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Original Article

Configurational forces in magnetism with application to the dynamics of a small-scale ferromagnetic shape memory cantilever

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This paper has two related parts. In the first part we derive using classical methods a formula for the configurational force on a magnetoelastic wall. In the second part we formulate the problem of the motion of a small-scale cantilever made of ferromagnetic shape memory material and driven by a remotely applied field. The motion is caused by the movement of twin boundaries, and, consequently, the energy stored by the cantilever scales as the thickness, rather than the thickness cubed. We derive equations of motion for the cantilever. The usual balances of linear and rotational motion are found to be insufficient to determine the motion, and this is remedied by the postulation of a kinetic law that relies on the formula for the driving force on a magnetoelastic wall given in the first part.

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Dedicated to Professor Ingo Müller on the occasion of his 65th birthday

1 Configurational forces in magnetism

1.1 Introduction

The first part of this paper (Sect. 1) concerns the formulas in magnetoelasticity for real forces and for configurational forces, the latter understood in the sense of Eshelby. In Sect. 2 we use these formulas in the derivation of equations of motion for an unusual concept for a MEMS (micro-electro-mechanical systems) cantilever. The main result of Sect. 1 is the formula (92) for the configurational force on an interface in a deforming solid, where the interface is a surface of discontinuity of deformation gradient and magnetization.

The main motivation for considering such singularities is the emergence of ferromagnetic shape memory (and related) materials [26]. These materials have interfaces that exhibit large jumps of both magnetization and deformation gradient. The motion of these interfaces can give rise to large macroscopic shape changes that are some 50 times larger then those of giant magnetostrictive materials. As these materials are currently understood, these shape changes arise primarily from configurational forces on the internal interfaces (rather than real electromagnetic forces).

According to the usual formulation of micromagnetics (without deformation), the magnetization is in H^1 and therefore does not admit discontinuities of magnetization across a plane. Point singularities are allowed, and these play an important role in some studies of wall substructure. However, exchange constants are relatively small, and a common situation is to have nearly constant regions of magnetization separated by relatively sharp domain walls (see DeSimone [12] for a discussion of the limiting variational principle predicted by micromagnetics as the exchange constant goes to zero). Thus for reasons of simplicity and accessibility to simulation, it is desirable to treat the interfaces as sharp, and to lump all the complexity of wall structure into laws for the motion and energy of idealized sharp interfaces. The basic idea goes back to Gibbs [19], and the need for configurational forces in related situations was recognized by Eshelby [16], [17]. Most importantly, as discussed by Abeyaratne and Knowles [1], [2], formulas for configurational force can then be used to formulate dynamic laws of motion that 'close the equations' and move domain walls in an energetically favorable direction, while avoiding the complexity of determining wall structure. When deformation is introduced, the situation with regard to the smoothness of minimizers of the total energy of magnetoelasticity is less clear. In the known ferromagnetic shape memory materials, it is believed that the jump discontinuities of deformation gradient are atomically sharp.

A similar situation recurs throughout materials science – fracture mechanics, phase transformations, dislocation mechanics, grain growth – and there are various approaches to the formulation of dynamic equations. The approach discussed above relies on a formula for the configurational force f on the singularity (following Eshelby). One writes a kinetic law of the form $v = -\mu f$, where v is an appropriate measure of the velocity of the singularity and μ is the mobility. Despite the widespread use of this recipe, or its obvious generalizations, there are few direct experimental or other tests of such kinetic laws (Since magnetic configurational forces can be applied in many different ways, it is hoped that magnetism might finally provide such tests).

An alternative approach is the formulation of an additional kinetic law based on the microscopic physics. In the case without deformation there is a reputable dynamic law for magnetism: the Landau-Lifschitz-Gilbert

(LLG) equation. This potentially could be used to derive the kinetic law by passing to a sharp interface limit, at least for a magnetization having a steady one–dimensional profile. It is not yet clear whether this would give a definitive derivation of an Eshelby-type kinetic law (though it is certainly worth exploring), because of the interaction of the wall with other defects (pinning) and other metallurgical factors not taken into account by LLG[1]. In ordinary magnetic materials the kinetics of 180° walls, which have no jump in deformation gradient, has been studied theoretically using LLG, and there is beautiful experimental confirmation of some of the results in (nonconductive) oxide materials (see O'Dell [39] for a summary of this work[2]). Curiously, this work seems unaware of Eshelby's, and the results are not cast in the form of kinetic laws for interfaces, if indeed that is possible.

Section 1 is strongly influenced by the fundamental works of Toupin [46] and especially Brown [4], [5], [6] (It is interesting to note that the latter work was written before the discovery in the late 1960s of giant magnetostrictive materials, so there were no examples of materials at the time which exhibited a large field–induced shape change.) Our treatment of magnetic forces is an updated and simplified treatment of Brown's, and has some points of contact with recent work of Desimone and Podio-Guidugli [13], [14]. As in all of these works, our arguments make lots of assumptions of regularity of the deformation and magnetization, smoothness of interfaces of discontinuity, etc. However, we have made a considerable effort to streamline and simplify the arguments, and to avoid unnecessary constructions like extending the deformation outside of the body. The treatment is also self-contained.

A key step in the development of a continuum theory for magnetoelasticity is the calculation of the force on a subregion of a magnetized body exerted by a surrounding region. As discussed by Brown, this can be broken down into an explicit long range part and a remaining short range part. In Sect. 1.2.1 we refer to the long range part as Brown's formula. In forthcoming work, Müller and Schlömerkemper [36] have investigated the magnetic forces between intimately neighboring subregions of a magnetoelastic body from the atomic viewpoint, i.e., when the body is viewed as a collection of dipoles on a lattice and forces are calculated using dipole formulas. This work is interesting in that it gives explicit formulas for the long and short range parts of the magnetic force.

In Sect. 1 we treat the magnetism using magnetostatic formulas. This should be acceptable when conduction is negligible and the typical time a light wave takes to travel from one part of the system to another is much less than the the time it takes for the system to undergo a negligible change of state. It would be extremely useful to generalize this work to allow for conduction; in conductive ordinary ferromagnetic films there is good evidence that the kinetics of 180° walls is dominated[3] by conductivity (summarized by O'Dell [39], again, with good experimental confirmation). Finally, Sect. 1 is also purely mechanical: a fully thermodynamic theory would seem to be relatively easy to construct, especially with the help of the treatment of Müller [35] of thermodynamic driving forces; see Sect. 1.4 and James and Rizzoni [25] for additional comments/results in this direction.

There is considerable previous work on the derivation of configurational forces on interfaces in non-conductive ferromagnetic materials. Maugin and his coworkers ([31], [32], [33], [34]) have derived an expression for configurational forces based on a balance equation of "pseudomomentum" applied to "material quasi-inhomogeneities". This balance is explained by Fomethe and Maugin ([33]) as follows (italics of the authors): "This is the complete projection onto $\mathscr{M}^3$ of the balance of linear momentum. It has *co-vectorial* components on $\mathscr{M}^3$ and is thus completely *intrinsic*, having nothing to do with the representation in physical space. All terms in the right-hand side may be viewed as sources of *material* quasi-*inhomogeneities* as it was originally shown that this balance (here, *un*-balance as we like to say) equation reflects the invariance or lack of invariance of the physical system under study with respect to the *material point* $\boldsymbol{X}$, and *not* the physical placement $\boldsymbol{x}$". After making a considerable effort, the present author does not understand this theory. However, the final expression for the configurational force can be compared with ours and they disagree: the difference between the two expressions is the term $\langle \det\nabla y\rangle [\![\frac{1}{2}|\mathbf{m}|^2]\!] + [\![\det\nabla y]\!]\langle|\mathbf{m}|^2\rangle$, where $\langle a\rangle = \frac{1}{2}(a^+ + a^-)$

[1] There is an emerging understanding of of how pinning might affect kinetic laws of the type $v = F(f)$, i.e., the form of F that might be appropriate in that case (Bhattacharya [3]).

[2] Regarding 90° and other domain walls it is interesting that O'Dell ([39], p. 68) states, "The dynamics of walls which separate domains of different magnetic crystal orientations, that is other than simple 180° domain walls, should really be considered as the dynamics of phase boundaries."

[3] In conductors the contribution to kinetic laws from LLG is considered negligible.

denotes the average across the interface, $[\![a]\!] = a^+ - a^-$ denotes the jump, $\mathbf{m}$ is the magnetization and $\nabla\mathbf{y}$ is the deformation gradient. This term vanishes if both $\det\nabla\mathbf{y}$ and $|\mathbf{m}|$ do not jump across the interface. Part of the origin of the discrepancy may be the *spin-precession equation* used by Fomethe and Maugin (e.g., [33], (2.12) and (iii) on p. 148), of which the LLG equation is a special case. This equation is claimed to be equivalent to the balance of moment of momentum, but it implies a condition of saturation. That is, it implies that $|(\det\nabla\mathbf{y})\mathbf{m}|$ depends on the particle only. This condition is significantly violated in ferromagnetic shape memory materials: some near-stoichiometric compositions of the alloy Ni_2MnGa exhibit the co-existence of (and transformation between) strongly ferromagnetic martensite and nonferromagnetic austenite. Thus, the spin-pression equation does not apply to these materials. The present work relies completely on the theory of magnetoelasticity as presented in 1966 by Brown [6][4], together with the concept of the force on a defect presented a decade earlier by Eshelby.

Jiang [28], in a paper that is similar in approach to Sect. 1, derived a formula for the configurational force on a magnetoelastic interface. His formula also disagrees with ours. Apparently, this is because he omitted magnetic forces in the balance of linear and angular momentum.

Many other works treat the configurational forces on interfaces by neglecting parts of the energy or some of the forces. It is quite common to neglect anisotropy and/or demagnetization energies. For example, the recent work of Müllner and Ullakko [38] on dynamics of twin boundaries takes the driving force to be that arising from the applied field energy only ($\mathbf{h}_0 \cdot [\![\mathbf{m}]\!]$ where $\mathbf{h}_0$ is the applied field and $[\![\mathbf{m}]\!] = \mathbf{m}^+ - \mathbf{m}^-$ denotes the jump of magnetization across the interface). O'Handley [40] takes the driving force to arise from applied field and anisotropy energies, together with a kind of intrinsic elasticity of the twin boundary that does not fall into the format of Eshelby. The nice study of the influence of magnetic field on the growth of grains in Bi by Mullins [37] takes the magnetic driving force to be that arising from anisotropy energy alone. In these works the omitted terms are in some cases significant; for example, the term in the driving force arising from demagnetization energy, $\frac{\gamma}{2}(\det\nabla\mathbf{y})(\mathbf{m}\cdot\mathbf{n})^2$, can be comparable to $\mathbf{h}_0 \cdot [\![\mathbf{m}]\!]$ at fields of the order of the saturation magnetization.

The formulas here have ferroelectric counterparts (free charges excluded), but, to simplify the presentation we confine attention to the magnetic case.

1.2 Real forces

1.2.1 Force on a subregion and Brown's formula

To formulate dynamic laws in continuum mechanics, one needs a formula for the force exerted on any subregion of a body. In this section we consider the magnetic contribution to this force.

Let $\Omega' \subset \mathbb{R}^3$ be a given domain and let $\mathscr{P} \subset \Omega'$ be a typical smooth subdomain. Let Ω' be magnetized: $\mathbf{m} : \Omega' \to \mathbb{R}^3$. We assume that $\mathbf{m} \in L^\infty(\Omega)$. In writing some formulas we will tacitly assume that $\mathbf{m}$ has been extended to all of $\mathbb{R}^3$ with $\mathbf{m} = 0$ outside Ω'.

Let us first suppose that a field $\mathbf{h}_0 : \mathbb{R}^3 \to \mathbb{R}^3$ is produced by sources, e.g., coils or other magnetized matter, that are far from the body ($\mathbf{h}_0$ is the field that would be present if the body Ω' were removed). Then the force exerted on all of Ω' is given by the formula,

$$\int_{\Omega'} (\mathbf{m} \cdot \nabla\mathbf{h}_0)\, d\mathbf{x}. \tag{1}$$

Here the integrand is in components $(m_j h_{0\,i,j})$. In the simplest view the derivation of (1) follows by an elementary atomic–to–continuum argument based on the formula for the force on an elementary dipole that goes like this. A dipole can be conceived as either the limiting case of two charges $+q$ and $-q$ separated by a distance r (where $qr = const.$) or else as a small current loop of area A and current I (where $IA = const.$). In the former case, to first approximation in $r = |\mathbf{r}|$, the total force on the dipole is zero ($= +q\mathbf{h}_0 - q\mathbf{h}_0$), and to second approximation in r it is $+q\mathbf{h}_0(\mathbf{x}+\mathbf{r}) - q\mathbf{h}_0(\mathbf{x}) = (q\mathbf{r})\cdot\nabla\mathbf{h_0} + \ldots$. This term survives in the dipole limit and the limiting value of the quantity $q\mathbf{r}$ is the dipole moment. Given that $\mathbf{m}$ represents the dipole moment

[4] Brown's balance of moment of momentum is a macroscopic equation and does not imply a condition of saturation

per unit volume, it is an easy averaging argument to see that (1) emerges as the formula for the force. The usual practice in magnetism is to base this derivation not on the charge–picture but rather on the current loop–picture of a dipole. In this case the force is given by the Lorentz formula $q\mathbf{v} \times \mathbf{b}_0$[5], and the argument is very similar to the above.

When we wish to apply the formula to a subset $\mathscr{P} \subset \Omega'$, we have to account for the fact that $\Omega' \setminus \mathscr{P}$ is magnetized and therefore also produces a field on $\mathscr{P}$. But in this case Brown feels uncomfortable applying the formula (1). His objections concern the fact that the derivation of (1) sketched above relies on the fact that the field varies slowly over the local region of the dipole. In fact, the local field in the charge–picture and in the current loop–pictures are very different, though the remote fields are very nearly the same. With contributions to the dipole of an atom from spin (highly localized) and rotation around the nucleus (usually the smaller contribution in ferromagnetic materials, but not localized on the scale of an atom), one could question whether the formula (1) is accurate when the field–producing dipoles are in intimate contact with the dipoles that contribute to the force, such as occurs near $\partial\mathscr{P}$. To avoid such issues, Brown does a very reasonable thing. He takes a region $\mathscr{P}_\delta \to \mathscr{P}$ that contains strictly $\mathscr{P}$, calculates the force on $\mathscr{P}$ due to the field produced by $\Omega' \setminus \mathscr{P}_\delta$ and then takes the limit $\delta \to 0$. The region $\Omega' \setminus \mathscr{P}_\delta$ is considered to be "much further from $\mathscr{P}$ than atomic dimensions". He calls the result the *long range contribution to the magnetic force on* $\mathscr{P}$.

Remark. This force does not represent the total force having magnetic origins of $\Omega' \setminus \mathscr{P}$ on $\mathscr{P}$. Brown [6] joins a local traction, that may also partly have magnetic origins, to the long range force as discussed below in Sect. 1.2.2. Explicit formulas for the long and short range part, arising from purely lattice dipole interactions, are given by Müller and Schlömerkemper [36].

So, following Brown, we are interested in the long range contribution to the magnetic force on $\mathscr{P}$, i.e., the limiting value of the quantity:

$$\mathscr{F}^{(\delta)} = \int_{\mathscr{P}} (\mathbf{m} \cdot \nabla \mathbf{h}^{(\delta)})\, d\mathbf{x}, \quad \mathbf{h}^{(\delta)} = -\nabla \phi^{(\delta)}, \quad \operatorname{div}(\mathbf{h}^{(\delta)} + \gamma \mathbf{m} \chi_{\Omega' \setminus \mathscr{P}_\delta}) = 0. \tag{2}$$

Here, $\mathscr{P}$ is a smooth subdomain of Ω', the closure of $\mathscr{P}$ is contained in Ω', $\mathscr{P}_\delta = \{\mathbf{x} \in \Omega' : \operatorname{dist}(\mathbf{x}, \mathscr{P}\} < \delta\}$, and $\chi_{\mathscr{D}}$ is the characteristic function of a set $\mathscr{D}$:

$$\chi_{\mathscr{D}}(\mathbf{x}) = \begin{cases} 1, & \mathbf{x} \in \mathscr{D}, \\ 0, & \mathbf{x} \notin \mathscr{D}. \end{cases} \tag{3}$$

We have omitted the part of the force arising from the applied field $\mathbf{h}_0$ which just contributes the harmless additive term

$$\int_{\mathscr{P}} (\mathbf{m} \cdot \nabla \mathbf{h}_0)\, d\mathbf{x}. \tag{4}$$

The differential equations in (2) are the magnetostatic equations, representing Maxwell's equations $\operatorname{div} \mathbf{b} = 0$, $\operatorname{curl} \mathbf{h} = 0$ on $\mathbb{R}^3$, $\mathbf{b} = \mathbf{h} + \gamma \mathbf{m}$, $\mathbf{b}$ is the *magnetic induction*, $\mathbf{h}$ is the *magnetic field*, and γ is a constant that depends on the system of units ($\gamma = 4\pi$ in the c.g.s. units that are still widely used in magnetism). As discussed for example by James and Kinderlehrer [22], given $\mathbf{m} \in L^2(\mathbb{R}^3)$ this system has a unique solution $\mathbf{h} = -\nabla\varphi \in L^2(\mathbb{R}^3)$. The magnetostatic potential φ is therefore unique up to a constant of integration which can be fixed in any convenient way; we do this by assuming that solutions φ of the magnetostatic equation always lie in

$$\mathscr{V} = \{\varphi \in L^2(\mathbb{R}^3) : \nabla\varphi \in L^2(\mathbb{R}^3)\}, \tag{5}$$

which can always be done. If a magnetization exhibits a classical jump at an interface with unit normal $\mathbf{n}$, then, from the magnetostatic equation we have, first, that $\mathbf{h}$ also jumps at this interface and $[\![\mathbf{h} + \gamma\mathbf{m}]\!] \cdot \mathbf{n} = 0$ and, second, from the fact that $\mathbf{h} = -\nabla\varphi$ is a gradient, $[\![\nabla\varphi]\!] = \alpha\mathbf{n}$. Putting these two facts together, we have that $[\![\nabla\varphi]\!] = \gamma[\![\mathbf{m} \cdot \mathbf{n}]\!]\mathbf{n}$.

[5] Note: $\mathbf{b}_0 = \mathbf{h}_0$ "between the atoms", that is, for the purpose of calculating this force. The formula $\mathbf{b} = \mathbf{h} + \gamma\mathbf{m}$ is a macroscopic formula that emerges from homogenizing the field of dipoles in a manner that preserves the equation $\operatorname{div}\mathbf{b} = 0$ (for a modern treatment, see James and Müller [24]).

