s\,\mathrm{d}X_3\\ &= \int_0^t\int_0^1 \mathbf{A}\left(\mathbf{X}(s), X_3\right) \cdot \left(\mathbf{A}\mathbf{X}(s)_{,s} \wedge \mathbf{A}\mathbf{e}_3\right)\mathrm{d}s\,\mathrm{d}X_3\end{aligned}$$

$$= \int_0^t \int_0^1 \mathbf{A}\big(\mathbf{X}(s), X_3\big) \cdot (\mathrm{Adj}\mathbf{A})^{\mathrm{T}}\big(\mathbf{X}(s)_{,s} \wedge \mathbf{e}_3\big)\, \mathrm{d}s\, \mathrm{d}X_3$$
$$= \int_0^t \int_0^1 (\mathrm{Adj}\mathbf{A})\mathbf{A}\big(\mathbf{X}(s), X_3\big) \cdot \big(\mathbf{X}(s)_{,s} \wedge \mathbf{e}_3\big)\, \mathrm{d}s\, \mathrm{d}X_3$$
$$= \det\mathbf{A} \int_0^t \int_0^1 (\mathbf{X}(s), X_3) \cdot \big(\mathbf{X}(s)_{,s} \wedge \mathbf{e}_3\big)\, \mathrm{d}s\, \mathrm{d}X_3. \tag{A.6}$$

By the divergence theorem,

$$\int_0^1 \big(\mathbf{X}(s), X_3\big) \cdot \big(\mathbf{X}(s)_{,s} \wedge \mathbf{e}_3\big)\, \mathrm{d}s = \int_S \mathrm{div}\mathbf{X}\, \mathrm{d}\mathbf{X} = 2|S|, \tag{A.7}$$

which, substituted into (A.6), gives

$$\int_{\partial S \times \{0,t\}} \big(\mathrm{Adj}\nabla\mathbf{y}^n\big)\mathbf{y}^n \cdot \mathbf{N}\, \mathrm{d}a = 2t|S|\det A. \tag{A.8}$$

This, together with (A.4) and the identity $(\mathrm{Adj}\nabla\mathbf{y})\mathbf{y}\cdot\mathbf{e}_3 = \mathbf{y}\cdot(\mathbf{y}_{,1}\wedge\mathbf{y}_{,2})$, gives (A.1) for the approximating sequence $\mathbf{y}^n$. Letting $n \to \infty$, by the continuity in $W^{2,2}$ of the volume functional and of the last term in (A.1), we obtain (A.1) for $\mathbf{y}$. □

Appendix B

We prove a convenient 2-dimensional version of the isoperimetric inequality.

LEMMA B.2. *Let $\gamma > 1$ and $\bar{\lambda} \in [0, 1]$ be given. Then, for any couple $(u, v) \in (W^{1,3}(0, l_2))^2$ satisfying the boundary conditions (5.24), (5.25) and the constraint (5.29),*

$$\int_0^{l_2} u'(Z_2)v(Z_2)\, \mathrm{d}Z_2 \leqslant \begin{cases} \dfrac{1}{2}l_2R(\gamma\bar{\lambda}+1-\bar{\lambda}) - \dfrac{l_2}{2}\sqrt{R^2 - \dfrac{l_2^2}{4}}, \\ \quad \text{for } 1 < \gamma\bar{\lambda}+1-\bar{\lambda} \leqslant \dfrac{\pi}{2}, \\ \dfrac{1}{2}l_2R(\gamma\bar{\lambda}+1-\bar{\lambda}) + \dfrac{l_2}{2}\sqrt{R^2 - \dfrac{l_2^2}{4}}, \\ \quad \text{for } \gamma\bar{\lambda}+1-\bar{\lambda} > \dfrac{\pi}{2}. \end{cases} \tag{B.1}$$

