Problèmes mathématiques de la mécanique/*Mathematical Problems in Mechanics*

# Rigorous derivation of nonlinear plate theory and geometric rigidity

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Note presented by John M. Ball.

**Abstract**  We show that nonlinear plate theory arises as a $\Gamma$-limit of three-dimensional nonlinear elasticity. A key ingredient in the proof is a sharp rigidity estimate for maps $v : (0, 1)^3 \to \mathbb{R}^3$. We show that the L$^2$ distance of $\nabla v$ from a single rotation is bounded by a multiple of the L$^2$ distance from the set SO(3) of all rotations. To cite this article: G. Friesecke et al., C. R. Acad. Sci. Paris, Ser. I 334 (2002) 173–178. © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

### Dérivation rigoureuse de la théorie non linéaire des plaques et rigidité géométrique

**Résumé**  Nous montrons que la théorie non linéaire des plaques émerge comme $\Gamma$-limite de la théorie de l'élasticité tridimensionnelle. La démonstration repose sur un résultat de rigidité pour des fonctions $v : (0, 1)^3 \to \mathbb{R}^3$. Nous montrons que la distance L$^2$ de $\nabla v$ d'une rotation est bornée par un multiple de la distance L$^2$ à l'ensemble SO(3) des rotations. Pour citer cet article : G. Friesecke et al., C. R. Acad. Sci. Paris, Ser. I 334 (2002) 173–178. © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

## Version française abrégée

Un problème fondamental dans la théorie de l'élasticité non linéaire est de comprendre la relation entre les théories tridimensionelles et bidimensionelles pour des domaines minces $\Omega_h = S \times \left(-\frac{1}{2}, \frac{1}{2}\right)$. La dérivation de ces théories a une longue tradition. En l'absence de résultats rigoureux (*voir* [2] pour un résumé des exceptions) on s'est servi d'hypothèses a priori (petitesse, ansatz spéciaux) qui débouchent sur une variété de théories de plaques et de coques qui fréquemment ne sont pas mutuellement consistantes.

Le point de départ de notre approche rigoureuse est l'énergie élastique $E^h(v) = \int_{\Omega_h} W(\nabla v(z)) \, \mathrm{d}z$ d'une déformation $v : \Omega_h = S \times \left(-\frac{h}{2}, \frac{h}{2}\right) \to \mathbb{R}^3$. Heuristiquement, on s'attend à ce que des déformations avec $E^h \sim h$ correspondant à une élongation du plan médian conduisent à une théorie de membrane, alors que

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des déformations avec $E^h \sim h^3$ correspondant à une flexion (sans élongation du plan médian) donnent une théorie de plaques non linéaire. La théorie de membrane a été justifiée rigoureusement dans [12–14], au sens de la $\Gamma$-convergence, [4,3]. Dans cette Note nous donnons une dérivation rigoureuse de la théorie de plaques non linéaire. Cela est plus difficile parce que le problème limite dépend des dérivées d'ordre supérieur et donc la limite $h \to 0$ correspond à une limite singulière.

Pour énoncer notre résultat, on se ramène à une région fixe par le changement de coordonnées $x = \left(z_1, z_2, \frac{z_3}{h}\right)$, puisque $y : \Omega = S \times \left(-\frac{1}{2}, \frac{1}{2}\right) \to \mathbb{R}^3$. Introduisant la notation $x' = (x_1, x_2)$ et $\nabla' y = y_{,1} \otimes e_1 + y_{,2} \otimes e_2$ pour les coordonnées et le gradient dans le plan, on a alors $\nabla v = \left(\nabla' y, \frac{1}{h} y_{,3}\right)$ et

$$\frac{1}{h} E(v) = I^h(y) := \int_\Omega W\left(\nabla' y, \frac{1}{h} y_{,3}\right) dx.$$

THÉORÈME 1. – *Soit $S \subset \mathbb{R}^2$ un ouvert borné connexe lipschitzien. On suppose que la fonction $W$ vérifie* : (i) $W \in C^0(M^{3 \times 3})$, $W$ est $C^2$ dans une voisinage de $SO(3)$ ; (ii) $W(QF) = W(F)$, pour tout $Q \in SO(3)$ ; (iii) $W(F) \geqslant c\, \mathrm{dist}^2(F, SO(3))$, $c > 0$, $W = 0$ sur $SO(3)$. Alors, lorsque $h \to 0$, les fonctionnelles $\frac{1}{h^2} I^h$ $\Gamma$-convergent (dans la topologie faible ou forte de $W^{1,2}(\Omega, \mathbb{R}^3)$) vers $I^0$ définie par :

$$I^0(y) = \begin{cases} \frac{1}{24} \int_S Q_2(II)\, dx' & \text{si } y \text{ est indépendant de } x_3 \text{ et } y \in \mathcal{A}, \\ +\infty & \text{sinon.} \end{cases}$$