Another fact about the magnetostatic equation that we shall use is the following. If $\varphi \in \mathscr{V}$ satisfies $\mathrm{div}(-\nabla\varphi + \gamma\mathbf{m}) = 0$ where $\mathbf{m} \in L^2$, then,

$$\|\nabla\varphi\|_{L^2} \leq \gamma\|\mathbf{m}\|_{L^2}. \tag{6}$$

This follows by writing the magnetostatic equation in weak form, $\int(-\nabla\varphi + \gamma\mathbf{m}) \cdot \nabla\xi \, d\mathbf{x} = 0$ for all $\xi \in L^2$, choosing $\xi = \varphi$, then using Schwarz' inequality on the second term.

The main goal of this section is to get an alternative expression for (2) that involves quantities (unlike $\mathbf{h}^{(\delta)}$) that can be calculated easily from the given data $\mathbf{m} : \Omega' \to \mathbb{R}^3$ and $\mathscr{P}$.

A useful quantity for discussions of magnetic forces is the Maxwell stress $\mathbf{T}^M$. Suppose that on a domain $\mathscr{D} \in \mathbb{R}^3$ we have a smooth magnetization $\mathbf{m} : \mathscr{D} \to \mathbb{R}^3$ and a solution φ of the magnetostatic equation, $\mathrm{div}(\nabla\varphi + \gamma\mathbf{m}\chi_{\mathscr{D}}) = 0$. The Maxwell stress is defined by,

$$\mathbf{T}^M := -\frac{1}{\gamma}\left(\nabla\varphi \otimes \gamma\mathbf{m} - \nabla\varphi \otimes \nabla\varphi + \frac{1}{2}(\nabla\varphi \cdot \nabla\varphi)\mathbf{I}\right). \tag{7}$$

The useful properties of the Maxwell stress are given in the following Lemma.

Lemma 1.1 *Let $\varphi \in \mathscr{V}$ be the magnetostatic potential of $\mathbf{m} \in L^2$, $\mathrm{div}(-\nabla\varphi + \gamma\mathbf{m}\chi_\Omega) = 0$. If $\mathbf{m}$ is smooth on a region $\mathscr{D} \in \Omega$, then*

$$\mathrm{div}\mathbf{T}^M = -\mathbf{m} \cdot \nabla^2\varphi \tag{8}$$

holds on $\mathscr{D}$. In addition, assume the classical kinematics of a jump in magnetization discussed above, for which φ satisfies the classical jump conditions, $[\![\nabla\varphi]\!] = \gamma[\![\mathbf{m} \cdot \mathbf{n}]\!]\mathbf{n}$. Then,

$$[\![\mathbf{T}^M\mathbf{n}]\!] = -\frac{\gamma}{2}[\![(\mathbf{m} \cdot \mathbf{n})^2]\!]\mathbf{n}. \tag{9}$$

Proof. Integrate the right hand side of (8) by parts:

$$\begin{aligned} m_i\varphi_{,ij} &= (m_i\varphi_{,j})_{,i} - m_{i,i}\varphi_{,j}, \\ &= (m_i\varphi_{,j})_{,i} - \frac{1}{\gamma}(\varphi_{,ii}\varphi_{,j}), \\ &= (m_i\varphi_{,j})_{,i} - \frac{1}{\gamma}\left((\varphi_{,i}\varphi_{,j})_{,i} - \varphi_{,i}\varphi_{,ij}\right), \\ &= (m_i\varphi_{,j})_{,i} - \frac{1}{\gamma}\left((\varphi_{,i}\varphi_{,j})_{,i} - \frac{1}{2}(\varphi_{,i}\varphi_{,i})_{,j}\right), \end{aligned} \tag{10}$$

which gives (8) The jump condition (9) follows by straightforward application of the jump condition $[\![\nabla\varphi]\!] = \gamma[\![\mathbf{m} \cdot \mathbf{n}]\!]\mathbf{n}$. □

Since discontinuities of $\mathbf{m} \cdot \mathbf{n}$ give rise to discontinuities of the field, we have to be careful with formulas like (2) in that case, because then $\nabla\mathbf{h}^{(\delta)}$ can contain hidden Dirac masses[6]. The derivation of this formula suggests in that case that we should smooth $\mathbf{m}$ slightly. So, we consider $\mathbf{m} \in L^{(2)}(\Omega')$ and let $\mathbf{m}^{(\varepsilon)} \to \mathbf{m}$ be a smooth approximation. Let the magnetostatic potentials for various regions be given by

$$\begin{aligned} \mathrm{div}(-\nabla\psi^{(\varepsilon)} + \gamma\mathbf{m}^{(\varepsilon)}\chi_{\mathscr{P}}) &= 0, \\ \mathrm{div}(-\nabla\psi^{(\varepsilon,\delta)} + \gamma\mathbf{m}^{(\varepsilon)}\chi_{\mathscr{P}_\delta}) &= 0, \\ \mathrm{div}(-\nabla\varphi^{(\varepsilon)} + \gamma\mathbf{m}^{(\varepsilon)}\chi_{\Omega'}) &= 0, \end{aligned} \tag{11}$$

$\psi^{(\varepsilon)}, \psi^{(\varepsilon,\delta)}, \varphi^{(\varepsilon)} \in \mathscr{V}$. Using linearity, the formula for the force is then, assuming existence of the limits,

$$\mathscr{F} = \lim_{\delta\to 0}\lim_{\varepsilon\to 0}\int_{\mathscr{P}} \mathbf{m}^{(\varepsilon)} \cdot \nabla^2(\psi^{(\varepsilon,\delta)} - \varphi^{(\varepsilon)}) \, d\mathbf{x}, \tag{12}$$

$$= \lim_{\delta\to 0}\lim_{\varepsilon\to 0}\int_{\mathscr{P}} \mathbf{m}^{(\varepsilon)} \cdot \nabla^2\psi^{(\varepsilon,\delta)} \, d\mathbf{x} - \lim_{\varepsilon\to 0}\int_{\mathscr{P}} \mathbf{m}^{(\varepsilon)} \cdot \nabla^2\varphi^{(\varepsilon)} \, d\mathbf{x}. \tag{13}$$

[6] A hidden Dirac mass is a Dirac mass (delta function) that arises in an integrand on passing to a sharp interface limit.

Note that the inner limit in (12) exists because of the positive minimum distance between $\Omega' \setminus \mathscr{P}_\delta$ and $\mathscr{P}$) and the fact that $\triangle(\psi^{(\varepsilon,\delta)} - \varphi^{(\varepsilon)}) = 0$ on $\mathscr{P}$. The existence of the other limits needs to be proved.

In (13) the last term is relatively convenient since it involves the overall (smoothed) field. The goal is to calculate the first term.

Lemma 1.2 *We have,*

$$\int_{\mathscr{P}} m_i^{(\varepsilon)} \psi_{,ij}^{(\varepsilon,\delta)} d\mathbf{x} = \frac{1}{\gamma} \int_{\partial\mathscr{P}} \left(\gamma \psi_j^{(\varepsilon,\delta)} m_i^{(\varepsilon)} - \psi_{,j}^{(\varepsilon,\delta)} \psi_{,i}^{(\varepsilon,\delta)} + \frac{1}{2} (\psi_{,k}^{(\varepsilon,\delta)} \psi_{,k}^{(\varepsilon,\delta)}) \delta_{ij} \right) n_i \, da. \tag{14}$$

Proof. This follows immediately from the first property of the Maxwell stress above.

Lemma 1.3 *Assume that $\partial\mathscr{P} \in C^2$ and let $\nabla\psi^{(\varepsilon),o}$ denote the outer trace of $\nabla\psi^{(\varepsilon)}$ on $\partial\mathscr{P}$. Then, in components*

$$\int_{\partial\mathscr{P}} \left(\psi_{,i}^{(\varepsilon),o} \psi_{,j}^{(\varepsilon),o} - \frac{1}{2} (\psi_{,k}^{(\varepsilon),o} \psi_{,k}^{(\varepsilon),o}) \delta_{ij} \right) n_i \, da = 0. \tag{15}$$

Proof. Regularity theory for elliptic equations (Gilbarg and Trudinger [20], Theorem 8.2) shows that if $\mathscr{P}$ has a C^2 boundary then $\psi^{(\varepsilon)} \in W^{2,2}(\Omega \setminus \mathscr{P})$, which in turn implies that $\nabla\psi^{(\varepsilon)}$ has an outer trace (i.e. boundary values from the outside) on $\partial\mathscr{P}$. Using the divergence theorem (cf., Evans and Gariepy [18], p. 133) and decay properties of $\nabla\psi^{(\varepsilon)}$, the left hand side of (15) is

$$\int_{\mathbb{R}^3 \setminus \mathscr{P}} \left(\psi_{,i}^{(\varepsilon)} \psi_j^{(\varepsilon)} - \frac{1}{2} (\psi_{,k}^{(\varepsilon)} \psi_{,k}^{(\varepsilon)}) \delta_{ij} \right)_{,i} \, d\mathbf{x} \, . \tag{16}$$

Since $\psi_{,ii}^{(\varepsilon)} = 0$ on $\mathbb{R}^3 \setminus \mathscr{P}$, this is

$$\int_{\mathbb{R}^3 \setminus \mathscr{P}} \left(\psi_{,ii}^{(\varepsilon)} \psi_j^{(\varepsilon)} + \psi_{,i}^{(\varepsilon)} \psi_{,ij}^{(\varepsilon)} - 2 \frac{1}{2} (\psi_{,kj}^{(\varepsilon)} \psi_{,k}^{(\varepsilon)}) \right) d\mathbf{x} = 0. \tag{17}$$

□

Formalizing the notation introduced above, quantities evaluated on a boundary not having the superscript ^{o} are always assumed to be inner traces. Now calculate (14) + $\frac{1}{\gamma}$(15):

$$\begin{aligned} \int_{\mathscr{P}} m_i^{(\varepsilon)} \psi_{,ij}^{(\varepsilon,\delta)} d\mathbf{x} = \frac{1}{\gamma} \int_{\partial\mathscr{P}} (& \gamma m_i^{(\varepsilon)} \psi_j^{(\varepsilon,\delta)} \quad - \\ & \psi_{,i}^{(\varepsilon,\delta)} \psi_j^{(\varepsilon,\delta)} + \psi_{,i}^{(\varepsilon),o} \psi_j^{(\varepsilon),o} \quad + \quad \frac{1}{2} (\psi_{,k}^{(\varepsilon,\delta)} \psi_{,k}^{(\varepsilon,\delta)} - \psi_{,k}^{(\varepsilon),o} \psi_{,k}^{(\varepsilon),o}) \delta_{ij}) \, n_i \, da \end{aligned} \tag{18}$$

Let $\eta^{(\varepsilon,\delta)} = \psi^{(\varepsilon,\delta)} - \psi^{(\varepsilon)}$, so that

$$\operatorname{div}(-\nabla\eta^{(\varepsilon,\delta)} + \gamma \mathbf{m}^{(\varepsilon)} \chi_{\mathscr{P}_\delta \setminus \mathscr{P}}) = 0. \tag{19}$$

In view of (19), the L^∞ bound on $\mathbf{m}$, and the bound (6), we have

$$||\nabla\eta^{(\varepsilon,\delta)}||_{L^2} \quad \leq \quad \gamma \, ||\mathbf{m}^{(\varepsilon)} \chi_{\mathscr{P}_\delta \setminus \mathscr{P}}||_{L^2} \quad \leq \quad \mathrm{O}(\delta), \tag{20}$$

independent of ε. In this setting we have the classical jump conditions for $\psi^{(\varepsilon)}$ at the boundary of $\mathscr{P}$:

$$\nabla\psi^{(\varepsilon)} - \nabla\psi^{(\varepsilon),\mathrm{o}} = \gamma(\mathbf{m}^\varepsilon \cdot \mathbf{n})\mathbf{n}. \tag{21}$$

Eliminate $\psi^{(\varepsilon)}$ from (21) using the definition of $\eta^{(\varepsilon,\delta)}$,

$$\nabla\psi^{(\varepsilon,\delta)} = \nabla\psi^{(\varepsilon),\mathrm{o}} \; + \; \gamma(\mathbf{m}^\varepsilon \cdot \mathbf{n})\mathbf{n} \; + \; \nabla\eta^{(\varepsilon,\delta)}. \tag{22}$$

Substitute (22) into (18), and simplify:

$$\int_{\mathscr{P}} m_i^{(\varepsilon)}\psi_{,ij}^{(\varepsilon,\delta)}d\mathbf{x} = \frac{\gamma}{2}\int_{\partial\mathscr{P}}(\mathbf{m}^{(\varepsilon)}\cdot\mathbf{n})^2 n_j\,da \quad + \\ \frac{1}{\gamma}\int_{\partial\mathscr{P}}\left(-\psi_{,i}^{(\varepsilon),o}\eta_{,j}^{(\varepsilon,\delta)} - \eta_{,i}^{(\varepsilon,\delta)}\psi_{,j}^{(\varepsilon),o} - \eta_{,i}^{(\varepsilon,\delta)}\eta_{,j}^{(\varepsilon,\delta)} + \psi_{,k}^{(\varepsilon),o}\eta_{,k}^{(\varepsilon,\delta)}\delta_{ij} + \frac{1}{2}\eta_{,k}^{(\varepsilon,\delta)}\eta_{,k}^{(\varepsilon,\delta)}\delta_{ij}\right) n_i\,da \qquad (23)$$

Now extend $\nabla\psi^{(\varepsilon),o}$ to the interior of $\mathscr{P}$ as a $W^{1,2}(\mathscr{P})$ function (Lemma 1.3 and, e.g., Evans and Gariepy [18], p. 135). Let the extension be called $\mathbf{v}^{(\varepsilon)}$. Use the divergence theorem on the second integral on the right hand side of (23) and simplify:

$$\int_{\mathscr{P}} m_i^{(\varepsilon)}\psi_{,ij}^{(\varepsilon,\delta)}d\mathbf{x} = \frac{\gamma}{2}\int_{\partial\mathscr{P}}(\mathbf{m}^{(\varepsilon)}\cdot\mathbf{n})^2 n_j\,da \quad + \frac{1}{\gamma}\int_{\mathscr{P}}\left(-v_{i,i}^{(\varepsilon)}\eta_{,j}^{(\varepsilon,\delta)} + v_{i,j}^{(\varepsilon)}\eta_{,i}^{(\varepsilon,\delta)} - v_{j,i}^{(\varepsilon)}\eta_{,i}^{(\varepsilon,\delta)}\right)d\mathbf{x} \qquad (24)$$

The original $W^{2,2}$ bound on $\psi^{(\varepsilon)}$ is uniform in ε on $(\mathbb{R}^3\setminus\mathscr{P})$. Thus $\mathbf{v}^{(\varepsilon,o)}$ inherits a $W^{(1,2)}$ bound uniform in ε. In that case the last term in (24) has a double limit $lim_{\delta\to 0}\ lim_{\varepsilon\to 0}$ which is zero in view of (20).

Given the existence of the limits, we get the following version of Brown's formula:

$$\mathscr{F} = \lim_{\varepsilon\to 0}\left(\frac{\gamma}{2}\int_{\partial\mathscr{P}}(\mathbf{m}^{(\varepsilon)}\cdot\mathbf{n})^2\,\mathbf{n}\,da \quad - \quad \int_{\mathscr{P}}\mathbf{m}^{(\varepsilon)}\cdot\nabla^2\varphi^{(\varepsilon)}\,d\mathbf{x}\right) \quad + \quad \int_{\mathscr{P}}(\mathbf{m}\cdot\nabla\mathbf{h}_0)\,d\mathbf{x}. \qquad (25)$$

Here, we have also included the applied field.

The remaining issue is the existence of limits in (25). The double limit of the second term in (24) exists and vanishes as explained above, under suitable hypotheses. If the limit as $\varepsilon\to 0$ of the first term in (24) exists, together with the limit as $\varepsilon\to 0$ of the second term in (25), then all limits exist. Simple sufficient conditions are given below.

Lemma 1.4 *Let the smoothed magnetization $\mathbf{m}^{(\varepsilon)}\to\mathbf{m}$ in L^2 be as above, and let $\varphi^{(\varepsilon)}$ be its magnetoelastic potential. Assume that $\|\,\mathbf{m}^{(\varepsilon)}\,\|_{L^2(\partial\mathscr{P})}$ and $\|\,\nabla\varphi^{(\varepsilon)}\,\|_{L^2(\partial\mathscr{P})}$ are bounded independent of ε, and that $\mathbf{m}^{(\varepsilon)}\to\mathbf{m}$ and $\nabla\varphi^{(\varepsilon)}\to\nabla\varphi$ pointwise a.e. on $\partial\mathscr{P}$. Then the two limits,*

$$lim_{\varepsilon\to 0}\int_{\mathscr{P}}\mathbf{m}^{(\varepsilon)}\cdot\nabla^2\varphi^{(\varepsilon)}\,d\mathbf{x} \quad\text{and}\quad lim_{\varepsilon\to 0}\int_{\partial\mathscr{P}}(\mathbf{m}^{(\varepsilon)}\cdot\mathbf{n})^2\mathbf{n}\,da \qquad (26)$$

exist.

Remark. As is seen from the lemma, the essential point for the existence of the limits is that domain walls can intersect $\partial\mathscr{P}$ transversely, but they cannot coincide with $\partial\mathscr{P}$ on a region with positive area. Of course, in order that the formula for $\mathscr{F}$ be acceptable, the limits have to exist, and therefore the hypotheses would have to hold, for all such approximating sequences. This would not be true for an arbitrary smooth approximation in L^2.

Proof. The first integral in (26) is a boundary integral. That is, the Maxwell stress calculation (Lemma 1.1) applies, with $\psi^{(\varepsilon,\delta)}$ replaced by $\varphi^{(\varepsilon)}$. The limit of the resulting integral over $\partial\mathscr{P}$ (and also the limit of the second integral in (26)) exists by the bounded convergence theorem. □

From (25) the long range magnetic force on $\mathscr{P}$ has a contribution that mimics a hydrostatic pressure applied at $\partial\mathscr{P}$. Unlike usual hydrostatic pressures, the value of the pressure depends on the normal.

Brown's formula has a weird feature. The long range force evaluated for a sequence of domains with a wiggly boundary that approaches uniformly a smooth domain is not equal to the force evaluated for the smooth domain. This is because of weak convergence and the cubic dependence on the normal in the force formula. It would be interesting to explore the physical implications of this observation.

Brown assumes that the total force on $\mathscr{P}$ is the sum of the long range magnetic force $\mathscr{F}$ and a local force that he assumes has the form of a surface traction,

$$\mathscr{F}_{loc} = \int_{\partial\mathscr{P}}\mathbf{t}(\mathbf{x},\mathbf{n})\,da\ . \qquad (27)$$

This must be recognized as quite a strong assumption.