Proof. We recall that for any two planar curves parametrized by the couples (u, v) and (ϕ, ψ) in $(W^{1,3}(0, l_2))^2$ satisfying the same boundary conditions, the following 2-dimensional version of the isoperimetric inequality (2.19) holds:

$$\left|\int_0^{l_2}\big(u'(Z_2)v(Z_2)-\phi'(Z_2)\psi(Z_2)\big)\,\mathrm{d}Z_2\right|$$
$$\leqslant \frac{1}{4\pi}\left[\int_0^{l_2}\Big(\sqrt{(u')^2(Z_2)+(v')^2(Z_2)}+\sqrt{(\phi')^2(Z_2)+(\psi')^2(Z_2)}\Big)\mathrm{d}Z_2\right]^2 \tag{B.2}$$

[24], from which, using the triangle inequality, we have

$$\int_0^{l_2} u'(Z_2)v(Z_2)\,\mathrm{d}Z_2$$
$$\leqslant \int_0^{l_2}\phi'(Z_2)\psi(Z_2)\,\mathrm{d}Z_2+\frac{1}{4\pi}\left[\int_0^{l_2}\Big(\sqrt{(u')^2(Z_2)+(v')^2(Z_2)}\right.$$
$$\left.+\sqrt{(\phi')^2(Z_2)+(\psi')^2(Z_2)}\Big)\,\mathrm{d}Z_2\right]^2. \tag{B.3}$$

Let now us choose

$$\phi(Z_2)=\frac{\pi-\alpha}{\sin\alpha}\int_0^{Z_2}\cos\left[\frac{2}{l_2}(\pi-\alpha)s-(\pi-\alpha)\right]\mathrm{d}s, \tag{B.4}$$

$$\psi(Z_2)=\frac{\pi-\alpha}{\sin\alpha}\int_0^{Z_2}\sin\left[\frac{2}{l_2}(\pi-\alpha)s-(\pi-\alpha)\right]\mathrm{d}s, \tag{B.5}$$

with α the solution in $(0,2\pi)$ to the equation (5.32). The couple (ϕ,ψ) is a parametric representation of a circular arch joining the origin with the point $(0,l_2)$, lying in the negative halfplane, and having length

$$\int_0^{l_2}\sqrt{(\phi')^2(Z_2)+(\psi')^2(Z_2)}\,\mathrm{d}Z_2=2\pi R-l_2(\gamma\bar\lambda+1-\bar\lambda), \tag{B.6}$$

with the radius of the arch R satisfying (5.34). The (algebraic) area enclosed by (ϕ,ψ) is given by

$$\int_0^{l_2}\phi'(Z_2)\psi(Z_2)\,\mathrm{d}Z_2$$
$$=\begin{cases}\dfrac{1}{2}l_2R(\gamma\bar\lambda+1-\bar\lambda)-\pi R^2-\dfrac{l_2}{2}\sqrt{R^2-\dfrac{l_2^2}{4}}, & \text{if } \gamma\in\left(1,\dfrac{\pi}{2}\right],\\[2ex] \dfrac{1}{2}l_2R(\gamma\bar\lambda+1-\bar\lambda)-\pi R^2+\dfrac{l_2}{2}\sqrt{R^2-\dfrac{l_2^2}{4}}, & \text{if } \gamma\in\left(\dfrac{\pi}{2},+\infty\right).\end{cases} \tag{B.7}$$

Substituting (B.6) and (B.7) into the isoperimetric inequality (B.3) gives (B.1). □

A straightforward calculation shows that the upper value in (B.1) is achieved by the circular arch parametrized by the couple (U, V) with U and V given by (5.30), (5.31).

Appendix C

LEMMA C.3. *Let $\gamma > 1$ and $\bar{\lambda} \in [0, 1]$ be given. Then, the system of equations*

$$\begin{cases} \int_0^{l_2} r_n(t)\cos(\beta_n t + \alpha_n)\,\mathrm{d}t = l_2, \\ \int_0^{l_2} r_n(t)\sin(\beta_n t + \alpha_n)\,\mathrm{d}t = 0, \end{cases} \tag{C.1}$$

with r_n defined as in (5.39), admits at least a solution (α_n^*, β_n^*) for each integer n. Moreover,

$$\alpha_n^* \to \alpha, \tag{C.2}$$

$$\beta_n^* \to -\frac{2\alpha}{l_2}, \tag{C.3}$$

up to a subsequence, with α the solution in $(0, 2\pi)$ to (5.32).