*Ici la classe de fonctions admissibles consiste en les isométries $\mathcal{A} = \left\{y \in W^{2,2}(S; \mathbb{R}^3) : |y_{,1}| = |y_{,2}| = 1, y_{,1} \cdot y_{,2} = 0\right\}$ et $II$ est la seconde forme fondamentale $II_{ij} = \nu_{,i} \cdot y_{,j} = -\nu \cdot y_{,ij}$, $\nu = y_{,1} \wedge y_{,2}$. La forme quadratique $Q_2$ est définie par $Q_2(G) = \min_{a \in \mathbb{R}^3} Q_3(G + a \otimes e_3 + e_3 \otimes a)$, où $Q_3(F) = \frac{\partial^2 W}{\partial F^2}(\mathrm{Id})(F, F)$ est la forme quadratique d'élasticité linéaire.*

Dans le cas de l'élasticité isotropique on a $Q_3(F) = 2\mu \left|\frac{F + F^{\mathrm{T}}}{2}\right|^2 + \lambda(\mathrm{tr}\, F)^2$, $Q_2(G) = 2\mu \left|\frac{G + G^{\mathrm{T}}}{2}\right|^2 + \frac{2\mu\lambda}{2\mu + \lambda}(\mathrm{tr}\, G)^2$. Donc $I^0$ est identique à l'expression proposée dans le travail original de Kirchhoff [9], équation (9), mais pas à l'expression $\frac{1}{24} \int_S Q_2(D^2 y_3)\, dx'$ obtenue en supprimant la non-linéarité géométrique, qui, dans la littérature postérieure, a souvent été associée au nom de Kirchhoff.

Les points essentiels de la preuve sont les deux résultats suivants, qui permettent de linéariser le problème dans des directions transverses de $SO(3)$. Le théorème 2 est un résultat de rigidité pour des fonctions proches d'un mouvement rigide qui améliore les résultats classiques de John [7,8]. Le corollaire 3 est un résultat de compacité précis qui perd sa validité si le facteur $1/h^2$ avant l'intégrale $I^h$ est remplacé par n'importe quel facteur convergent plus lentement vers l'infini.

THÉORÈME 2. – *Soit $n \geqslant 2$ et soit $Q$ un cube $n$-dimensionnel. Il existe alors une constante $C$ telle que, pour toutes fonctions $v \in W^{1,2}(Q, \mathbb{R}^n)$, il existe une rotation $R \in SO(n)$ avec*

$$\|\nabla v - R\|_{L^2(Q)} \leqslant C \big\|\mathrm{dist}\big(\nabla v, SO(n)\big)\big\|_{L^2(Q)}.$$

COROLLAIRE 3. – *On suppose que $S \subset \mathbb{R}^2$ est un ouvert borné connexe lipschitzien et que $W(F) \geqslant c\, \mathrm{dist}^2(F, SO(3))$, $c > 0$. On suppose aussi que $y^{(h)} : \Omega = S \times \left(-\frac{1}{2}, \frac{1}{2}\right) \to \mathbb{R}^3$ vérifie*

$$\limsup_{h \to 0} \frac{1}{h^2} I^h\big(y^{(h)}\big) < \infty.$$

*Alors, lorsque $h \to 0$, pour une sous-suite, $y^{(h)} \to y$ dans $W^{1,2}(\Omega; \mathbb{R}^3)$, $\frac{1}{h} y_{,3}^{(h)} \to b$ dans $L^2(\Omega; \mathbb{R}^3)$, $y \in W^{2,2}(\Omega; \mathbb{R}^3)$, $b \in W^{1,2}(\Omega; \mathbb{R}^3)$, $|y_{,1}| = |y_{,2}| = 1$, $y_{,1} \cdot y_{,2} = 0$, $b = y_{,1} \wedge y_{,2}$.*

## 1. Introduction

A fundamental problem in nonlinear elasticity is to understand the relation between three-dimensional and two-dimensional theories for thin domains $\Omega_h = S \times \left(-\frac{h}{2}, \frac{h}{2}\right)$. The derivation of such theories has a long history with major contributions from Euler, J. Bernoulli, Cauchy, Kirchhoff, Love, E. and F. Cosserat, von Kármán and a great many modern authors [16]. The derivations usually involve certain ansatzes and smallness assumptions leading to a variety of plate/shell theories which are often not consistent with each other. Thus there has been a great interest in precise convergence results comparing three-dimensional solutions and their two-dimensional counterparts (*see* [2] for a recent survey).