Having a formula for the total force on $\mathscr{P}$, namely $\mathscr{F} + \mathscr{F}_{loc}$, it is possible to do continuum mechanics in the standard way. In particular, put the total force equal to the inertia and apply to the Cauchy tetrahedron. Of course, the Cauchy argument has to give a stress that is linear in the normal. The result is that there is a piece of the traction $\mathbf{t}(\mathbf{x}, \mathbf{n})$ that splits off and that exactly cancels the first term in (25). Specifically, the argument gives that $\mathbf{t}(\mathbf{x}, \mathbf{n}) + \frac{\gamma}{2}(\mathbf{m} \cdot \mathbf{n})^2 \mathbf{n} =: \mathbf{T}\mathbf{n}$ is linear in the normal. So, if the assumptions are justified, the first term in (25) is in a way a red herring. However, as shown below, this term also arises in several places in configurational force arguments, and in the final form of the configurational force, so that the possible weirdness associated with the cubic dependence on the normal deserves further study.

We do not repeat these arguments, which are straightforward and can be found in Brown ([6], Chapter II) and Toupin [46], but summarize the results in the next section.

1.2.2 Summary of equations of motion, constitutive equations and boundary conditions

Let $\mathbf{y} : \Omega \times [0, \infty) \to \mathbb{R}^3$ be the motion, $\mathbf{h} : \mathbb{R}^3 \times [0, \infty) \to \mathbb{R}^3$ be the magnetic field and $\mathbf{m} : \mathbb{R}^3 \times [0, \infty) \to \mathbb{R}^3$ be the magnetization. The magnetization is supported on the deformed configuration $\mathbf{y}(\Omega, t)$; sometimes we emphasize this by writing $\mathbf{m}\chi_{\mathbf{y}(\Omega,t)}$ in place of $\mathbf{m}$. To connect with the preceding section, $\Omega' = \mathbf{y}(\Omega, t)$ and $\mathbf{h} = -\nabla\varphi + \mathbf{h}_0$ where $\varphi \in \mathscr{V}$ is the solution of

$$\operatorname{div}(-\nabla\varphi + \gamma \mathbf{m}\chi_{\mathbf{y}(\Omega,t)}) = 0, \tag{28}$$

To avoid confusion between gradients in the reference and deformed configurations, we reserve the symbol ∇ for the gradient of a function in the deformed configuration; the notation ∇_x will denote the gradient with respect to $\mathbf{x}$ of a function of $(\mathbf{x}, t)$. The superimposed dot denotes the time derivative of a function of $(\mathbf{x}, t)$, while a subscript $,_t$ denotes the partial derivative with respect to t of a function in the deformed configuration.

There is assumed to be a free energy per unit reference volume,

$$W(\mathbf{A}, \mathbf{m}), \tag{29}$$

where $\mathbf{A}$ $(= \nabla_x \mathbf{y})$ is the deformation gradient and $\mathbf{m}$ is the magnetization. A detailed discussion of the restrictions on W arising from frame–indifference and symmetry are given by James and Kinderlehrer [23] (the setting of this work is the study of giant magnetostrictive materials, but the treatment of invariance applies also to ferromagnetic shape memory materials). Brown assumed a saturation condition on the magnetization of the form $|(\det\mathbf{A})\mathbf{m}| = m_s$, where m_s was assumed to be a scalar function of temperature only, reflecting the constancy of the dipole moment per "atom". There are to my knowledge no experimental tests of the form of this relation, but as explained above it is significantly violated in ferromagnetic shape memory materials (Typical data in Ni_2MnGa give approximately $m_s = 0.11$ for martensite and 0.14 for austenite in arbitrary units (e.g., Pan and James [41])). There might be a tendency for $|(\det\mathbf{A})\mathbf{m}| = m_s$ to be nearly satisfied (with appropriate values of m_s) for $\mathbf{A}$ near the energy wells of austenite (resp., martensite). In any case, in this paper we will not assume any condition of saturation; if such a condition is operative in some subdomain of W, we will impose it through strong growth of W away from the constraint set. This is a reasonable physical assumption, given that even conventional ferromagnetic materials noticeably violate the constraint of saturation under sufficiently large fields. In all ferromagnetic materials W has a temperature dependence that we have suppressed.

It is useful to introduce the magnetization per unit reference volume $\mathbf{m}_o$, defined by,

$$\mathbf{m}_o = (\det\mathbf{A})\mathbf{m} \, . \tag{30}$$

(When $\mathbf{m}_o$ occurs in various equations, it is viewed as a function of $(\mathbf{x}, t)$ defined on $\Omega \times [0, \infty)$. It is then calculated in the obvious way $\mathbf{m}_o(\mathbf{x}, t) = (\det\nabla_{\mathbf{x}}\mathbf{y}(\mathbf{x}, t))\mathbf{m}(\mathbf{y}(\mathbf{x}, t), t)$). An alternative form of the free energy per unit reference volume is given by:

$$W_o(\mathbf{A}, \mathbf{m}_o) = W(\mathbf{A}, (\det\mathbf{A})^{-1}\mathbf{m}_o). \tag{31}$$

Using a virtual work argument Toupin gets that

$$\mathbf{T}_o = \frac{\partial W_o}{\partial \mathbf{A}}, \qquad \mathbf{h} = \frac{\partial W_o}{\partial \mathbf{m}_o}. \tag{32}$$

Like Brown, we shall carry these formulas over to dynamics.

Following the procedure sketched above (and given in detail in Chapter II of Brown), the balance of linear momentum is,

$$\frac{d}{dt}\int_{\mathscr{P}} \rho_o \dot{\mathbf{y}}\, d\mathbf{x} = \int_{\mathbf{y}(\mathscr{P},t)} (\mathbf{m}\cdot\nabla\mathbf{h}) d\mathbf{y} \; + \; \int_{\mathscr{P}} \mathrm{div}\mathbf{T}_o d\mathbf{x}, \tag{33}$$

with local form,

$$\rho_o \ddot{\mathbf{y}} = (\mathbf{m}\cdot\nabla\mathbf{h})(\det\nabla\mathbf{y}) \; + \; \mathrm{div}\mathbf{T}_o. \tag{34}$$

Here, $\rho_o : \Omega \to \mathbb{R}$ is the assigned reference density. The traction on $\partial\mathbf{y}(\mathscr{P},t)$ is

$$\mathbf{t} = \mathbf{T}\mathbf{n} \; - \; \frac{\gamma}{2}(\mathbf{m}\cdot\mathbf{n})^2\mathbf{n}. \tag{35}$$

The relation between $\mathbf{T}$ and $\mathbf{T}_o$ is the usual relation: $\mathbf{T} = (\det\mathbf{A})^{-1}\mathbf{T}_o\mathbf{A}^T$. If, as part of the specification of boundary conditions, mechanical loads were applied to $\partial\mathbf{y}(\Omega,t)$, then these would correspond to assigning $\mathbf{t}$ there. There is the obvious referential version of (35), i.e.,

$$\mathbf{t}_o = \mathbf{T}_o\mathbf{n}_o - \frac{\gamma}{2}\frac{(\mathbf{m}_o\cdot\mathbf{A}^{-T}\mathbf{n}_o)^2\mathbf{A}^{-T}\mathbf{n}_o}{(\mathbf{n}_o\cdot\mathbf{C}^{-1}\mathbf{n}_o)(\det\mathbf{A})}, \tag{36}$$

where $\mathbf{C} = \mathbf{A}^T\mathbf{A}$. Equation (36) would be used if the force per unit reference area $\mathbf{t}_o$ were assigned.

The following calculation indicates briefly the relation between the right hand side of (33), the boundary conditions and the formula for the long range force $\mathscr{F}$ given in Sect. 1.2.1. The right hand side of (33) is the force on $\mathscr{P}$:

$$\text{Force on } \mathscr{P} = \int_{\mathbf{y}(\mathscr{P},t)} (\mathbf{m}\cdot\nabla\mathbf{h}) d\mathbf{y} + \int_{\partial\mathbf{y}(\mathscr{P},t)} \mathbf{T}\mathbf{n} da. \tag{37}$$

Write $\mathbf{h} = \mathbf{h}_0 - \nabla\varphi$ and introduce the boundary conditions (35):

$$= \int_{\mathbf{y}(\mathscr{P},t)} (\mathbf{m}\cdot\nabla\mathbf{h}_0) d\mathbf{y} - \int_{\mathbf{y}(\mathscr{P},t)} (\mathbf{m}\cdot\nabla^2\varphi) d\mathbf{y} + \int_{\partial\mathbf{y}(\mathscr{P},t)} (\mathbf{t} + \frac{\gamma}{2}(\mathbf{m}\cdot\mathbf{n})^2\mathbf{n})\, da. \tag{38}$$

Introduce into (38) the formula (25) for the long range force, assuming everything is smooth:

$$= \mathscr{F} + \int_{\partial\mathbf{y}(\mathscr{P},t)} \mathbf{t}\, da. \tag{39}$$

This treatment requires a bit more care when we have discontinuities.

The second part of the Cauchy tetrahedron argument usually gives the symmetry of the stress $\mathbf{T}$. The formula for the torque on $\mathscr{P}$ contains an additional contribution due ultimately to the torque on a dipole being given by $\mathbf{d}\times\mathbf{h}$ where $\mathbf{d}$ is the dipole moment. There is a lengthy but straightforward argument of the type given in Sect. 1.2.1, with the following outline: 1) develop a formula like (1) for the torque on $\mathscr{P}$, 2) calculate the long range magnetic torque analogously to the argument in Sect. 1.2.1, 3) balance torques on the tetrahedron. The result is that $\mathbf{T}$ is not generally symmetric, and its antisymmetric part is given by the explicit formula,

$$(\mathbf{T} - \mathbf{T}^T) = (\mathbf{m}\otimes\mathbf{h} - \mathbf{h}\otimes\mathbf{m}). \tag{40}$$

1.3 Rate of change of demagnetization energy in the smooth case

We are heading for an expression for the balance of energy of a magnetoelastic body, first in the case that fields and motions are smooth (note that this may entail quite strong assumptions about smoothness of the regions considered, in view of the magnetostatic equation). One term in the energy equation is expected to be the rate of change of demagnetization energy, so here we consider only this term. The main tool is Reynolds' Transport Theorem.

Let E_m denote the demagnetization energy:

$$E_m \;=\; \frac{1}{2}\int_{\mathbf{y}(\Omega,t)} \mathbf{m}\cdot\nabla\varphi \; d\mathbf{y} \;=\; \frac{1}{2}\int_{\Omega} \mathbf{m}_{\mathrm{o}}\cdot\nabla\varphi \; d\mathbf{x} \;=\; \frac{1}{2\gamma}\int_{\mathbb{R}^3} |\nabla\varphi|^2 d\mathbf{y}. \tag{41}$$

Here, as usual, the magnetostatic equation is $\mathrm{div}(-\nabla\varphi + \gamma\mathbf{m}\chi_{\mathbf{y}(\Omega,t)}) = 0$ and the first and third terms of (41) are related through this equation. The result we seek is:

Lemma 1.5 (Brown). *Let* $\mathbf{v}(\mathbf{y}(\mathbf{x},t),t) = \dot{\mathbf{y}}(\mathbf{x},t)$ *be the spatial description of velocity and let* φ *be the magnetostatic potential.*

$$\frac{dE_m}{dt} = \int_{\Omega} \nabla\varphi\cdot\dot{\mathbf{m}}_{\mathrm{o}}\; d\mathbf{x} + \int_{\mathbf{y}(\Omega,t)} \mathbf{m}\cdot\nabla^2\varphi\,\mathbf{v}\; d\mathbf{y} - \frac{\gamma}{2}\int_{\partial\mathbf{y}(\Omega,t)} (\mathbf{m}\cdot\mathbf{n})^2(\mathbf{v}\cdot\mathbf{n})\; da. \tag{42}$$

Proof. Use Reynolds' Transport Theorem on the third form of (41):

$$\begin{aligned}
\frac{dE_m}{dt} &= \frac{1}{2\gamma}\frac{d}{dt}\int_{\mathbb{R}^3} |\nabla\varphi|^2 d\mathbf{y} \\
&= \frac{1}{2\gamma}\frac{d}{dt}\int_{\mathbf{y}(\Omega,t)} |\nabla\varphi|^2 d\mathbf{y} + \frac{1}{2\gamma}\frac{d}{dt}\int_{\mathbb{R}^3\setminus\mathbf{y}(\Omega,t)} |\nabla\varphi|^2 d\mathbf{y} \\
&= \frac{1}{2\gamma}\int_{\mathbf{y}(\Omega,t)} \frac{\partial}{\partial t}|\nabla\varphi|^2 d\mathbf{y} + \frac{1}{2\gamma}\int_{\mathbb{R}^3\setminus\mathbf{y}(\Omega,t)} \frac{\partial}{\partial t}|\nabla\varphi|^2 d\mathbf{y} - \frac{1}{2\gamma}\int_{\partial\mathbf{y}(\Omega,t)} [\![|\nabla\varphi|^2]\!](\mathbf{v}\cdot\mathbf{n})\; da.
\end{aligned} \tag{43}$$

On $\mathbf{y}(\Omega,t)$ we have a kind of temporal version of the Maxwell stress calculation,

$$\begin{aligned}
\mathbf{m}\cdot\nabla\varphi_{,t} &= m_i\varphi_{,it} \\
&= (m_i\varphi_{,t}\,)_{,i} - m_{i,i}\varphi_{,t} \\
&= (m_i\varphi_{,t}\,)_{,i} - \frac{1}{\gamma}\varphi_{,ii}\;\varphi_{,t} \\
&= (m_i\varphi_{,t}\,)_{,i} - \frac{1}{\gamma}\left((\varphi_{,i}\;\varphi_{,t}\,)_{,i} - \frac{1}{2}(\varphi_{,i}\;\varphi_{,i}\,)_{,t}\right),
\end{aligned} \tag{44}$$

that is,

$$\frac{1}{2\gamma}\frac{\partial}{\partial t}|\nabla\varphi|^2 = \mathbf{m}\cdot\nabla\varphi_{,t} \;-\; \mathrm{div}(\varphi_{,t}\mathbf{m} - \frac{1}{\gamma}\varphi_{,t}\,\nabla\varphi). \tag{45}$$

On $\mathbb{R}^3\setminus\mathbf{y}(\Omega,t)$ we have similarly,

$$(\varphi_{,i}\;\varphi_{,t}\,)_{,i} = \varphi_{,ii}\;\varphi_{,t} + \varphi_{,i}\;\varphi_{,it} = \frac{1}{2}(\varphi_{,k}\;\varphi_{,k}\,)_{,t}\;\;, \tag{46}$$

that is,

$$\frac{1}{2\gamma}\frac{\partial}{\partial t}|\nabla\varphi|^2 = \frac{1}{\gamma}\mathrm{div}(\varphi_{,t}\,\nabla\varphi) \tag{47}$$

Insert (45) and (47) into (43):

$$\begin{aligned}
\frac{dE_m}{dt} = &\int_{\mathbf{y}(\Omega,t)} \mathbf{m}\cdot\nabla\varphi_{,t}\, d\mathbf{y} - \int_{\partial\mathbf{y}(\Omega,t)} \left(\varphi_{,t}\,\mathbf{m} - \frac{1}{\gamma}\varphi_{,t}\,\nabla\varphi\right)\cdot\mathbf{n}\, da \\
&-\frac{1}{\gamma}\int_{\partial\mathbf{y}(\Omega,t)} \varphi^o{}_{,t}\,\nabla\varphi^o\cdot\mathbf{n}\, da - \frac{1}{2\gamma}\int_{\partial\mathbf{y}(\Omega,t)} [\![|\nabla\varphi|^2]\!]\mathbf{v}\cdot\mathbf{n}\, da.
\end{aligned} \tag{48}$$

Here, the superscript ^{o} denotes the outer trace on $\partial\mathbf{y}(\Omega,t)$. Now return to (41) but use the second form:

$$\begin{aligned}\frac{dE_m}{dt} &= \frac{1}{2}\frac{d}{dt}\int_\Omega \mathbf{m}_o(\mathbf{x},t)\cdot\nabla\varphi(\mathbf{y}(\mathbf{x},t),t)d\mathbf{x},\\ &= \frac{1}{2}\int_\Omega \left(\dot{\mathbf{m}}_o\cdot\nabla\varphi+\mathbf{m}_o\cdot\nabla^2\varphi\,\dot{\mathbf{y}}+\mathbf{m}_o\cdot\nabla\varphi_{,t}\right)d\mathbf{x}.\end{aligned}\tag{49}$$

Calculate $2\times(49)-(48)$:

$$\begin{aligned}\frac{dE_m}{dt} = \int_\Omega \left(\dot{\mathbf{m}}_o\cdot\nabla\varphi+\mathbf{m}_o\cdot\nabla^2\varphi\,\dot{\mathbf{y}}\right)d\mathbf{x} &- \int_{\partial\mathbf{y}(\Omega,t)}[\![\varphi_{,t}\,\mathbf{m}-\frac{1}{\gamma}\varphi_{,t}\,\nabla\varphi]\!]\cdot\mathbf{n}da\\ &+ \frac{1}{2\gamma}\int_{\partial\mathbf{y}(\Omega,t)}[\![|\nabla\varphi|^2]\!]\mathbf{v}\cdot\mathbf{n}da.\end{aligned}\tag{50}$$

Now invoke the kinematic jump conditions for $(\varphi_{,t}\,,\nabla\varphi)$ at $\partial\mathbf{y}(\Omega,t)$, and combine them with the jump conditions coming from the magnetostatic equation, i.e.,

$$[\![\nabla\varphi]\!]=-\gamma(\mathbf{m}\cdot\mathbf{n})\mathbf{n},\quad [\![\varphi_{,t}\,]\!]=\gamma(\mathbf{m}\cdot\mathbf{n})\mathbf{n}\cdot\mathbf{v},\quad \text{(see Sect. 1.5).}\tag{51}$$

Substitute these jump conditions into the right hand side of (50) and do a little manipulation: the integrand of the boundary terms in (50) collapses to $-\frac{\gamma}{2}(\mathbf{m}\cdot\mathbf{n})^2(\mathbf{v}\cdot\mathbf{n})$.

□

Evidently, the first term of (42) represents the change of demagnetization energy expected when the magnetization rotates (but, e.g., the body is not allowed to change shape). The remaining terms are working terms arising from changes of shape occurring within field produced by the body's own magnetization.