Proof. We change variables to reduce the system (C.1) to the form

$$\begin{cases} \int_0^{1} q_n(t)\cos(\beta_n l_2 s + \alpha_n)\,\mathrm{d}s = 1, \\ \int_0^{1} q_n(t)\sin(\beta_n l_2 s + \alpha_n)\,\mathrm{d}s = 0, \end{cases} \tag{C.4}$$

with $q_n(s) := r_n(l_2 s)$. The system (C.4) is equivalent to

$$\begin{cases} \cos\alpha_n \displaystyle\int_0^1 q_n(s)\cos(\beta_n l_2 s)\,\mathrm{d}s - \sin\alpha_n \displaystyle\int_0^1 q_n(s)\sin(A_n s)\,\mathrm{d}s = 1, \\ \cos\alpha_n \displaystyle\int_0^1 q_n(s)\sin(\beta_n l_2 s)\,\mathrm{d}s + \sin\alpha_n \displaystyle\int_0^1 q_n(s)\cos(\beta_n l_2 s)\,\mathrm{d}s = 0. \end{cases} \tag{C.5}$$

Take the square, sum and use the trigonometric identity $\sin^2\alpha_n + \cos^2\alpha_n = 1$ to get

$$1 = \left[\int_0^1 q_n(s)\cos(\beta_n l_2 s)\,\mathrm{d}s\right]^2 + \left[\int_0^1 q_n(s)\sin(\beta_n l_2 s)\,\mathrm{d}s\right]^2 =: f^n(\beta_n). \tag{C.6}$$

Note that $f^n(0) = 0$. Besides, because

$$\lim_{n\to\infty} f^n(2\pi) = 0, \tag{C.7}$$

there exists an integer, say n_0, such that for each $n > n_0$

$$f^n(2\pi) < \frac{1}{2}. \tag{C.8}$$

Therefore, using the continuity of f^n, we conclude that there exists a $\beta_n^* \in [0, 2\pi]$ which solves (C.6). Besides, in view of (C.6), we may define α_n^* in $[0, 2\pi]$ such that

$$\cos \alpha_n^* = \int_0^1 q_n(s) \cos(\beta_n l_2 s)\, \mathrm{d}s. \tag{C.9}$$

From (C.6) and (C.9), we find

$$\left[\int_0^1 q_n(s) \sin(\beta_n l_2 s)\, \mathrm{d}s\right]^2 = \sin^2 \alpha_n^*. \tag{C.10}$$

Since there are two solutions of (C.10), without loss of generality we choose

$$\left[\int_0^1 q_n(s) \sin(\beta_n l_2 s)\, \mathrm{d}s\right] = -\sin \alpha_n^*. \tag{C.11}$$

On using (C.10), (C.11), and (C.6), the equations in (C.5) are identically satisfied for $\alpha_n^*,\ \beta_n^*$. Besides, because $\alpha_n^*, \beta_n^* \in [0, 2\pi]$, we have

$$\alpha_n^* \to \alpha, \tag{C.12}$$
$$\beta_n^* \to \beta, \tag{C.13}$$

up to a subsequence. The limit problem associated with (C.1)

$$\begin{cases} (\gamma\bar{\lambda} + 1 - \bar{\lambda}) \displaystyle\int_0^{l_2} \cos(\beta t + \alpha)\, \mathrm{d}t = l_2, \\ (\gamma\bar{\lambda} + 1 - \bar{\lambda}) \displaystyle\int_0^{l_2} \sin(\beta t + \alpha)\, \mathrm{d}t = 0, \end{cases} \tag{C.14}$$

is equivalent to

$$\begin{cases} \beta = -\dfrac{2\alpha}{l_2}, \\ \sin\alpha = \dfrac{\alpha}{(\bar{\lambda}\gamma + 1 - \bar{\lambda})}. \end{cases} \tag{C.15}$$

This system admits a unique solution in $(0, 2\pi)$, since $(\gamma\bar{\lambda} + 1 - \bar{\lambda}) > 1$ by hypothesis. □

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