Our approach starts from the elastic energy

$$E^h(v) = \int_{\Omega_h} W\big(\nabla v(z)\big)\, dz$$

of a deformation

$$v : \Omega_h = S \times \left(-\frac{h}{2}, \frac{h}{2}\right) \;\to\; \mathbb{R}^3.$$

Heuristically, one expects that deformations with $E^h \sim h$ correspond to a stretching of the midplane $S$ leading to a membrane theory, while $E^h \sim h^3$ corresponds to a bending deformation (where $S$ remains unstretched) leading to nonlinear plate theory. Membrane theory was rigorously justified in [12–14] in the sense of $\Gamma$-convergence [4,3]. In this Note we rigorously derive nonlinear plate theory. This is more delicate since the limit problem involves higher derivatives and hence the limit $h \to 0$ corresponds to a singular perturbation.

To state our result it is convenient to work in a fixed domain $\Omega = S \times \left(-\frac{1}{2}, \frac{1}{2}\right)$, change variables $x = \left(z_1, z_2, \frac{z_3}{h}\right)$ and rescale deformations according to $y(x) = v(z(x))$ so that $y : \Omega \to \mathbb{R}^3$. We abbreviate $x' = (x_1, x_2)$ and use the notation $\nabla' y = y_{,1} \otimes e_1 + y_{,2} \otimes e_2$ for the in-plane gradient so that

$$\nabla v = \left(\nabla' y, \tfrac{1}{h}\, y_{,3}\right)$$

and

$$\frac{1}{h}\, E^h(v) = I^h(y) := \int_\Omega W\left(\nabla' y, \frac{1}{h}\, y_{,3}\right) dx.$$

Recall that a sequence of functionals $I^{(h)} : X \to \mathbb{R} \cup \{\infty\}$ on a Banach space $X$ is said to be $\Gamma$-convergent to $I^0 : X \to \mathbb{R} \cup \{\infty\}$ with respect to the weak (respectively strong) topology on $X$ if: (i) (Ansatz-free lower bound) for all sequences $y^{(h)}$ converging weakly (resp. strongly) to $y$, $\liminf_{h \to 0} I^{(h)}(y^{(h)}) \geqslant I^0(y)$; (ii) (Attainment of lower bound) for each $y \in X$ there exists a sequence $y^h$ converging weakly (resp. strongly) to $y$ such that $\lim_{h \to 0} I^{(h)}(y^{(h)}) = I^0(y)$.

THEOREM 1. – *Suppose that $S \subset \mathbb{R}^2$ is a bounded Lipschitz domain and that the stored energy $W$ satisfies*: (i) $W \in C^0(M^{3 \times 3})$, $W \in C^2$ *in a neighbourhood of* $SO(3)$; (ii) $W(QF) = W(F)$, *for all $Q \in SO(3)$*; (iii) $W(F) \geqslant c\,\mathrm{dist}^2(F, SO(3))$ *for some $c > 0$, $W = 0$ on $SO(3)$. Then, for $h \to 0$ the functionals $\frac{1}{h^2} I^h$ are $\Gamma$-convergent (with respect to the strong or the weak $W^{1,2}$ topology) to $I^0$ given by*

$$I^0(y) = \begin{cases} \frac{1}{24} \int_S Q_2(II)\, dx' & \text{if } y \text{ is independent of } x_3 \text{ and } y \in \mathcal{A}, \\ +\infty & \text{else.} \end{cases}$$