1.4 Balance of energy in the smooth case

In the usual way begin with the local form of the balance of linear momentum (34), take the inner product with the velocity, and integrate over Ω :

$$\begin{aligned}\frac{d}{dt}\int_\Omega\frac{\rho_o}{2}|\dot{\mathbf{y}}|^2d\mathbf{x} &= \int_{\mathbf{y}(\Omega,t)}(\mathbf{m}\cdot\nabla\mathbf{h})\cdot\mathbf{v}\,d\mathbf{y}+\int_\Omega\left(\mathrm{div}(\mathbf{T}_o^T\dot{\mathbf{y}})-\mathbf{T}_o\cdot\nabla\dot{\mathbf{y}}\right)d\mathbf{x}\\ &= \int_{\mathbf{y}(\Omega,t)}(\mathbf{m}\cdot\nabla\mathbf{h})\cdot\mathbf{v}\,d\mathbf{y}+\int_{\partial\Omega}\dot{\mathbf{y}}\cdot\mathbf{T}_o\mathbf{n}_o da_o-\int_\Omega\frac{\partial W_o}{\partial\mathbf{A}}\cdot\nabla\dot{\mathbf{y}}\,d\mathbf{x}.\end{aligned}\tag{52}$$

Rearrange this, bringing the free energy density to the left hand side by using the relations (32) and also use the boundary conditions (35):

$$\frac{d}{dt}\int_\Omega\left(\frac{\rho_o}{2}|\dot{\mathbf{y}}|^2+W_o\right)d\mathbf{x}=\int_\Omega\mathbf{h}\cdot\dot{\mathbf{m}}_o\,d\mathbf{x}+\int_{\mathbf{y}(\Omega,t)}(\mathbf{m}\cdot\nabla\mathbf{h})\cdot\mathbf{v}\,d\mathbf{y}+\int_{\partial\mathbf{y}(\Omega,t)}\left(\mathbf{t}\cdot\mathbf{v}+\frac{\gamma}{2}(\mathbf{m}\cdot\mathbf{n})^2\mathbf{v}\cdot\mathbf{n}\right)da.\tag{53}$$

Separate out the applied and demag. fields, $\mathbf{h}=-\nabla\varphi+\mathbf{h}_0$ in (53):

$$\begin{aligned}\frac{d}{dt}\int_\Omega\left(\frac{\rho_o}{2}|\dot{\mathbf{y}}|^2+W_o\right)d\mathbf{x} &= -\left(\int_\Omega\nabla\varphi\cdot\dot{\mathbf{m}}_o d\mathbf{x}+\int_{\mathbf{y}(\Omega,t)}(\mathbf{m}\cdot\nabla^2\varphi)\cdot\mathbf{v}\,d\mathbf{y}-\frac{\gamma}{2}\int_{\partial\mathbf{y}(\Omega,t)}(\mathbf{m}\cdot\mathbf{n})^2\mathbf{v}\cdot\mathbf{n}\,da\right)\\ &+ \int_\Omega\mathbf{h}_0\cdot\dot{\mathbf{m}}_o\,d\mathbf{x}+\int_{\mathbf{y}(\Omega,t)}(\mathbf{m}\cdot\nabla\mathbf{h}_0)\cdot\mathbf{v}\,d\mathbf{y}+\int_{\partial\mathbf{y}(\Omega,t)}\mathbf{t}\cdot\mathbf{v}\,da\end{aligned}\tag{54}$$

Notice that the terms in parentheses in (54) are exactly the terms on the right hand side of (42). Hence, combining (42) with (54), we get,

$$\frac{d}{dt}\left[\int_\Omega\left(\frac{\rho_o}{2}|\dot{\mathbf{y}}|^2+W_o\right)d\mathbf{x}+E_m\right]=\int_\Omega\mathbf{h}_0\cdot\dot{\mathbf{m}}_o d\mathbf{x}+\int_{\mathbf{y}(\Omega,t)}(\mathbf{m}\cdot\nabla\mathbf{h}_0)\cdot\mathbf{v}\,d\mathbf{y}+\int_{\partial\mathbf{y}(\Omega,t)}\mathbf{t}\cdot\mathbf{v}\,da.\tag{55}$$

This is a form of the energy equation that has the desirable feature of having the energy of micromagnetics on the left hand side, except that the applied field energy is on the right. For later purposes, it is worth examining this equation for hidden Dirac masses. Thus we imagine that this balance of energy has been derived for smooth fields and we now contemplate taking a limit as the magnetization approximates a sharp domain wall. The only problematic term is the first term on the right hand side of (55). To remove the hidden Dirac mass we introduce the applied field (or Zeeman) energy:

$$E_z \;=\; -\int_{\Omega} \mathbf{h}_0 \cdot \mathbf{m}_{\mathrm{o}}\, d\mathbf{x} \;=\; -\int_{\mathbf{y}(\Omega,t)} \mathbf{h}_0 \cdot \mathbf{m}\, d\mathbf{y}. \tag{56}$$

Then we integrate the first term on the right hand side of (55) by parts (in time) and introduce the applied field energy:

$$\frac{d}{dt}\left[\int_{\Omega}\left(\frac{\rho_{\mathrm{o}}}{2}|\dot{\mathbf{y}}|^2 + W_{\mathrm{o}}\right) d\mathbf{x} + E_m + E_z\right] = -\int_{\Omega} \dot{\mathbf{h}}_0 \cdot \mathbf{m}_{\mathrm{o}}\, d\mathbf{x} + \int_{\mathbf{y}(\Omega,t)} \mathbf{m} \cdot \nabla \mathbf{h}_0 \mathbf{v}\, d\mathbf{y} + \int_{\partial\mathbf{y}(\Omega,t)} \mathbf{t} \cdot \mathbf{v}\, da. \tag{57}$$

This is the final form of the energy equation in the smooth case. Note that there are no hidden Dirac masses. That is, even though the energy equation was derived under strong assumptions of smoothness, it also is well-defined for velocities, deformation gradients and magnetizations that have moving sharp discontinuities, at least with the typical strong assumptions of regularity of the interfaces, limiting values, etc.

The usual thinking is the following. There may be sources of dissipation that have been disallowed by the derivation above. These can be kinematic in nature, like the sharp domain walls just mentioned (excluded from the derivation by smoothness), or else they can be thermodynamic (excluded from the derivation by the fact that we ignored the temperature dependence of W_{o}). The common practice in the case that dissipation is present is to postulate that (57) is satisfied with $\leq$. In this case the l.h.s. – r.h.s. of (57) is termed the *dissipation function*. The same inequality would result from a full thermodynamic treatment. From this inequality it is seen that (formally!), the usual energy of micromagnetics is a Lyapunov function as long as the applied field and traction are suitable (as a simple example, no applied loads and constant applied field).

1.5 Configurational forces

1.5.1 Kinematics of moving interfaces across which magnetization and deformation gradient may jump

In the following two subsections we recall the standard description of moving surfaces of discontinuity of velocity, deformation gradient and magnetization, mainly to set the notation. We will need to describe the interface motion in both deformed and reference configuration.

1.5.2 Moving discontinuity in the reference configuration

We assume that the interface is describable as the zero level set of a smooth function f_{o} defined on all of $\mathbb{R}^3 \times [0,\infty)$:

$$f_{\mathrm{o}}(\mathbf{x},t) = 0, \quad \text{with} \quad |\nabla_x f_{\mathrm{o}}| \neq 0. \tag{58}$$

Jumps are defined $[\![a]\!] = a^+ - a^-$ where a^+ (resp. a^-) are limiting values from the region $f_{\mathrm{o}} > 0$ (resp. $f_{\mathrm{o}} < 0$). Slices of this surface at fixed t, restricted to Ω, are denoted by $\mathscr{S}_{\mathrm{o}}^t$, and the spacelike reference normal is denoted by,

$$\mathbf{n}_{\mathrm{o}} = \frac{\nabla_x f_{\mathrm{o}}}{|\nabla_x f_{\mathrm{o}}|}. \tag{59}$$

We use the notation,

$$\Omega_+^t = \Omega \cap \{f_{\mathrm{o}} > 0\}, \quad \Omega_-^t = \Omega \cap \{f_{\mathrm{o}} < 0\}. \tag{60}$$

The surface locally admits a parametric representation $\tilde{\mathbf{x}}(u_1,u_2,t)$, the consistency condition being,

$$f_{\mathrm{o}}(\tilde{\mathbf{x}}(u_1,u_2,t),t) = 0 \quad \Longrightarrow \quad \nabla f_{\mathrm{o}} \cdot \dot{\tilde{\mathbf{x}}} + f_{\mathrm{o},t} = 0. \tag{61}$$

The spacetime reference normal is denoted

$$\nu_{\rm o} = (\nabla f_{\rm o}, f_{{\rm o},t}) \propto (\mathbf{n}_{\rm o}, -\dot{\tilde{\mathbf{x}}} \cdot \mathbf{n}_{\rm o}), \tag{62}$$

where we have used the second of (61). The normal reference velocity is

$$s_n^{\rm o} = \dot{\tilde{\mathbf{x}}} \cdot \mathbf{n}_{\rm o}. \tag{63}$$

1.5.3 Moving discontinuity in the deformed configuration

Under appropriate conditions of smoothness and invertibility of the motion, the interface is describable as the zero level set of a smooth function f defined on all of $\mathbb{R}^3 \times [0,\infty)$:

$$f(\mathbf{y},t) = 0, \quad \text{with} \quad |\nabla f| \neq 0. \tag{64}$$

Slices of this surface at fixed t, restricted to $\mathbf{y}(\Omega,t)$ are denoted by $\mathscr{S}^t$, and the spacelike normal is denoted by,

$$\mathbf{n} = \frac{\nabla f}{|\nabla f|}. \tag{65}$$

The surface locally admits a parametric representation $\tilde{\mathbf{y}}(u_1,u_2,t)$, the consistency condition being, as above,

$$f(\tilde{\mathbf{y}}(u_1,u_2,t),t) = 0 \quad \Longrightarrow \quad \nabla f \cdot \dot{\tilde{\mathbf{y}}} + f_{,t} = 0. \tag{66}$$

The spacetime normal is denoted

$$\nu = (\nabla f, f_{,t}) \propto (\mathbf{n}, -\dot{\tilde{\mathbf{y}}} \cdot \mathbf{n}), \tag{67}$$

where we have used the second of (66). The normal velocity is

$$s_n = \dot{\tilde{\mathbf{y}}} \cdot \mathbf{n}. \tag{68}$$

1.5.4 Various jump conditions

If the motion is represented as usual by $\mathbf{y}(\mathbf{x},t), \mathbf{x} \in \Omega$, then the relation between the referential and deformed parametric representations is,

$$\tilde{\mathbf{y}}(u_1,u_2,t) = \mathbf{y}(\tilde{\mathbf{x}}(u_1,u_2,t),t) \tag{69}$$

By definition, $\mathscr{S}_{\rm o}^t$ is a material surface if $f_{\rm o}$ is independent of t. Then, $\tilde{\mathbf{x}}(u_1,u_2)$ can be chosen independent of t and by differentiation of (69) with respect to t, we get (for material surfaces), $s_n = \mathbf{v} \cdot \mathbf{n}$.

If φ is a magnetostatic potential whose derivatives suffer a jump across $\mathscr{S}^t$, then Hadamard's jump condition applied to the space-time surface $f = 0$ gives,

$$[\![\nabla\varphi]\!] = \alpha\mathbf{n}, \quad [\![\varphi_{,t}]\!] = -s_n\alpha, \tag{70}$$

for some $\alpha : \mathscr{S}^t \to \mathbb{R}, t \geq 0$. This can be combined with the jump conditions coming from the magnetostatic equation to yield,

$$[\![\nabla\varphi]\!] = \gamma[\![\mathbf{m}\cdot\mathbf{n}]\!]\mathbf{n}, \quad [\![\varphi_{,t}]\!] = -\gamma s_n[\![\mathbf{m}\cdot\mathbf{n}]\!]. \tag{71}$$

These jump conditions, specialized to the material surface $\partial\mathbf{y}(\Omega,t)$, were used in Sect. 1.3.

The jump conditions for the motion are, by Hadamard's Theorem,

$$[\![\nabla\mathbf{y}]\!] = \mathbf{a} \otimes \mathbf{n}_{\rm o} \quad [\![\dot{\mathbf{y}}]\!] = -s_n^{\rm o}\mathbf{a}. \tag{72}$$

Formally differentiate (69) with respect to t on each side of the interface:

$$\dot{\tilde{\mathbf{y}}} = \dot{\mathbf{y}}^{\pm} + \nabla\mathbf{y}^{\pm}\,\dot{\tilde{\mathbf{x}}}. \tag{73}$$

(Take jumps and use (72) to verify the consistency of $\pm$ in (73)).

Lemma 1.6 *Let $g(\cdot,t):\mathbf{y}(\Omega,t)\to\mathbb{R}^3$ be smooth. Then,*

$$\int_{\mathscr{S}^t} g\, s_n da = \int_{\mathscr{S}^t} g\, \mathbf{v}^{\pm}\cdot\mathbf{n} da + \int_{\mathscr{S}_{\mathrm{o}}^t} (\det\nabla\mathbf{y})^{\pm} g\, s_n^{\mathrm{o}} da_{\mathrm{o}}, \tag{74}$$

the signs taken in parallel.

Proof. Use the parametric form, the parameter space denoted by $\mathscr{U}$ and (73):

$$\begin{aligned}\int_{\mathscr{S}^t} g\,\dot{\tilde{\mathbf{y}}}\cdot\mathbf{n}\, da &= \int_{\mathscr{U}} g\,\dot{\tilde{\mathbf{y}}}\cdot(\tilde{\mathbf{y}}_{,1}\times\tilde{\mathbf{y}}_{,2})du_1du_2\\ &= \int_{\mathscr{U}} g\,(\dot{\mathbf{y}}^{\pm}+\nabla\mathbf{y}^{\pm}\dot{\tilde{\mathbf{x}}})\cdot(\det\nabla\mathbf{y}^{\pm})(\nabla\mathbf{y}^{-T})^{\pm}(\tilde{\mathbf{x}}_{,1}\times\tilde{\mathbf{x}}_{,2})du_1du_2\\ &= \int_{\mathscr{S}^t} g\,\dot{\mathbf{y}}^{\pm}\cdot\mathbf{n}\, da + \int_{\mathscr{S}_{\mathrm{o}}^t} g\,(\det\nabla\mathbf{y}^{\pm})s_n^{\mathrm{o}}\, da_{\mathrm{o}}.\end{aligned} \tag{75}$$

□

Take jumps to get,

Lemma 1.7

$$\int_{\mathscr{S}^t}[\![g]\!]\, s_n da = \int_{\mathscr{S}^t}[\![g\,\mathbf{v}\cdot\mathbf{n}]\!]da + \int_{\mathscr{S}_{\mathrm{o}}^t}[\![\det\nabla\mathbf{y})g]\!]\, s_n^{\mathrm{o}} da_{\mathrm{o}}, \tag{76}$$

Begin with the integral form of the balance of linear momentum (33), introduce the Maxwell stress and use the divergence theorem.

$$\frac{d}{dt}\int_{\mathscr{P}}\rho_{\mathrm{o}}\dot{\mathbf{y}}\, d\mathbf{x} = \int_{\partial\mathbf{y}(\mathscr{P},t)}(\mathbf{T}^M+\mathbf{T})\mathbf{n}\, da + \int_{\mathbf{y}(\mathscr{P},t)}(\mathbf{m}\cdot\nabla\mathbf{h}_0)\, d\mathbf{y}. \tag{77}$$

This form of the balance of linear momentum is well-defined for motions with propagating discontinuities and we shall regard it as the statement of a weak solution. Consider the kinematics given above but specialized to a region $\mathscr{P}$ containing a point $\mathbf{x}_{\mathrm{o}}$ on the discontinuity at time t_{o}. Let $\mathscr{P}=\mathscr{P}_-^t\cup\mathscr{P}_+^t$ as above. Let $\mathbf{S}=\mathbf{T}^M+\mathbf{T}$. The balance of linear momentum is in this case,

$$\begin{aligned}\frac{d}{dt}\left(\int_{\mathscr{P}_-^t}\rho_{\mathrm{o}}\dot{\mathbf{y}}\, d\mathbf{x}+\int_{\mathscr{P}_+^t}\rho_{\mathrm{o}}\dot{\mathbf{y}}\, d\mathbf{x}\right) &= \int_{\partial\mathbf{y}(\mathscr{P}_-^t,t)}\mathbf{S}\mathbf{n}\, da+\int_{\partial\mathbf{y}(\mathscr{P}_+^t,t)}\mathbf{S}\mathbf{n}\, da+\int_{\mathscr{S}^t\cap\mathbf{y}(\mathscr{P},t)}[\![\mathbf{S}\mathbf{n}]\!]\, da\\ &+\int_{\mathbf{y}(\mathscr{P}_-^t,t)}(\mathbf{m}\cdot\nabla\mathbf{h}_0)\, d\mathbf{y}+\int_{\mathbf{y}(\mathscr{P}_+^t,t)}(\mathbf{m}\cdot\nabla\mathbf{h}_0)\, d\mathbf{y}.\end{aligned} \tag{78}$$

Use Reynolds' transport theorem and the smoothness of quantities away from the discontinuity.

$$\begin{aligned}\int_{\mathscr{P}_-^t}\rho_{\mathrm{o}}\ddot{\mathbf{y}}\, d\mathbf{x} &+ \int_{\mathscr{P}_+^t}\rho_{\mathrm{o}}\ddot{\mathbf{y}}\, d\mathbf{x}-\int_{\mathscr{S}_{\mathrm{o}}^t\cap\mathscr{P}}[\![\rho_{\mathrm{o}}\dot{\mathbf{y}}]\!]s_n^{\mathrm{o}}\, da_{\mathrm{o}}=\int_{\mathbf{y}(\mathscr{P}_-^t,t)}\mathrm{div}\mathbf{S}\, d\mathbf{x}+\int_{\mathbf{y}(\mathscr{P}_+^t,t)}\mathrm{div}\mathbf{S}\, d\mathbf{x}\\ &+\int_{\mathscr{S}^t\cap\mathbf{y}(\mathscr{P},t)}[\![\mathbf{S}\mathbf{n}]\!]\, da+\int_{\mathbf{y}(\mathscr{P}_-^t,t)}(\mathbf{m}\cdot\nabla\mathbf{h}_0)\, d\mathbf{y}+\int_{\mathbf{y}(\mathscr{P}_+^t,t)}(\mathbf{m}\cdot\nabla\mathbf{h}_0)\, d\mathbf{y}.\end{aligned} \tag{79}$$

Cancel the obvious terms using the smooth form of the balance of linear momentum. The remaining terms are,

$$-\int_{\mathscr{S}_{\mathrm{o}}^t\cap\mathscr{P}}[\![\rho_{\mathrm{o}}\dot{\mathbf{y}}]\!]s_n^{\mathrm{o}}\, da_{\mathrm{o}}=\int_{\mathscr{S}^t\cap\mathbf{y}(\mathscr{P},t)}[\![\mathbf{S}\mathbf{n}]\!]\, da=\int_{\mathscr{S}^t\cap\mathbf{y}(\mathscr{P},t)}[\![(\mathbf{T}^M+\mathbf{T})\mathbf{n}]\!]\, da. \tag{80}$$

Introduce the jump conditions for the Maxwell stress given in Lemma 1.1 and the boundary conditions $[\![\mathbf{T}\mathbf{n}]\!]=\frac{\gamma}{2}[\![(\mathbf{m}\cdot\mathbf{n})^2]\!]+[\![\mathbf{t}]\!]$ (i.e., take jumps across $\mathscr{S}^t$ of (35)). All the terms that arose from nonlocal magnetism (including the integral of $\frac{\gamma}{2}[\![(\mathbf{m}\cdot\mathbf{n})^2]\!]$) cancel. We get,

$$-\int_{\mathscr{S}_{\mathrm{o}}^t\cap\mathscr{P}}[\![\rho_{\mathrm{o}}\dot{\mathbf{y}}]\!]s_n^{\mathrm{o}}\, da_{\mathrm{o}}=\int_{\mathscr{S}^t\cap\mathbf{y}(\mathscr{P},t)}[\![\mathbf{t}]\!]\, da=\int_{\mathscr{S}_{\mathrm{o}}^t\cap\mathscr{P}}[\![\mathbf{t}_{\mathrm{o}}]\!]\, da_{\mathrm{o}}. \tag{81}$$

Now localize by normalizing and shrinking $\mathscr{P}$ to $\mathbf{x}_o$,

$$-[\![\rho_o \dot{\mathbf{y}}]\!] s_n^o = [\![\mathbf{t}_o]\!]. \tag{82}$$

Evidently, long range magnetic forces cannot drive a magnetoelastic wall.

1.5.5 Contribution to the configurational force from the demagnetization energy alone

Since it is a lengthy calculation in itself, and it has independent interest, we calculate separately the configurational force arising from the demagnetization energy alone (41). This calculation parallels that of Sect. 1.3, but there are additional terms that arise from the moving jump of magnetization.

Proposition 1.1 *Assume the kinematics and notation of Subsections 1.5.2 – 1.5.3.*

$$\begin{aligned}\frac{dE_m}{dt} &= \int_\Omega \left(\nabla\varphi\cdot\dot{\mathbf{m}}_o + \mathbf{m}_o\cdot\nabla^2\varphi\,\dot{\mathbf{y}}\right) d\mathbf{x} - \frac{\gamma}{2}\int_{\partial\mathbf{y}(\Omega,t)} (\mathbf{m}\cdot\mathbf{n})^2\,(\mathbf{v}\cdot\mathbf{n})da \\ &+ \frac{\gamma}{2}\int_{\mathscr{S}^t} [\![(\mathbf{m}\cdot\mathbf{n})^2]\!] s_n da - \int_{\mathscr{S}_o^t} [\![\mathbf{m}_o\cdot\nabla\varphi]\!] s_n^o da_o.\end{aligned} \tag{83}$$

Proof. As before, use Reynolds' transport theorem on the third form of (41), accounting now for three regions of smoothness:

$$\begin{aligned}\frac{dE_m}{dt} &= \frac{1}{2\gamma}\frac{d}{dt}\int_{\mathbb{R}^3}|\nabla\varphi|^2 d\mathbf{y} \\ &= \frac{1}{2\gamma}\left(\frac{d}{dt}\int_{\mathbf{y}(\Omega_+^t,t)}|\nabla\varphi|^2 d\mathbf{y} + \frac{d}{dt}\int_{\mathbf{y}(\Omega_-^t,t)}|\nabla\varphi|^2 d\mathbf{y} + \frac{d}{dt}\int_{\mathbb{R}^3\setminus\mathbf{y}(\Omega,t)}|\nabla\varphi|^2 d\mathbf{y}\right) \\ &= \frac{1}{2\gamma}\left(\int_{\mathbf{y}(\Omega_+^t,t)}\frac{\partial}{\partial t}(|\nabla\varphi|^2) d\mathbf{y} + \int_{\mathbf{y}(\Omega_-^t,t)}\frac{\partial}{\partial t}(|\nabla\varphi|^2) d\mathbf{y} + \int_{\mathbb{R}^3\setminus\mathbf{y}(\Omega,t)}\frac{\partial}{\partial t}(|\nabla\varphi|^2) d\mathbf{y}\right. \\ &- \left.\int_{\partial\mathbf{y}(\Omega,t)}[\![|\nabla\varphi|^2]\!]\mathbf{v}\cdot\mathbf{n}da - \int_{\mathscr{S}^t}[\![|\nabla\varphi|^2]\!] s_n da\right)\end{aligned} \tag{84}$$

On $\mathbf{y}(\Omega_+^t,t)$ and on $\mathbf{y}(\Omega_-^t,t)$ the magnetostatic potential is, say, smooth. Referring to the earlier argument ((44) and (46)), we have,

$$\begin{aligned}&\text{On} \quad \mathbf{y}(\Omega_\pm^t,t), \quad \frac{1}{2\gamma}\frac{\partial}{\partial t}|\nabla\varphi|^2 = \mathbf{m}\cdot\nabla\varphi_{,t} - \operatorname{div}(\varphi_{,t}\,\mathbf{m} - \frac{1}{\gamma}\varphi_{,t}\,\nabla\varphi), \\ &\text{On} \quad (\mathbb{R}^3\setminus\mathbf{y}(\Omega,t)), \quad \frac{1}{2\gamma}\frac{\partial}{\partial t}|\nabla\varphi|^2 = \frac{1}{\gamma}\operatorname{div}(\varphi_{,t}\,\nabla\varphi).\end{aligned} \tag{85}$$

Substitute (85) into (84), and recall that the superscript ^{o} indicates the outer trace:

$$\begin{aligned}\frac{dE_m}{dt} &= \int_{\mathbf{y}(\Omega,t)}\mathbf{m}\cdot\nabla\varphi_{,t}\; d\mathbf{y} \\ &+ \frac{1}{\gamma}\int_{\partial\mathbf{y}(\Omega,t)}\left(-\gamma\varphi_{,t}\,\mathbf{m} + \varphi_{,t}\,\nabla\varphi - \varphi^o{}_{,t}\,\nabla\varphi^o - \frac{1}{2}[\![|\nabla\varphi|^2]\!]\mathbf{v}\right)\cdot\mathbf{n}\,da \\ &+ \frac{1}{\gamma}\int_{\mathscr{S}^t}[\![\gamma\varphi_{,t}\,\mathbf{m} - \varphi_{,t}\,\nabla\varphi - \frac{1}{2}|\nabla\varphi|^2\mathbf{n}s_n]\!]\cdot\mathbf{n}\,da.\end{aligned} \tag{86}$$

Just as in the smooth case, the integral over $\partial\mathbf{y}(\Omega,t)$ can be simplified using the jump conditions (71). In exactly the same way, the integral over $\mathscr{S}^t$ simplifies. The result is,

$$\frac{dE_m}{dt} = \int_{\mathbf{y}(\Omega,t)}\mathbf{m}\cdot\nabla\varphi_{,t}\; d\mathbf{y} + \frac{\gamma}{2}\int_{\partial\mathbf{y}(\Omega,t)}(\mathbf{m}\cdot\mathbf{n})^2(\mathbf{v}\cdot\mathbf{n})\,da + \frac{\gamma}{2}\int_{\mathscr{S}^t}[\![(\mathbf{m}\cdot\mathbf{n})^2]\!] s_n da. \tag{87}$$

Following the previous line of argument used in the smooth case (Sect. 1.3), now differentiate the second form of (41) using Reynolds' transport theorem:

$$\frac{d}{dt}\left\{\frac{1}{2}\int_{\Omega_+^t}\mathbf{m}_\mathrm{o}\cdot\nabla\varphi\,d\mathbf{x}+\frac{1}{2}\int_{\Omega_-^t}\mathbf{m}_\mathrm{o}\cdot\nabla\varphi\,d\mathbf{x}\right\}=$$
$$\frac{1}{2}\int_\Omega\left(\dot{\mathbf{m}}_\mathrm{o}\cdot\nabla\varphi+\mathbf{m}_\mathrm{o}\cdot\nabla^2\varphi\,\dot{\mathbf{y}}+\mathbf{m}_\mathrm{o}\cdot\nabla\varphi_{,t}\right)d\mathbf{x}-\frac{1}{2}\int_{\mathscr{S}_\mathrm{o}^t}[\![\mathbf{m}_\mathrm{o}\cdot\nabla\varphi]\!]s_n^\mathrm{o}da_\mathrm{o}. \tag{88}$$

We emphasize the following important point concerning this equation. The terms $\dot{\mathbf{m}}_\mathrm{o}$, $\nabla^2\varphi$ and $\nabla\varphi_{,t}$ in (88) are ordinary (not generalized) functions; they do not contain Dirac masses. To keep track of this point, the reader may wish to rewrite the first term on the right hand side of (88) as the sum of the separate integrals over the regions $\Omega_\pm^t$.

As before, we now calculate $2\times$ (88) $-$ (87). This gives,

$$\begin{aligned}\frac{dE_m}{dt} &= \int_\Omega\left(\dot{\mathbf{m}}_\mathrm{o}\cdot\nabla\varphi+\mathbf{m}_\mathrm{o}\cdot\nabla^2\varphi\,\dot{\mathbf{y}}\right)d\mathbf{x}-\frac{\gamma}{2}\int_{\partial\mathbf{y}(\Omega,t)}(\mathbf{m}\cdot\mathbf{n})^2(\mathbf{v}\cdot\mathbf{n})da\\ &- \int_{\mathscr{S}_\mathrm{o}^t}[\![\mathbf{m}_\mathrm{o}\cdot\nabla\varphi]\!]s_n^\mathrm{o}da_\mathrm{o}+\frac{\gamma}{2}\int_{\mathscr{S}^t}[\![(\mathbf{m}\cdot\mathbf{n})^2]\!]s_n da.\end{aligned} \tag{89}$$

□

1.5.6 The general case

We evaluate the dissipation function (see Sect. 1.4) for a moving discontinuity of the deformation gradient, velocity and magnetization. The result is expressible as an integral over the discontinuity, whose integrand is then be proportional to the configurational force.

Theorem 1.1 *Assume the kinematics of Subsections 1.5.2–1.5.3, the magnetostatic equation, and the equations of motion and boundary conditions given in Subection 1.2.2. Define the dissipation function $\mathscr{D}$ by,*

$$\begin{aligned}\mathscr{D} &= \frac{d}{dt}\left[\int_\Omega(\frac{\rho_\mathrm{o}}{2}|\dot{\mathbf{y}}^2|+W_\mathrm{o})\,d\mathbf{x}+E_m+E_z\right]\\ &+\int_\Omega\dot{\mathbf{h}}_0\cdot\mathbf{m}_\mathrm{o}\,d\mathbf{x}-\int_{\mathbf{y}(\Omega,t)}(\mathbf{m}\cdot\nabla\mathbf{h}_0)\,\mathbf{v}\,d\mathbf{y}-\int_{\partial\mathbf{y}(\Omega,t)}\mathbf{t}\cdot\mathbf{v}\,da.\end{aligned} \tag{90}$$

Then,

$$\mathscr{D}=-\int_{\mathscr{S}_\mathrm{o}^t}[\![W_\mathrm{o}-\mathbf{h}\cdot\mathbf{m}_\mathrm{o}-\frac{\gamma}{2}(\det\nabla\mathbf{y})(\mathbf{m}\cdot\mathbf{n})^2-\langle\mathbf{t}_\mathrm{o}\rangle\cdot\nabla\mathbf{y}\,\mathbf{n}_\mathrm{o}]\!]s_n^\mathrm{o}\,da_\mathrm{o}. \tag{91}$$

The configurational force is identified as,

$$\mathscr{C}=[\![W_\mathrm{o}-\mathbf{h}\cdot\mathbf{m}_\mathrm{o}-\frac{\gamma}{2}(\det\nabla\mathbf{y})(\mathbf{m}\cdot\mathbf{n})^2-\langle\mathbf{t}_\mathrm{o}\rangle\cdot\nabla\mathbf{y}\,\mathbf{n}_\mathrm{o}]\!]. \tag{92}$$

Proof. Begin with the expression for $\mathscr{D}$, break down the first integral on the right to a sum of integrals over $\Omega_\pm^t$ and introduce the expression (83) for dE_m/dt:

$$\begin{aligned}\mathscr{D} &= \frac{d}{dt}\left\{\int_{\Omega_+^t}(\frac{\rho_\mathrm{o}}{2}|\dot{\mathbf{y}}|^2+W_\mathrm{o}-\mathbf{h}_0\cdot\mathbf{m}_\mathrm{o})d\mathbf{x}+\int_{\Omega_-^t}(\frac{\rho_\mathrm{o}}{2}|\dot{\mathbf{y}}|^2+W_\mathrm{o}-\mathbf{h}_0\cdot\mathbf{m}_\mathrm{o})d\mathbf{x}\right\}\\ &+\int_\Omega(\dot{\mathbf{m}}_\mathrm{o}\cdot\nabla\varphi+\mathbf{m}_\mathrm{o}\cdot\nabla^2\varphi\,\dot{\mathbf{y}})\,d\mathbf{x}-\frac{\gamma}{2}\int_{\partial\mathbf{y}(\Omega,t)}(\mathbf{m}\cdot\mathbf{n})^2(\mathbf{v}\cdot\mathbf{n})da\\ &+\frac{\gamma}{2}\int_{\mathscr{S}^t}[\![(\mathbf{m}\cdot\mathbf{n})^2]\!]s_n\,da-\int_{\mathscr{S}_\mathrm{o}^t}[\![\mathbf{m}_\mathrm{o}\cdot\nabla\varphi]\!]s_n^\mathrm{o}\,da_\mathrm{o}\\ &+\int_\Omega\dot{\mathbf{h}}_0\cdot\mathbf{m}_\mathrm{o}\,d\mathbf{x}-\int_{\mathbf{y}(\Omega,t)}(\mathbf{m}\cdot\nabla\mathbf{h}_0)\mathbf{v}\,d\mathbf{y}-\int_{\partial\mathbf{y}(\Omega,t)}\mathbf{t}\cdot\mathbf{v}\,da\end{aligned}$$

$$
\begin{aligned}
&= \int_{\Omega^t_+} \left(\rho_o \dot{\mathbf{y}}\cdot\ddot{\mathbf{y}} + \mathbf{T}_o\cdot\nabla\dot{\mathbf{y}} + \mathbf{h}\cdot\dot{\mathbf{m}}_o - \dot{\mathbf{h}}_0\cdot\mathbf{m}_o - \mathbf{h}_0\cdot\dot{\mathbf{m}}_o\right) d\mathbf{x}\\
&+ \int_{\Omega^t_-} \left(\rho_o \dot{\mathbf{y}}\cdot\ddot{\mathbf{y}} + \mathbf{T}_o\cdot\nabla\dot{\mathbf{y}} + \mathbf{h}\cdot\dot{\mathbf{m}}_o - \dot{\mathbf{h}}_0\cdot\mathbf{m}_o - \mathbf{h}_0\cdot\dot{\mathbf{m}}_o\right) d\mathbf{x}\\
&+ \int_{\mathscr{S}_o^t} -[\![\frac{\rho_o}{2}|\dot{\mathbf{y}}|^2 + W_o - \mathbf{h}_0\cdot\mathbf{m}_o]\!]s_n^o\, da_o\\
&+ \int_\Omega (\dot{\mathbf{m}}_o\cdot\nabla\varphi + \mathbf{m}_o\cdot\nabla^2\varphi\,\dot{\mathbf{y}})\, d\mathbf{x} - \frac{\gamma}{2}\int_{\partial\mathbf{y}(\Omega,t)} (\mathbf{m}\cdot\mathbf{n})^2(\mathbf{v}\cdot\mathbf{n}) da\\
&+ \frac{\gamma}{2}\int_{\mathscr{S}^t} [\![(\mathbf{m}\cdot\mathbf{n})^2]\!]s_n\, da - \int_{\mathscr{S}_o^t} [\![\mathbf{m}_o\cdot\nabla\varphi]\!]s_n^o\, da_o\\
&+ \int_\Omega \dot{\mathbf{h}}_0\cdot\mathbf{m}_o\, d\mathbf{x} - \int_{\mathbf{y}(\Omega.t)} (\mathbf{m}\cdot\nabla\mathbf{h}_0)\mathbf{v}\, d\mathbf{y} - \int_{\partial\mathbf{y}(\Omega,t)} \mathbf{t}\cdot\mathbf{v}\, da. \qquad (93)
\end{aligned}
$$

Decompose the field $\mathbf{h} = -\nabla\varphi + \mathbf{h}_0$, integrate by parts the terms $\mathbf{T}_o\cdot\nabla\dot{\mathbf{y}}$, introduce the local form of the balance of linear momentum, and note that there is a lot of cancellation here (Note that the terms being cancelled do not contain hidden Dirac masses). In particular, all of the volume terms cancel. The remaining surface terms are,

$$
\begin{aligned}
\mathscr{D} &= \int_{\partial\Omega} \dot{\mathbf{y}}\cdot\mathbf{T}_o\mathbf{n}_o\, da_o - \int_{\mathscr{S}_o^t} [\![\dot{\mathbf{y}}\cdot\mathbf{T}_o\mathbf{n}_o]\!]\, da_o\\
&+ \frac{\gamma}{2}\int_{\mathscr{S}^t} [\![(\mathbf{m}\cdot\mathbf{n})^2]\!]s_n\, da - \int_{\partial\mathbf{y}(\Omega,t)} \left(\frac{\gamma}{2}(\mathbf{m}\cdot\mathbf{n})^2(\mathbf{v}\cdot\mathbf{n}) + \mathbf{t}\cdot\mathbf{v}\right) da\\
&+ \int_{\mathscr{S}_o^t} -[\![\frac{\rho_o}{2}|\dot{\mathbf{y}}|^2 + W_o + \mathbf{m}_o\cdot\nabla\varphi - \mathbf{h}_0\cdot\mathbf{m}_o]\!]s_n^o\, da_o. \qquad (94)
\end{aligned}
$$