*Here the class $\mathcal{A}$ of admissible maps consist of isometries $\mathcal{A} = \big\{ y \in W^{2,2}(S; \mathbb{R}^3) : |y_{,1}| = |y_{,2}| = 1,\ y_{,1} \cdot y_{,2} = 0 \big\}$ and $II$ is the second fundamental form $II_{ij} = v_{,i} \cdot y_{,j} = -v \cdot y_{,ij}$, $v = y_{,1} \wedge y_{,2}$. The quadratic form $Q_2$ is defined by: $Q_2(G) = \min_{a \in \mathbb{R}^3} Q_3(G + a \otimes e_3 + e_3 \otimes a)$, where $Q_3(F) = \frac{\partial^2 W}{\partial F^2}(\mathrm{Id})(F, F)$ is twice the linearized energy.*

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*Remarks*. – 1. For isotropic elasticity we have

$$Q_3(F) = 2\mu \left| \frac{F + F^{\mathrm{T}}}{2} \right|^2 + \lambda (\mathrm{tr}\, F)^2, \qquad Q_2(G) = 2\mu \left| \frac{G + G^{\mathrm{T}}}{2} \right|^2 + \frac{2\mu\lambda}{2\mu + \lambda} (\mathrm{tr}\, G)^2$$

Thus $I^0$ agrees with the expression proposed in the original work of Kirchhoff [9], equation (9), but not with the expression obtained by suppressing the geometric nonlinearity and replacing $II_{ij}$ by $(y_3)_{,ij}$ which much of the subsequent literature associated with Kirchhoff's name.

2. In particular, if $W(F) = \mathrm{dist}^2(F, \mathrm{SO}(3))$, then the limit energy is $\frac{1}{12} \int_S |II|^2 \, \mathrm{d}x$ which agrees, up to a factor, with the Willmore functional [20].

3. Similar results can be proved if one imposes suitable boundary conditions which are consistent with $\mathcal{A}$. The effect of more general boundary conditions can be rather subtle, see [1,6] for an example related to the formation of blisters in thin films under compression for which one can prove $\min I^h \sim h$.

4. Condition (i) can be relaxed. It suffices that $W$ has a second order Taylor expansion at the identity and that $W$ is continuous in a neighbourhood of $\mathrm{SO}(3)$ (such as $\{F \in \mathrm{M}^{3\times 3} \mid \det F > 0\}$), taking the value $+\infty$ elsewhere.

Detailed proofs and the extensia to boundary value problems will appear elsewhere.

In independent work Pantz [18] establishes an interesting partial $\Gamma$-convergence result using strong a priori assumptions on the 3D deformations similar to those made by John [7,8] (see below) which do not follow from smallness of the elastic energy $I^h$.

## 2. Geometric rigidity

The key ingredient in the proof of Theorem 1 is the following rigidity result which guarentees that low energy maps are close to a rigid motion.

THEOREM 2. – *Let $n \geqslant 2$ and let $Q \subset \mathbb{R}^n$ be a cube. Then there exists a constant $C$ such that for all $v \in \mathrm{W}^{1,2}(Q; \mathbb{R}^n)$ there exists a rotation $R \in \mathrm{SO}(n)$ such that*

$$\|\nabla v - R\|_{\mathrm{L}^2(Q)} \leqslant C \big\| \mathrm{dist}\big(\nabla v, \mathrm{SO}(n)\big) \big\|_{\mathrm{L}^2(Q)}. \tag{1}$$

*Remarks*. – 1. The celebrated results of John [7,8] establish Theorem 2 under the additional restrictions that $y \in \mathrm{C}^1(Q; \mathbb{R}^n)$ and $\mathrm{dist}(\nabla y(x), \mathrm{O}(n)) \leqslant \delta$ for all $x \in Q$ and some sufficiently small $\delta > 0$ (*see* also [10]). For the application to plate theory it is crucial to remove these restrictions, since they do not follow from smallness of the elastic energy.

2. Likewise, it is important to obtain an estimate which is linear in $\varepsilon = \| \mathrm{dist}(\nabla v, \mathrm{SO}(n))\|_{\mathrm{L}^2(Q)}$ since we need to sum over many small cubes of size $h$. An estimate in terms of $\sqrt{\varepsilon} + \varepsilon$ is much easier to prove but useless for the present purpose.