Now the goal is to consolidate the surface terms. We do this by substituting for the first two terms on the right hand side of (94). For the first term we begin with the boundary conditions (35) in referential form, $\mathbf{T}_o\mathbf{n}_o = \mathbf{t}_o + \frac{\gamma}{2}(\mathbf{m}\cdot\mathbf{n})^2(\det\nabla\mathbf{y})(\nabla\mathbf{y})^{-T}\mathbf{n}_o$. Introduce the boundary conditions into the second term on the right hand side of (94):

$$
\int_{\mathscr{S}_o^t} [\![\dot{\mathbf{y}}\cdot\mathbf{T}_o\mathbf{n}_o]\!]\, da_o = \int_{\mathscr{S}_o^t} [\![\dot{\mathbf{y}}\cdot\mathbf{t}_o]\!]\, da_o + \frac{\gamma}{2}\int_{\mathscr{S}^t} [\![(\mathbf{m}\cdot\mathbf{n})^2(\mathbf{v}\cdot\mathbf{n})]\!]\, da. \qquad (95)
$$

Repeat the analogous calculation on $\partial\Omega$:

$$
\int_{\partial\Omega} \dot{\mathbf{y}}\cdot\mathbf{T}_o\mathbf{n}_o\, da_o = \int_{\partial\mathbf{y}(\Omega,t)} \left(\dot{\mathbf{y}}\cdot\mathbf{t} + \frac{\gamma}{2}(\mathbf{m}\cdot\mathbf{n})^2(\mathbf{v}\cdot\mathbf{n})\right) da. \qquad (96)
$$

Substitute (95) and (96) for the first two terms of (94) and simplify.

$$
\begin{aligned}
\mathscr{D} &= \int_{\mathscr{S}_o^t} \left(-[\![\frac{\rho_o}{2}|\dot{\mathbf{y}}|^2 + W_o + \mathbf{m}_o\cdot\nabla\varphi - \mathbf{h}_0\cdot\mathbf{m}_o]\!]s_n^o - [\![\dot{\mathbf{y}}\cdot\mathbf{t}_o]\!]\right) da_o\\
&+ \frac{\gamma}{2}\int_{\mathscr{S}^t} \left([\![(\mathbf{m}\cdot\mathbf{n})^2]\!]s_n - [\![(\mathbf{m}\cdot\mathbf{n})^2(\mathbf{v}\cdot\mathbf{n})]\!]\right) da. \qquad (97)
\end{aligned}
$$

Now use Lemma 1.7 with $g = (\mathbf{m}\cdot\mathbf{n})^2$ to simplify the last two terms of (97):

$$
\mathscr{D} = \int_{\mathscr{S}_o^t} \left(-[\![\frac{\rho_o}{2}|\dot{\mathbf{y}}|^2 + W_o + \mathbf{m}_o\cdot\nabla\varphi - \mathbf{h}_0\cdot\mathbf{m}_0 - \frac{\gamma}{2}(\det\nabla\mathbf{y}(\mathbf{m}\cdot\mathbf{n})^2)]\!]s_n^o - [\![\dot{\mathbf{y}}\cdot\mathbf{t}_o]\!]\right) da_o. \qquad (98)
$$

In the standard way, use the identities $\frac{1}{2}[\![a^2]\!] = [\![a]\!]\langle a\rangle$ and $[\![ab]\!] = [\![a]\!]\langle b\rangle + [\![b]\!]\langle a\rangle$, where $\langle a\rangle := \frac{1}{2}(a_+ + a_-)$.

$$\begin{aligned}\mathscr{D} &= \int_{\mathscr{S}_o^t} \left(-\rho_o[\![\dot{\mathbf{y}}]\!]\cdot\langle\dot{\mathbf{y}}\rangle\, s_n^o - [\![W_o + \mathbf{m}_o\cdot\nabla\varphi - \mathbf{h}_0\cdot\mathbf{m}_o - \frac{\gamma}{2}(\det\nabla\mathbf{y}(\mathbf{m}\cdot\mathbf{n})^2)]\!]s_n^o\right) da_o \\ &- \int_{\mathscr{S}_o^t} \left([\![\dot{\mathbf{y}}]\!]\cdot\langle\mathbf{t}_o\rangle + [\![\mathbf{t}_o]\!]\cdot\langle\dot{\mathbf{y}}\rangle\right) da_o. \end{aligned} \tag{99}$$

Now use the kinematic jump conditions (72) and the dynamic jump conditions (82). We get finally,

$$\mathscr{D} = \int_{\mathscr{S}_o^t} -[\![W_o - \mathbf{h}\cdot\mathbf{m}_o - \frac{\gamma}{2}(\det\nabla\mathbf{y}(\mathbf{m}\cdot\mathbf{n})^2) - \langle\mathbf{t}_o\rangle\cdot\nabla\mathbf{y}\mathbf{n}_o]\!]s_n^o\, da_o. \tag{100}$$

□

1.5.7 No inertia

In this and the next subsection we discuss special cases of the formula for the configurational force. Send ρ_o to zero, so the dynamic jump conditions imply that

$$[\![\mathbf{t}_o]\!] = 0, \quad \langle\mathbf{t}_o\rangle = \mathbf{t}_o. \tag{101}$$

In this case,

$$\begin{aligned}\langle\mathbf{t}_o\rangle\cdot\nabla\mathbf{y}^{\pm}\mathbf{n}_o &= \mathbf{t}_o\cdot\nabla\mathbf{y}^{\pm}\mathbf{n}_o \\ &= \left(\mathbf{T}_o^{\pm}\mathbf{n}_o - \frac{\gamma}{2}(\mathbf{m}^{\pm}\cdot\mathbf{n})^2(\det\nabla\mathbf{y}^{\pm})(\nabla\mathbf{y}^{\pm})^{-T}\mathbf{n}_o\right)\cdot\nabla\mathbf{y}^{\pm}\mathbf{n}_o, \\ &= \mathbf{n}_o\cdot(\mathbf{T}_o^{\pm})^T\nabla\mathbf{y}\mathbf{n}_o - \frac{\gamma}{2}(\mathbf{m}^{\pm}\cdot\mathbf{n})^2(\det\nabla\mathbf{y}^{\pm}),\end{aligned} \tag{102}$$

so the dissipation function (100) becomes,

$$\begin{aligned}\mathscr{D} &= -\int_{\mathscr{S}_o^t} [\![W_o - \mathbf{h}\cdot\mathbf{m}_o - \mathbf{n}_o\cdot\mathbf{T}_o^T\nabla\mathbf{y}\mathbf{n}_o]\!]s_n^o\, da_o, \\ &= -\int_{\mathscr{S}_o^t} \mathbf{n}_o\cdot[\![(W_o - \frac{\partial W_o}{\partial\mathbf{m}_o}\cdot\mathbf{m}_o)\mathbf{I} - \mathbf{T}_o^T\nabla\mathbf{y}]\!]\mathbf{n}_o\, s_n^o\, da_o.\end{aligned} \tag{103}$$

This gives a generalization of the Eshelby tensor.

1.5.8 No deformation

Consider a rigid body ($\mathbf{y}(\mathbf{x},t) = \mathbf{x}$ for simplicity) and assume that $W_o(\mathbf{m}_o)$ only. Drop the $\mathbf{t}_o$ term, as this arises from working terms that are zero for a rigid body. The dissipation function in this case becomes,

$$\mathscr{D} = \int_{\mathscr{S}_o^t} -[\![W_o - \mathbf{h}\cdot\mathbf{m}_o - \frac{\gamma}{2}((\mathbf{m}\cdot\mathbf{n})^2)]\!]s_n^o\, da_o. \tag{104}$$

Here, the sub and superscripts $_o$ and ^{o} could be dropped. The formula (104) is the same formula as one obtains by doing the static Eshelby calculation on the usual energy of micromagnetics with exchange energy omitted (so that sharp discontinuities of magnetization are allowed).

2 Analysis of a microscale cantilever

2.1 Introduction

In the following subsections we derive equations of motion of a small-scale cantilever under certain simplifying assumptions. We begin by explaining the thinking behind the particular design that is chosen.

The larger problem being considered here is, how does one produce motion of a vehicle through a fluid (air or water) at scales of a mm down to a micron, or perhaps smaller? We begin with various elementary facts. From the scaling laws of the (incompressible) Navier-Stokes equations of fluid mechanics[7], viscous forces are greatly exaggerated at small scales. (As an example, the motion of a typical commercial aircraft, scaled down just to 1 m scale, is equivalent to the flight of the same aircraft at conventional scale, at the same velocity, in glycerine). Thus the forces required to overcome viscous forces at small scales are enormous. For such applications actuators that have large work output per unit volume (of the actuator) are desired. A second fact is that as a vehicle is scaled smaller and smaller at fixed shape, its surface area/volume tends to infinity. This seems to argue against the use of an on-board fuel system delivering power based on chemical change, since the drag on such a vehicle scales approximately as the surface area, while the total energy available to the fuel undergoing chemical change is proportional to its volume. Insects and microorganisms have evolved a number of fascinating mechanisms – both in their structure and their mode of propulsion – that overcome to some extent these physical constraints (Bray [9]).

Shape-memory materials are among the actuator materials with the largest work output per unit volume, per cycle, Krulevitch et al. [29]. The actuation is caused by a temperature change. As a graphic example, the NiTi shape memory material can generate stresses as high as 500 MPa during the shape memory cycle; hence a film 1mm wide by 10μm thick is capable of lifting a mass of nearly 1/2 kilogram! Thus these materials are attractive for applications to MEMS (micro-electro-mechanical systems), but for the problem of small-scale flight, the difficulty of supplying temperature pulses to a moving vehicle seems hard to overcome[8]. There is a new class of related actuator materials termed ferromagnetic shape memory materials ([26], [45], [40], [48]) with high power density, and there are related ferroelectric materials that are also under development. Like shape memory materials, the ferromagnetic shape memory materials exhibit a reversible martensitic phase transformation, but they are also ferromagnetic. By applying a magnetic field to the martensitic state, they can be made to undergo a large shape change, comparable to that produced by the best shape memory materials. However, they open up the possibility for remote actuation via the use of an applied magnetic field. The usual fluids air and water being nonferromagnetic, the only energy being extracted from the field is that used by the MEMS device, except for the losses associated with field generation. Our scheme for producing motion in the cantilever makes use of ferromagnetic shape memory (or related ferroelectric) materials actuated remotely by an oscillating magnetic field. For the analysis presented here we use data (lattice parameters, easy axes of magnetization) measured for a Ni_2MnGa ferromagnetic shape memory alloy.

As in nearly all of MEMS, the attraction of using thin film methods of synthesis is compelling. Recently, single crystal films of Ni_2MnGa have been synthesized using molecular beam epitaxy by Dong et al. [15]. Single crystals have the particular advantage of avoiding fighting between the grains that compromises the actuation strain. This effect can be significant: in untextured polycrystals of Ni_2MnGa, for example, no ferromagnetic shape memory effect has been observed even under fields of 1.5 T, and this is understood theoretically [8] as arising from the interaction of crystallography – too few variants of martensite, with unfavorable transformation strains – and compatibility between grains. For these reasons we focus here on cantilevers made of single crystal films.

Now we discuss structural aspects. At first, a cantilever seems to be a particularly unfavorable choice for a small scale actuator from a structural perspective. That is, a bent cantilever of thickness h stores (or produces) energy that scales[9] as h^3 . This follows for classical elastic beams from Euler-Bernoulli theory,

[7] The predictions of the Navier-Stokes equations exhibit divergence from experimental results at micron scale in air [30], [42], so the reasoning given here would require modification at those scales

[8] However, a drawback of shape memory materials in bulk – namely, their slow response – caused by the necessity of having to wait until they cool during the shape-memory cycle, is greatly improved at small scales [29].

[9] More precisely, according to Euler-Bernoulli theory, beams of thickness h and width b, with identical bent shapes of the centerline, store energy ch^3, where c depends on the shape but not the thickness h.

but it is a much more general result [7]. Milliactuators have been built ([29]) with actuation produced by bilayers of NiTi shape memory alloy and Si that work in bending. These show work output far less than expected based on the theoretical values quoted above exactly because of the prefactor h^3. In fact it was these observations about scaling that motivated Bhattacharya and James [7] to derive the limiting energy as $h \to 0$ for a martensitic material (Γ–convergence arguments are particularly suited to study this issue). This gave a certain Cosserat membrane theory with energy proportional to h, and suggested certain "tents" and "tunnels" as the basis of designs for large work output microactuators. However, these seem unsuitable for the problem of flight at small scales.

It is also known to experts in structural mechanics that all these scaling results rely on the shear stiffness of the beam being much greater than its bending stiffness, an assumption that automatically held in the Γ–convergence argument because the material was fixed while h was tending to zero. There are macroscale toys that illustrate the failure of this assumption: a beam of soft rubber (small shear stiffness) with thin strips of sheet metal glued to each face (to increase bending stiffness) has dramatically different behavior in bending than a conventional "Euler-Bernoulli" cantilever beam, because this composite forces a shear mode. It is also interesting to note that single-cell *paramecia* have cilia, with active regions of bending as explained by Bray ([9]; Fig. 16–10 of this reference indicates that the bending is produced by an active shear mode that is expected to have the favorable scaling properties discussed here).

Shape memory, and ferromagnetic shape memory, materials have an exceedingly low shear stiffness for certain modes of shear. Thus, designed properly, a ferromagnetic shape memory beam can be made to "bend" using an applied field, the mechanism being a shear deformation, and produce work output proportional to h. Our design is one of this type. The bending is directly produced by twin boundary motion.

Solving the equations of motion for such a cantilever in full generality would be exceedingly difficult. In fact, we know of no solved dynamic (and hardly any static) problems at all that entail the motion of magnetoelastic domains. In general such dynamic problems are difficult, and the constitutive equations that would be needed as input to the theory for Ni_2MnGa, or any other shape memory alloy, are unavailable. Fortunately, there is a simplifying feature of this alloy. It has both hard elastic moduli, and is also a hard magnetic material. Thus, for the fields envisioned here, the strain and magnetization present in the material is very close to certain "energy wells" which, fortunately, are known to great accuracy for this alloy. Below we introduce simplifications that rely on its hardness.

A conventional nonlinear elastic material, with a single energy well and with hard elastic moduli, should behave in dynamics, to first approximation, consistent with the assumption that the deformation gradient belongs to this well, e.g., $\nabla \mathbf{y}(\mathbf{x}, t) \in \mathrm{SO}(3)$. Here, using the condition of frame-indifference, the "well" consists of all rotation matrices. Treating the stress as an arbitrary constraint stress, and writing the conventional balances of linear and rotational momentum, we obtain from this assumption classical rigid body mechanics (Euler's laws of motion). Our idea for simplifying the equations of the cantilever is to generalize this idea to several energy wells and add magnetism to get *piecewise rigid magneto-mechanics* (PRMM). A full treatment of this theory, and its cousins *piecewise rigid body mechanics* and *piecewise rigid thermodynamics* will appear elsewhere ([25]). In PRMM the body is subdivided into an number of magnetoelastic domains, each of which acts as a rigid body, superimposed on a constant strain, these being taken from the energy wells. Each subbody also has a corresponding constant magnetization determined by the well. The various subbodies are separated by interfaces which from kinematics are planar. As we find below, and as expected, the balances of linear and rotational momentum are not sufficient to determine the behavior of such a body subject to the obvious initial conditions. Kinetics of the interfaces is missing. We deal with this by writing a kinetic law for each interface of the type discussed briefly above:

$$\text{normal velocity of the interface} \quad = \quad \mathrm{f}\,(\text{configurational force on the interface}). \tag{105}$$

Surely this kind of theory is too simple to capture all behavior of even hard materials, yet, given the incredible complexity of the general dynamic problem (3D, change-of-type, extreme nonlinearity, nonlocal relation between field and magnetization), this seems a reasonable way to proceed. A next approximation, not explored here, would keep the quadratic approximation of the energy about the well within the context of a Cosserat ansatz.

In this paper we treat the magnetism using magnetostatic formulas. This is not strictly applicable to the ferromagnetic shape memory material Ni_2MnGa, which is conductive, but we use this material as an example anyway.

2.2 Ferromagnetic shape memory and related materials

As explained above, we shall make a semi–inverse assumption that the deformation gradient and magnetization lie on certain energy wells. For definiteness we assume the energy–well structure of the ferromagnetic shape memory material Ni_2MnGa. This is a Heusler alloy with an $L2_1$ cubic structure of the austenite. It undergoes a ferromagnetic transition above room temperature and a cubic to tetragonal martensitic transformation below room temperature; both transition temperatures are sensitive to composition and can be made to lie above room temperature by changing composition. The martensitic transformation is accompanied by a shortening of the cubic unit cell in the [100] direction, and the corresponding easy axis of magnetization is in that direction (Tickle and James [44]).

Static micromagnetics for magnetoelastic materials is governed by an energy of the form (cf. Brown [6], James and Kinderlehrer [23]; see also Sect. 1)

$$\int_\Omega W_o(\nabla \mathbf{y}(\mathbf{x}), \mathbf{m}(\mathbf{y}(\mathbf{x})), \theta) - (\mathbf{h}_0 \cdot \mathbf{m}) \det \nabla \mathbf{y}\, d\mathbf{x} + \frac{1}{2\gamma} \int_{\mathbb{R}^3} |\nabla \varphi|^2 d\mathbf{y}. \tag{106}$$

Here, Ω is the reference configuration (chosen to represent undistorted austenite at the transformation temperature), W_o is the local free energy density, $\mathbf{y} : \Omega \to \mathbb{R}^3$ is the deformation, $\mathbf{m} : \mathbf{y}(\Omega) \to \mathbb{R}^3$ is the magnetization which lives on the deformed configuration, θ is the temperature, $\mathbf{h}_0$ is the applied field, γ is 4π (cgs units), and φ is the magnetostatic potential. As discussed in Sect. 1, the magnetostatic potential is the unique solution, up to an additive constant and among potentials that have a finite value of the magnetostatic energy (the last term of (106)), of the differential equation,

$$\operatorname{div}(-\nabla \varphi + \gamma \mathbf{m}) = 0 \quad \text{on } \mathbb{R}^3. \tag{107}$$

In solving the magnetostatic equation (107) on all of $\mathbb{R}^3$ it is assumed that $\mathbf{m}$ vanishes outside of $\mathbf{y}(\Omega)$. The expression (106) is missing strain-gradient and exchange energies, whose significance we shall discuss in Sect. 2.4.