3. For $\varepsilon = 0$ we recover the Liouville theorem which states that $\nabla v \in \mathrm{SO}(n)$ implies $\nabla v = \mathrm{const}$. In the $\mathrm{W}^{1,\infty}$ setting this was first proved by Reshetnyak [19]. More generally, he proved that a sequence $\nabla v^{(k)}$ bounded in $\mathrm{L}^2$ that converges to $\mathrm{SO}(3)$ in measure in fact converges (after extraction of a subsequence) strongly to a single matrix on $\mathrm{SO}(3)$. The latter, however, is not sufficient to prove strong convergence of $(\nabla' y, \frac{1}{h} y_{,3})$ for a sequence for which $\frac{1}{h^2} I^h$ is uniformly bounded. The stronger estimate of Theorem 2 is needed because of the explicit presence of the scales $1/h^2$ and $1/h$.

4. The result can be extended to general Lipschitz domains.

5. The estimate holds also in $\mathrm{L}^p$, for $1 < p < \infty$.

*Sketch of proof.* – Let $\varepsilon = \| \mathrm{dist}(\nabla v, \mathrm{SO}(n)\|_{\mathrm{L}^2(Q)}$. By scaling we may assume that $Q$ is the unit cube and it suffices to consider $\varepsilon \leqslant 1$. We first prove an interior estimate on a concentric subcube $Q'$.

*Step* 1 (truncation). – A truncation argument [15,21,5] shows that there exists a constant $M$ such that if the result holds for $v$ with $|\nabla v| \leqslant M$, then it is true in general.

*Step* 2 (bound by $\sqrt{\varepsilon}$). – For a matrix $F \in \mathrm{M}^{n \times n}$ let cof $F$ denote the matrix of $(n-1) \times (n-1)$ minors such that $F^{\mathrm{T}} \mathrm{cof}\, F = \det F\, \mathrm{Id}$. Then for all $v \in \mathrm{W}^{1,\infty}$ we have div cof $\nabla v = 0$. Since cof $F - F$ vanishes on SO($n$) we deduce that $-\Delta v = \mathrm{div}\, f$, $|f| \leqslant C\, \mathrm{dist}(\nabla v, \mathrm{SO}(n))$. Hence $v = w + z$ where $w$ is harmonic, $z \in \mathrm{W}_0^{1,2}(Q; \mathbb{R}^n)$ and

$$\|\nabla z\|_{\mathrm{L}^2(Q)} \leqslant \|f\|_{\mathrm{L}^2(Q)} \leqslant C\varepsilon, \tag{2}$$

$$\Delta\big(|\nabla w|^2 - n\big) = \nabla w \cdot \Delta \nabla w + |\nabla^2 w|^2 = |\nabla^2 w|^2. \tag{3}$$

Let $Q' \subset Q'' \subset Q$ be strictly increasing concentric cubes. Multiplying (3) by $\varphi \in \mathrm{C}_0^\infty(Q)$ with $\varphi = 1$ on $Q''$ and $\varphi \geqslant 0$ and using the fact that $|F|^2 - n = 0$ on SO($n$) and (2) we deduce $\int_{Q''} |\nabla^2 w|^2\, \mathrm{d}x \leqslant C\varepsilon$. Since $w$ is harmonic this yields $\sup_{Q'} |\nabla^2 w| \leqslant C\sqrt{\varepsilon}$ and

$$\sup_{Q'} |\nabla w - R| \leqslant C\sqrt{\varepsilon} \quad \text{for some } R \in \mathrm{SO}(n). \tag{4}$$

*Step* 3 (linearization of SO($n$)). – We may suppose that $R = \mathrm{Id}$ and we use the expansion

$$\mathrm{dist}\big(G, \mathrm{SO}(n)\big) = \big|\tfrac{1}{2}\big(G + G^{\mathrm{T}}\big) - \mathrm{Id}\big| + \mathrm{O}\big(|G - \mathrm{Id}|^2\big).$$

This yields $\big\|\tfrac{1}{2}\big(\nabla w + (\nabla w)^{\mathrm{T}}\big) - \mathrm{Id}\big\|_{\mathrm{L}^2(Q')} \leqslant C\varepsilon$ and by Korn's inequality

$$\|\nabla w - \mathrm{Id} - W\|_{\mathrm{L}^2(Q')} \leqslant C\varepsilon, \quad W^{\mathrm{T}} = -W. \tag{5}$$

Moreover, (4) with $R = \mathrm{Id}$ implies $|W| \leqslant C\sqrt{\varepsilon}$, dist(Id $+ W$, SO($n$)) $\leqslant C\varepsilon$ and this together with (2) finishes the proof of the interior estimate $\|\nabla v - R\|_{\mathrm{L}^2(Q')} \leqslant C\varepsilon$.

*Step* 4 (global estimates). – Cover $Q$ by cubes $Q(a_i, r_i) = a_i + r_i\big(-\tfrac{1}{2}, \tfrac{1}{2}\big)^n$ such that

$$2r_i \leqslant \mathrm{dist}(a_i, \partial Q) \leqslant C\, r_i \tag{6}$$

and such that each $x \in Q$ is contained in at most $N$ enlarged cubes $Q(a_i, 4r_i)$. Application of the interior estimate to $w$ and harmonicity of $w$ yield

$$\int_{Q(a_i, r_i)} r_i^2 |\nabla^2 w|^2\, \mathrm{d}x \leqslant C \int_{Q(a_i, 2r_i)} |\nabla w - R_i|^2\, \mathrm{d}x \leqslant C \int_{Q(a_i, 4r_i)} \mathrm{dist}^2\big(\nabla w, \mathrm{SO}(n)\big)\, \mathrm{d}x.$$

The desired assertion now follows from (6) and the weighted Poincaré inequality (*see* [17], Theorem 1.5 or [11], Theorem 8.8), $\int_Q |g|^2\, \mathrm{d}x \leqslant C_Q \int_Q (|g|^2 + |\nabla g|^2)\, \mathrm{dist}^2(x, \partial Q)\, \mathrm{d}x$ applied with $g = |\nabla w - R|$.

## 3. Compactness and $\Gamma$-convergence

COROLLARY 3. – *Suppose that $S \subset \mathbb{R}^2$ is a bounded Lipschitz domain and that $W(F) \geqslant c\, \mathrm{dist}^2(F, \mathrm{SO}(3))$, $c > 0$. Suppose further that $y^{(h)} : \Omega = S \times \big(-\tfrac{1}{2}, \tfrac{1}{2}\big) \to \mathbb{R}^3$ satisfies*

$$\limsup_{h \to 0} \frac{1}{h^2} I^h\big(y^{(h)}\big) < \infty. \tag{7}$$

*Then, for a subsequence*

$$y^{(h)} \to y \quad \text{in } \mathrm{W}^{1,2}\big(\Omega; \mathbb{R}^3\big), \quad \tfrac{1}{h} y_{,3}^{(h)} \to b \quad \text{in } \mathrm{L}^2\big(\Omega; \mathbb{R}^3\big),$$

$$y \in \mathrm{W}^{2,2}\big(\Omega; \mathbb{R}^3\big), \quad b \in \mathrm{W}^{1,2}\big(\Omega; \mathbb{R}^3\big), \quad |y_{,1}| = |y_{,2}| = 1, \quad y_{,1} \cdot y_{,2} = 0, \quad b = y_{,1} \wedge y_{,2}.$$

*Remark.* – The scaling in (7) is optimal. There exists a sequence $y^{(h)}$ such that $h^{-\beta} I(y^{(h)}) \to 0$ for all $\beta < 2$ such that $y^{(h)} \rightharpoonup y$ weakly in $\mathrm{W}^{1,2}$ but strong convergence fails.

*Idea of proof.* – Application of Theorem 2 to slightly overlapping cubes yields an $\mathrm{L}^2$ difference quotient estimate for $\nabla_h y := (\nabla' y, \tfrac{1}{h} y_{,3})$. This implies compactness and $\mathrm{W}^{2,2}$ regularity of the limit.

*Proof of Theorem* 1 (*sketch*). – The main point is to establish a lower bound on $\liminf_{h\to 0} h^{-2} I^h(y^{(h)})$ for $y^{(h)} \rightharpoonup y$. Using Theorem 1 we can find a map $Q^{(h)} : S \to \mathrm{SO}(3)$, piecewise constant on squares of side $h$, such that

$$\left(Q^{(h)}\right)^{\mathrm{T}} \frac{1}{h} \left(\left(\nabla' y^{(h)}, \tfrac{1}{h} y_{,3}\right) - Q^{(h)}\right) \;\rightharpoonup\; G \quad \text{in L}^2(\Omega),$$

and with the help of Corollary 3 one can conclude that $G_{ij}(x', x_3) = G^0_{ij}(x') + x_3 II_{ij}(x')$, $i, j \in \{1, 2\}$.

Careful expansion of the energy $W$, the definition of $Q_3$ and $Q_2$ and minimization over $G^0$ then yield the desired lower bound in terms of $I^0(y)$.

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