Fix the temperature θ below the martensitic transformation temperature, and let $SO(3)(\mathbf{F}, \mathbf{m})$ denote all pairs of (3×3 matrices, vectors in $\mathbb{R}^3$) given by $\{(\mathbf{RF}, \mathbf{Rm}) : \mathbf{R} \in SO(3)\}$; notice that, as a consequence of the condition of frame–indifference, the same rotation matrix goes in front of $\mathbf{F}$ and $\mathbf{m}$. The energy wells of W_o for Ni_2MnGa are all pairs $(\mathbf{F}, \mathbf{m})$ of (deformation gradient, magnetization) belonging to the set

$$\begin{aligned} &SO(3)(\mathbf{U}_1, \mathbf{m}_1) \quad \cup \quad SO(3)(\mathbf{U}_1, -\mathbf{m}_1) \ \cup \\ &SO(3)(\mathbf{U}_2, \mathbf{m}_2) \quad \cup \quad SO(3)(\mathbf{U}_2, -\mathbf{m}_2) \ \cup \\ &SO(3)(\mathbf{U}_3, \mathbf{m}_3) \quad \cup \quad SO(3)(\mathbf{U}_3, -\mathbf{m}_3), \end{aligned} \tag{108}$$

where, in the cubic basis,

$$\begin{aligned} \mathbf{U}_1 &= \begin{pmatrix} \eta_2 & 0 & 0 \\ 0 & \eta_1 & 0 \\ 0 & 0 & \eta_1 \end{pmatrix}, \quad \mathbf{m}_1 = m_s[100], \\ \mathbf{U}_2 &= \begin{pmatrix} \eta_1 & 0 & 0 \\ 0 & \eta_2 & 0 \\ 0 & 0 & \eta_1 \end{pmatrix}, \quad \mathbf{m}_2 = m_s[010], \\ \mathbf{U}_3 &= \begin{pmatrix} \eta_1 & 0 & 0 \\ 0 & \eta_1 & 0 \\ 0 & 0 & \eta_2 \end{pmatrix}, \quad \mathbf{m}_3 = m_s[001]. \end{aligned} \tag{109}$$

Typical values for Ni_2MnGa are $\eta_1 = 1.013$, $\eta_2 = 0.952$ and $m_s = 600$ emu/cm^3 (cgs units), [44].

Consider a continuous deformation $\mathbf{y}$ whose gradient assumes two values lying on the energy wells on each side of an interface, e.g., $\nabla\mathbf{y}(\mathbf{x}) = \mathbf{R}\mathbf{U}_2,\ \mathbf{x}\cdot\mathbf{n}_o \geq 0$ and $\nabla\mathbf{y}(\mathbf{x}) = \mathbf{U}_1,\ \mathbf{x}\cdot\mathbf{n}_o < 0$, $\mathbf{R}\in \mathrm{SO}(3)$. There is such a deformation if and only if the compatibility condition

$$\mathbf{R}\mathbf{U}_2 - \mathbf{U}_1 = \mathbf{a}\otimes\mathbf{n}_o \tag{110}$$

holds. Given the values of $\mathbf{U}_1$ and $\mathbf{U}_2$ above, there are precisely two solutions $(\mathbf{R}^{\mathrm{I}}, \mathbf{a}^{\mathrm{I}}\otimes\mathbf{n}_o^{\mathrm{I}})$ and $(\mathbf{R}^{\mathrm{II}}, \mathbf{a}^{\mathrm{II}}\otimes\mathbf{n}_o^{\mathrm{II}})$ of this equation. They are,

$$\mathbf{a}^{\mathrm{I}}\otimes\mathbf{n}_o^{\mathrm{I}} = \frac{\eta_1^2-\eta_2^2}{\eta_1^2+\eta_2^2}(\eta_2,-\eta_1,0)\otimes(1,1,0) \quad \text{and} \quad \mathbf{a}^{\mathrm{II}}\otimes\mathbf{n}_o^{\mathrm{II}} = \frac{\eta_1^2-\eta_2^2}{\eta_1^2+\eta_2^2}(\eta_2,\eta_1,0)\otimes(1,-1,0) \tag{111}$$

(Here, $\mathbf{R}^{\mathrm{I}}$ and $\mathbf{R}^{\mathrm{II}}$ follow from (110)). Either of these deformations produces a deformed configuration with a twin boundary[10] and the twinning in this case is compound.

Consider one of these deformations $\mathbf{y}(\mathbf{x})$ restricted to $\mathbf{x}\in\Omega$. Append to it a magnetization $\mathbf{m}$. Note that in the integrand of (106) the magnetization $\mathbf{m}(\mathbf{y}(\mathbf{x}))$ is composed with the deformation, reflecting the hypothesis that the magnetization is supported on the deformed configuration. Using the form above of $\mathbf{y}$ and of the energy wells, it follows that the pair $(\nabla\mathbf{y}(\mathbf{x}), \mathbf{m}(\mathbf{y}(\mathbf{x})))$ belongs to the wells if $\mathbf{m}(\mathbf{z}) = \pm\mathbf{R}\mathbf{m}_2$ for $\{\mathbf{z}\in\mathbf{y}(\Omega) : \mathbf{z}\cdot\mathbf{n}\geq 0\}$ and $\mathbf{m}(\mathbf{z}) = \pm\mathbf{m}_1$ for $\{\mathbf{z}\in\mathbf{y}(\Omega) : \mathbf{z}\cdot\mathbf{n} < 0\}$. Here $\mathbf{n}$ is the deformed normal, i.e., $\mathbf{n} = \mathbf{U}_1^{-T}\mathbf{n}_o$. By the magnetostatic equation (107) the source for $\nabla\varphi$ is $\operatorname{div}\mathbf{m}$; a divergence-free magnetization on $\mathbb{R}^3$ gives $\nabla\varphi = 0$ on $\mathbb{R}^3$ and therefore produces no magnetostatic energy. Therefore the twin boundary associated with $\mathbf{y}(\mathbf{x})$ produces no magnetostatic energy if and only if,

$$(\mathbf{R}\mathbf{m}_2 \pm \mathbf{m}_1)\cdot\mathbf{n} = 0. \tag{112}$$

Here, one of $\pm$ has been erased without loss of generality. Given that $\mathbf{R}$ and $\mathbf{n}$ have already been determined from the solution of (110), so that the only freedom in (112) is the choice of $\pm$, it is a remarkable accident of symmetry that in fact (112) does hold for a choice of $\pm$. In fact, this result holds in the more general case of any Type I or Type II twin[11]. All of these conclusions continue to hold when there are any number of parallel twin boundaries separating regions where $(\nabla\mathbf{y}(\mathbf{x}), \mathbf{m}(\mathbf{y}(\mathbf{x})))$ take values $(\mathbf{R}\mathbf{U}_2, \mathbf{R}\mathbf{m}_2)/(\mathbf{U}_1, \mathbf{m}_1)/(\mathbf{R}\mathbf{U}_2, \mathbf{R}\mathbf{m}_2)/(\mathbf{U}_1, \mathbf{m}_1), \ldots$.

In summary, for the (deformation, magnetization) constructed here, there is no contribution to the local free energy and the only contribution to the magnetostatic energy is from the jump in magnetization at $\partial\Omega$. With a favorable shape of Ω it seems that this deformation is of rather low energy. A deeper analysis of its stability is given in Sect. 2.4.

2.3 A family of shear motions of a cantilever

We consider a particularly simple example in piecewise rigid magneto-mechanics. Consider one of the solutions $(\mathbf{R}, \mathbf{a}, \mathbf{n}_o)$ of (110), and its corresponding pair of magnetizations satisfying (112). Choose an orthonormal coordinate basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{n}_o\}$ and let the reference configuration be a rectangular solid of length L and cross-section $\mathscr{A} = (-\frac{b}{2}, \frac{b}{2})\times(-\frac{h}{2}, \frac{h}{2})$, i.e., $\Omega = \mathscr{A}\times(0, L)$. Suppose the cantilever is divided by a set of parallel bands with reference normal $\mathbf{n}_o$ and alternating (deformation gradient, magnetization) given as above by $(\mathbf{R}\mathbf{U}_2, \mathbf{R}\mathbf{m}_2)/(\mathbf{U}_1, \pm\mathbf{m}_1)/(\mathbf{R}\mathbf{U}_2, \mathbf{R}\mathbf{m}_2)/(\mathbf{U}_1, \pm\mathbf{m}_1), \ldots$ with $\pm$ chosen to satisfy (112)

It is convenient to have formulas for motions and magnetizations of this type. To this end, let $0 = s_0 < s_1(t) < s_2(t) < \ldots < s_N(t) < s_{N+1} = L$ be functions defined for $t\geq 0$ (The $s_i(t)$ represent interfacial positions in the reference configuration, and it is convenient to include also the "artificial" interfaces s_0 and s_{N+1} to simplify subsequent formulas). Let $\chi(s,t)$ be a characteristic function of two variables defined for $s\in(0,L)$ and $t\geq 0$ as shown in Fig. 1, i.e., $\chi(s,t) = 1$ for $s\in[s_i(t), s_{i+1}(t))$, i odd, and $\chi(s,t) = 0$ otherwise.

Consider the following motion and magnetization:

[10] The connection with the crystallography of twinning can be found in, e.g., James and Hane [27]

[11] ibid. While magnetoelastic domain walls in conventional ferromagnetic materials are not usually thought of as twins, they are also in many cases precisely twins corresponding to a very small transformation strain.

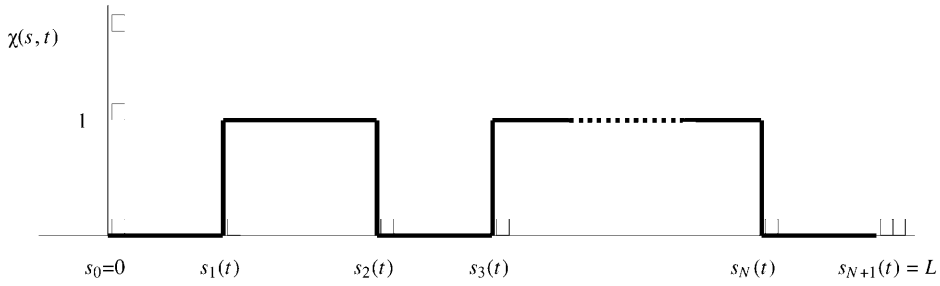

Fig. 1. The function $\chi(s,t)$ used in the ansatz (113)

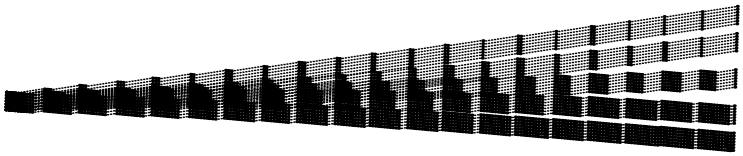

Fig. 2. A family of snapshots of shear motions of a cantilever, corresponding to a particular choice of the functions $s_1(t),\dots,s_N(t)$. Drawn with lattice parameters appropriate to Ni_2MnGa. The magnetization is approximately $\nearrow$ for the "dark" variant and $\searrow$ for the "light" variant

$$\begin{aligned}\mathbf{y}(\mathbf{x},t) &= \mathbf{U}_1\mathbf{x} + \mathbf{a}\int_0^{\mathbf{x}\cdot\mathbf{n}_o} \chi(s,t)\,ds,\\ \mathbf{m}(\mathbf{y}(\mathbf{x},t),t) &= \chi(\mathbf{x}\cdot\mathbf{n}_o,t)\,\mathbf{R}\mathbf{m}_2 + (1-\chi(\mathbf{x}\cdot\mathbf{n}_o,t))\,\mathbf{m}_1.\end{aligned} \tag{113}$$

The latter uniquely defines $\mathbf{m}(\mathbf{y},t)$ on the deformed configuration if $\mathbf{y}(\mathbf{x},t)$ is invertible, which of course is true if the cantilever is not so long that it wraps back on top of itself. It is easily verified, by differentiation of the first of (113) with respect to $\mathbf{x}$, that this motion and magnetization satisfies the conditions given above. Notice that they are uniquely determined by the assignment of the interfacial positions, $s_1(t) < s_2(t) < \ldots < s_N(t)$ for $t \geq 0$. The pictures of deformations shown in Fig. 2 were generated by plotting $\mathbf{y}(\mathbf{x},t)$ for five different values of t and for $\mathbf{x}$ on a fine periodic square lattice $\cap\ \Omega$.

In the spirit of "remote actuation" we now consider a scheme for causing the cantilever to bend by applying a field. Notice that while the motion given in (113) generates a great variety of deformed configurations, the magnetization only assumes the two constant values $\mathbf{R}\mathbf{m}_2$ and $\pm\mathbf{m}_1$ for all such configurations. Hence, it is tempting to assign a field which favors alternately these two magnetizations. The applied field energy is the second term of (106); thus, a natural choice for the applied field is,

$$\mathbf{h}_0 = \frac{h_0}{2m_s}\{(1-\sin(\omega t))\mathbf{R}\mathbf{m}_2 + (1+\sin(\omega t))(\pm\mathbf{m}_1)\}. \tag{114}$$

Here the choice of sin is somewhat arbitrary: it may be desirable to choose a square wave or some other periodic function. The assignment (114) of the applied field always has a positive component on both of the magnetizations $\mathbf{R}\mathbf{m}_2$ and $\pm\mathbf{m}_1$ and therefore does not have a tendency to cause 180° domains to be introduced, which would invalidate the ansatz (113).

There are many interesting generalizations of this family of motions and fields that may be desirable to study from the point of view of actuation or flight. First, the applied field is uniform in space, yet the ansatz (113) is flexible enough to describe beating motions like the movement of an eel through water. A little reflection on Fig. 2 suggests that this would be promoted by an applied field that is periodic in both space and time. Second the beam could have nonuniform cross-section, and this may also be desirable from the point of view of flight. Third, the ansatz (113) implies that the cantilever is fixed at its root $\mathbf{x}\cdot\mathbf{n}_o = 0$. If one premultiplies both the deformation and the magnetization in (113) by the same time-dependent rotation matrix $\hat{\mathbf{R}}(t)$ and adds a time-dependent translation $\hat{\mathbf{c}}(t)$ to the motion, then one retains the assumption that the (deformation gradient, magnetization) lies on the energy wells, and the resulting generalized ansatz is sufficiently flexible to describe a general motion of the cantilever, e.g., the motion that would ensue after "launching" the cantilever, in a time-varying ambient field, with arbitrary initial position and velocity consistent with the ansatz. Fourth, the nucleation and annihilation of interfaces, with suitable criteria for each, could be included ([25]).

2.4 Stability of the initial state

It would be useless to discuss this ansatz unless, for some assignment of interfacial positions $s_1, \ldots, s_N$, it describes a relatively stable equilibrium state with no applied field. That is, without some degree of stability, it is unlikely that such configurations could be achieved initially in a film, preceding the application of the time-dependent field. It is currently impossible to do a full calculation of stability because the complete W_o of (106) is unavailable. Here we give an approximate analysis of stability. We take two different approaches: 1) a pure micromagnetic analysis (deformation fixed) for a thin film, and 2) a domain theory calculation within the context of the ansatz (113).

A key issue is magnetic hardness. In a thin film there is a strong tendency arising from the magnetostatic energy for the magnetization to lie in the plane of the film. So a key question is: In a thin film of Ni_2MnGa with an easy axis that is not in-plane, is magnetic anisotropy sufficiently strong to keep the magnetization near the easy axis? To investigate this issue, consider a single variant of martensite (e.g., $\mathbf{y}(\mathbf{x}) = \mathbf{U}_1\mathbf{x}$), change variables to the deformed configuration $\mathbf{z} = \mathbf{y}(\mathbf{x})$ and consider the micromagnetic energy,

$$\int_{\mathbf{y}(\Omega)} \{\alpha|\nabla\mathbf{m}|^2 + \bar{W}(\mathbf{m})\}\, d\mathbf{z} + \frac{1}{2\gamma}\int_{\mathbb{R}^3} |\nabla\varphi|^2 d\mathbf{z} \tag{115}$$

with φ determined from $\mathbf{m}$ via (107) and $\bar{W}(\mathbf{m}) = W_o(\mathbf{U}_1, \mathbf{m})/\det\mathbf{U}_1$. Suppose $\mathbf{y}(\Omega)$ is a film of cross-section $\mathscr{S}$ and thickness h, and let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be an orthonormal basis with $\mathbf{e}_3$ normal to the film. Consider a family of minimizers $\mathbf{m}^{(h)}$ of (115) parameterized by the thickness h of the film. It can be shown that as $h \to 0$, $\mathbf{m}^{(h)}$ converges in a rather strong sense[12] to $\bar{\mathbf{m}}$, which minimizes the 2-D energy,

$$\int_{\mathscr{S}} \{\alpha|\nabla_p\mathbf{m}|^2 + \bar{W}(\mathbf{m}) + \frac{1}{2\gamma}(\mathbf{m}\cdot\mathbf{e}_3)^2\}\, dz_1 dz_2. \tag{116}$$

Here, ∇_p is the gradient in the plane of the film and $\alpha > 0$ is the exchange constant. In (116) the thin film energy is completely local and the magnetostatic equation no longer plays a role. It is a trivial matter to minimize the energy (116): all minimizers are constant magnetizations $\bar{\mathbf{m}}$ ($\nabla_p\bar{\mathbf{m}} = 0$) that minimize pointwise the function $\bar{W}(\mathbf{m}) + \frac{1}{2\gamma}(\mathbf{m}\cdot\mathbf{e}_3)^2$. The first term favors the easy axis and the second favors the plane of the film. Fortunately, the function $\bar{W}$ has been measured (Tickle and James [44]). These measurements are all consistent with the idea that single variant Ni_2MnGa martensite is a classic uniaxial magnetic material with anisotropy energy,

$$\bar{W}(\mathbf{m}) = \frac{K_u}{m_s^2}|\mathbf{m}\times\mathbf{e}|^2, \tag{117}$$

where $\mathbf{e}$ is a unit vector on the c-axis, $K_u = 2.45 \times 10^6$ ergs/cm^3 and $m_s = 600$ emu/cm^3 (In our notation the c-axis for variant 1 is simply (100) in the cubic basis). Recalling the geometry of the cantilever, we find by minimization of $\bar{W}(\mathbf{m}) + \frac{1}{2\gamma}(\mathbf{m}\cdot\mathbf{e}_3)^2$ with these experimental values that the minimizing magnetization differs from the easy axis by only $0.17°$.

Tickle [45] also found that the magnetic response for a specimen loaded with varying loads (all designed *not* to cause transformation to another variant) was very nearly independent of the load, indicating that the the assumption $\mathbf{y}(\mathbf{x}) = \mathbf{U}_1\mathbf{x}$ above is not too restrictive. In any case, a relaxed-deformation theory could be formulated along the lines above with effectively the same conclusions.

This calculation suggests that a very thin film will support a single domain with the magnetization differing very slightly from the easy axis. As the thickness h is increased, domains are expected, having again magnetization near the easy axes. To estimate the domain structure, we now consider an approximate domain theory calculation within the context of the ansatz (113).

For motions and magnetizations constrained by piecewise rigid magneto-mechanics the appropriate free energy (no applied field, k energy wells) is, schematically,

[12] The precise statement is that the rescaled magnetization $\mathbf{m}^{(h)}(z_1, z_2, \frac{1}{h}z_3)$ has a subsequence that converges to $\bar{\mathbf{m}}(z_1, z_2)$ in $H^1(\mathscr{S}\times(0,1))$ (Gioia and James [21]).

$$\left\{\text{interfacial energy} \quad + \quad \frac{1}{2\gamma}\int_{\mathbb{R}^3} |\nabla\varphi|^2 d\mathbf{y}\right\}$$
$$\text{subject to} \quad (\nabla\mathbf{y}(\mathbf{x}), \mathbf{m}(\mathbf{y}(\mathbf{x}))) \in \{\mathrm{SO}(3)(\mathbf{U}_1, \mathbf{m}_1) \cup \ldots \cup \mathrm{SO}(3)(\mathbf{U}_k, -\mathbf{m}_k)\},$$
$$\text{and} \qquad \mathrm{div}(-\nabla\varphi + \gamma\mathbf{m}) = 0 \quad \text{on } \mathbb{R}^3. \tag{118}$$

The interfacial energy is described schematically, because accurate expressions for it have not been investigated for strong jumps in both magnetization and deformation gradient. In pure magnetism interfacial energy arises from the presence of exchange energy which, together with anisotropy and applied field energies, produces reliable values for wall energies. In pure mechanics, there are less reliable expressions (variants of $\int_\Omega \kappa|\nabla^2\mathbf{y}|^2 d\mathbf{x}$) that are analogous to exchange energy, and model interfacial energy, but which do not generally get the details of twin boundary structure right. Hence, for our purposes we shall simply assume a constant energy per unit area Γ on the magnetoelastic discontinuity.

The goal is then to estimate the magnetostatic energy for the ansatz with a given choice of interfacial positions. We begin with the basic bound given in Sect. 1:

$$\frac{1}{2\gamma}\int_{\mathbb{R}^3} |\nabla\varphi|^2 d\mathbf{y} \le \frac{1}{2}\int_{\mathbb{R}^3} |\tilde{\mathbf{m}}|^2 d\mathbf{y}, \tag{119}$$

for any $\tilde{\mathbf{m}} \in L^2(\mathbb{R}^3)$ satisfying $\mathrm{div}\,\mathbf{m} = \mathrm{div}\,\tilde{\mathbf{m}}$ on $\mathbb{R}^3$. In fact, the bound is sharp: minimization of the right hand side over such $\tilde{\mathbf{m}}$ gives the left hand side. The bound has been used often in recent years (e.g., Choksi, Kohn, Otto [10]) and we are guided by some of these constructions which in related cases have been proven optimal. Note that $\tilde{\mathbf{m}}$ need not satisfy any constraints[13] like $|\tilde{\mathbf{m}}| = m_s$ and need not be supported on Ω. In Appendix A we construct two families of test functions for the right hand side of (119). Consider equal twin width s and a wall energy per unit area Γ. These bounds give a critical thickness $h_c = c_1\Gamma$ and the relations,

$$\begin{aligned} &\text{For} \quad h < h_c, \quad s \to \infty \quad \text{(no interfaces)},\\ &\text{For} \quad h \ge h_c, \quad s = \sqrt{c_2\Gamma h}. \end{aligned} \tag{120}$$

The constants $c_1 = 7.6 \times 10^{-6}\mathrm{cm}^3/\mathrm{erg}$ and $c_2 = 4.3 \times 10^{-7}\mathrm{cm}^3/\mathrm{erg}$ that emerge from the bounds depend only on the lattice parameters and the saturation magnetization. The dependence is given in Appendix A. They are probably not optimal in general, but are optimal within a class of test functions. If we assume a typical value for wall energy for ferromagnetic materials[14] $\Gamma = 1\,\mathrm{erg}/\mathrm{cm}^2$ (Cullity [11]), we get $h_c = 76\mathrm{nm}$ which is quite reasonable. As a check on this formula, we remark that when it is evaluated at a value of h appropriate to the mm scale specimen used in the MFM studies of Pan and James [41], it gives closely the twin spacing they observed[15].

The estimates in this subsection indicate that if the cantilever is not too thin, and with an appropriate relation between the wall spacing and the thickness of the cantilever, the ansatz (113) describes an attainable starting state.

2.5 The configurational force on a magnetoelastic domain wall and kinetic equations

It is well-known that in a nonlinear elastic material that has equilibrium states with surfaces of discontinuity of the deformation gradient, the corresponding dynamic laws – the balances of linear and angular momentum and entropy conditions together with the usual initial and boundary conditions – are insufficient to determine the motion. There is a large literature on this situation, reviewed briefly in Sect. 1, which reaches the conclusion that additional laws that describe the kinetics of interface motion are needed. This situation persists in the magnetoelastic case. As discussed in the introduction we tentatively follow the standard prescription for such kinetic laws: the normal velocity of the interface is a function of the configurational force on it.

[13] For the proof that (119) delivers the best bound, these additional restrictions cannot be assumed.

[14] However, there has been no study yet of the structure or energy of a domain wall that is also a strong discontinuity of the deformation gradient, so this value for the wall energy may not be reliable.

[15] It is too early to say that this supports the idea that the main interfacial energy at an intervariant wall in a ferromagnetic shape memory material has purely magnetic origins.

The dissipation function $\mathscr{D}$ has been derived in Sect. 1. For the kinematic setting described here, the dissipation function becomes an integral over $\mathscr{S}_o^t$, given by,

$$\mathscr{D} = \int_{\mathscr{S}_o^t} -[\![W_o - \mathbf{h}\cdot\mathbf{m}_o - \frac{\gamma}{2}(\det\nabla\mathbf{y}(\mathbf{m}\cdot\mathbf{n})^2) - \langle\mathbf{t}_o\rangle\cdot\nabla\mathbf{y}\mathbf{n}_o]\!] s_n^o \, da_o. \tag{121}$$

The term in double brackets is the configurational force. In general we consider kinetic laws of the form:

$$s_n^o = -\mathscr{F}([\![W_o - \mathbf{h}\cdot\mathbf{m}_o - \frac{\gamma}{2}(\det\nabla\mathbf{y}(\mathbf{m}\cdot\mathbf{n})^2) - \langle\mathbf{t}_o\rangle\cdot\nabla\mathbf{y}\mathbf{n}_o]\!]) \quad \text{on } \mathscr{S}_o^t, \tag{122}$$

where $\mathscr{F}$ is a monotone increasing function, so the nonpositivity of $\mathscr{D}$ is respected. Two typical forms of $\mathscr{F}$ are:

$$\begin{aligned} \mathscr{F}(\mathscr{C}) &= \mu\mathscr{C}, && \text{classical linear kinetics,} \\ \mathscr{F}(\mathscr{C}) &= \mu\begin{cases} -\sqrt{a\mathscr{C}^2 - b^2} & , \quad \mathscr{C} < -c, \\ 0 & , \quad -c \le \mathscr{C} \le c, \\ \sqrt{a\mathscr{C}^2 - b^2} & , \quad \mathscr{C} > c, \end{cases} && \text{“pinning” kinetics.} \end{aligned} \tag{123}$$

Here, $\mu > 0$ is the *mobility* and $a, b,\ c$ are positive constants with $ac^2 - b^2 = 0$.

2.6 Equations of motion: piecewise rigid magneto-mechanics

In this section we assume the basic ansatz (113) for the motion and magnetization of the cantilever. The equations of motion are expected to follow from three laws: balance of linear momentum, balance of rotational momentum and the kinetic equations. We take them one at a time.

2.6.1 Balance of linear momentum

Let us assume that no forces are applied to the cantilever except at its root, and except by the applied field (114). The balance of linear momentum for a subregion $\mathscr{A}\times(s,L)$ is,

$$\frac{d}{dt}\int_s^L \int_{\mathscr{A}} \rho_o \dot{\mathbf{y}}\, d\mathbf{x} = \mathbf{F}_{(s,L)}. \tag{124}$$

Here, the reference density ρ_o is assumed to be constant, and $\mathbf{F}_{(s,L)}$ is the total force applied to $\mathbf{y}(\mathscr{A}\times(s,L),t)$, given by the sum of the short and long range forces:

$$\begin{aligned} \mathbf{F}_{(s,L)} &= \int_{\mathscr{A}} \mathbf{t}_o(\mathbf{x},\mathbf{n}_o)|_{\mathbf{x}\cdot\mathbf{n}_o = s}\, da_o + \int_{\partial\mathbf{y}(\mathscr{A}\times(s,L),t)} \frac{\gamma}{2}(\mathbf{m}\cdot\mathbf{n})^2\mathbf{n}\, da \\ &- \int_{\mathbf{y}(\mathscr{A}\times(s,L),t)} (\mathbf{m}\cdot\nabla^2\varphi)\, d\mathbf{y} + \int_{\mathbf{y}(\mathscr{A}\times(s,L),t)} (\mathbf{m}\cdot\nabla\mathbf{h}_0)\, d\mathbf{y}. \end{aligned} \tag{125}$$

Here, we have written the magnetic terms in the deformed configuration but the traction in the reference configuration for simplicity.

Recall the definition of $\chi(s,t)$ and the convention regarding $\{s_0,\ldots,s_{(N+1)}\}$ described just before (113). Let $\lambda : (0,L)\times[0,\infty)\to\mathbb{R}$ be given by,

$$\lambda(s,t) = j \quad \text{for } s_j(t) < s \le s_{j+1}(t), \quad j \in \{0,\ldots,N\}. \tag{126}$$

Substitute the first of (113) into the left of (124). After a tedious calculation, we get for the left hand side of (124),

$$\frac{d}{dt}\int_s^L \int_{\mathscr{A}} \rho_o \dot{\mathbf{y}}\, d\mathbf{x} =$$
$$\rho_o bh\,\mathbf{a}\left\{\sum_{j=\lambda(s,t)}^{N}\left(\sum_{k=1}^{j}(-1)^k \ddot{s_k}(t)\right)(s_{j+1}(t)-s_j(t)) + \sum_{j=\lambda(s,t)}^{N}\left(\sum_{k=1}^{j}(-1)^k \dot{s_k}(t)\right)(\dot{s}_{j+1}(t)-\dot{s}_j(t))\right.$$
$$\left. -\left(\sum_{k=1}^{\lambda(s,t)}(-1)^k \ddot{s_k}(t)\right)(s-s_{\lambda(s,t)}(t)) - \left(\sum_{k=1}^{\lambda(s,t)}(-1)^k \dot{s_k}(t)\right)(-\dot{s}_{\lambda(s,t)}(t))\right\} \tag{127}$$

Consider the formula for the force (125) evaluated for s just to the right, or just to the left, of a discontinuity. The limiting values of traction are expected to jump across the discontinuity, but are subject to jump conditions arising from the balance of linear momentum. However, the formula for the configurational force (121) involves the average traction. The coupling between kinetic equations and the balance of linear momentum will therefore be clarified by exposing this jump condition, which has been derived in Sect. 1.5.4:

$$-[\![\rho_o \dot{\mathbf{y}}]\!] s_n^o = [\![\mathbf{t}_o]\!]. \tag{128}$$

Specialized to the ansatz (113), (128) is

$$(-1)^{k+1}\rho_o\,\mathbf{a}(\dot{s}_k)^2 = [\![\mathbf{t}_o]\!], \quad \text{at } \mathbf{x}\cdot\mathbf{n}_o = s_k(t) \tag{129}$$

2.6.2 Balance of rotational momentum

The balance of rotational momentum is,

$$\frac{d}{dt}\int_s^L \int_{\mathscr{A}} \rho_o \mathbf{y}\times\dot{\mathbf{y}}\, d\mathbf{x} = \mathbf{T}_{(s,L)}. \tag{130}$$

Here, $\mathbf{T}_{(s,L)}$ is the torque (about the origin) applied to $\mathbf{y}(\mathscr{A}\times(s,L),t)$.

$\mathbf{T}_{(0,L)}$ has contributions from the torques of magnetic and mechanical forces, as well as from magnetic torques (i.e., the "compass needle" body torque, $\int \mathbf{m}\times\mathbf{h}\, d\mathbf{y}$). One such contribution is the torque of the surface traction on a cross-section, the term

$$\int_s^L \int_{\mathbf{A}} \mathbf{y}\times\mathbf{t}_o\, da_o. \tag{131}$$

Since this torque on a cross-section can (for a rigid body) be assigned independent of the force, the presence of this term makes the balance of rotational momentum disposable: it just determines the value of this torque. Hence the balance of rotational momentum does not contribute to the equations of motion. In the general case, however, the balance of rotational momentum is an important equation that effectively determines the overall rotation (James and Rizzoni [25]).

2.6.3 Kinetic equations

We specialize the kinetic equations to the case of piecewise rigid magneto-mechanics. Since the ansatz (113) involves states from the energy wells, W_o vanishes in the formula for the configurational force[16]. In piecewise rigid magneto-mechanics, each open, connected region where the deformation gradient belongs to an energy well, $\{\mathbf{x}\in\Omega : \nabla\mathbf{y}(\mathbf{x},t)\in \mathrm{SO}(3)\mathbf{U}_i\}$, undergoes a rigid motion $\mathbf{y}(\mathbf{x},t) = \mathbf{R}(t)\mathbf{U}_i\mathbf{x}+\mathbf{c}(t)$, $\mathbf{R}(t)\in\mathrm{SO}(3)$, $\mathbf{c}(t)\in\mathbb{R}^3$, according to a classical theorem of continuum mechanics. At an interface between two neighboring regions of this type having reference normal $\mathbf{n}_o$, we have Hadamard's jump condition $\mathbf{R}_i(t)\mathbf{U}_i - \mathbf{R}_j(t)\mathbf{U}_j = \mathbf{a}\otimes\mathbf{n}_o$;

[16] As discussed above, the constrained theory is associated with hard moduli and large anisotropy constants, that is, the assumption that W_o grows steeply away from its energy wells. As in the elementary example $W(x) = (1/2)Ex^2 - \sigma x$, with minimizers $x = \sigma/E$ and minimum energy $\sigma^2/2E$, the large E limit ($\sigma^2/2E \ll \sigma$) is the case in which W_o in (121) is expected to be negligible compared to the the other three terms in the expression for the configurational force.

the left hand side of this condition being only a function of t, it is easily seen that $\mathbf{n}_o$ can also be at most a function of t, implying that the interface is a part of a plane. In martensitic materials that undergo reversible transformations, interfaces are often nearly planar, even in complex microstructures, though one might wish to generalize piecewise rigid magneto-mechanics to allow for fine phase mixtures.

It is then clear that the kinetic law (122) is rather highly constrained by its left hand side, especially when combined with the jump condition (128). Referring back to the kinematics of moving interfaces, Subsections 1.5.3, 1.5.4, an interface whose reference normal $\mathbf{n}_o$, $|\mathbf{n}_o| = 1$, is at most a function of time is expressible as the zero level set of the function,

$$f_o(\mathbf{x}, t) = (\mathbf{x} - \mathbf{x}_o(t)) \cdot \mathbf{n}_o(t), \tag{132}$$

where $\mathbf{x}_o(t)$ is an arbitrary smooth function of t. The normal referential velocity of propagation of such an interface is

$$s_n^o = \dot{\mathbf{x}}_o(t) \cdot \mathbf{n}_o(t) - (\mathbf{x} - \mathbf{x}_o(t)) \cdot \dot{\mathbf{n}}_o(t) \quad \text{on} \quad \{f_o(\mathbf{x}, t) = 0\}, \tag{133}$$

which is linear in $\mathbf{x}$. For the ansatz (113) we have simply $s_n^o = \dot{\mathbf{x}}_o(t) \cdot \mathbf{n}_o$. This will also be typical of general piecewise rigid magneto-mechanics, because there is typically a fixed, discrete set of normals associated with rank-one connections between energy wells. Noting that the only quantity on the right hand side of (122) that is not determined by the ansatz and applied field is $\langle \mathbf{t}_o \rangle$, we surmise that (122) places an unnecessarily strong restriction on the form of the limiting values of the traction.

An alternative approach that seems reasonable is to impose an average version of the kinetic law. In the case of the ansatz the dissipation function for the i^{th} interface is expressible in the form,

$$\begin{aligned} \mathscr{D} &= s_n^o \int_{\mathscr{A}} -[\![-\mathbf{h} \cdot \mathbf{m}_o - \frac{\gamma}{2}(\det \nabla \mathbf{y} (\mathbf{m} \cdot \mathbf{n})^2) - \langle \mathbf{t}_o \rangle \cdot \nabla \mathbf{y} \mathbf{n}_o]\!] \, da_o, \\ &= \dot{s}_i(t) \left\{ (\eta_1^2 \eta_2 \int_{\mathscr{A}} \mathbf{h} \, da_o)^+ \cdot \mathbf{R}\mathbf{m}_2 - (\eta_1^2 \eta_2 \int_{\mathscr{A}} \mathbf{h} \, da_o)^- \cdot \pm \mathbf{m}_1 + \left\langle \int_{\mathscr{A}} \mathbf{t}_o \, da_o \right\rangle \cdot \mathbf{a} \right\} \end{aligned} \tag{134}$$

Here, the choice of $\pm$ is the same that satisfies (112), and we have used several specifics about the ansatz: the vanishing of $[\![\det \nabla \mathbf{y} (\mathbf{m} \cdot \mathbf{n}_o)^2]\!]$, and equations (112), (110), and (109). The kinetic law is then of the form:

$$\dot{s}_i(t) = \mathscr{F} \left((\eta_1^2 \eta_2 \fint_{\mathscr{A}} \mathbf{h} \, da_o)^+ \cdot \mathbf{R}\mathbf{m}_2 - (\eta_1^2 \eta_2 \fint_{\mathscr{A}} \mathbf{h} \, da_o)^- \cdot \pm \mathbf{m}_1 + \left\langle \fint_{\mathscr{A}} \mathbf{t}_o \, da_o \right\rangle \cdot \mathbf{a} \right), \tag{135}$$

with $\mathscr{F}$ chosen as in (123). Here, $\fint_{\mathscr{A}}$ is $\frac{1}{\text{Area A}} \int_{\mathscr{A}} \ldots da$, this being done to remove an unphysical area dependence.

A general study of the well-posedness of piecewise rigid body mechanics is given in [25].

2.6.4 Summary of equations of motion

We can now summarize the equations of motion. The balance of rotational momentum plays no role. The remaining equations are the balance of linear momentum (124), the jump condition (128) and the kinetic law (135). The balance of linear momentum is:

$$\begin{aligned} \rho_o bh \, \mathbf{a} \quad & \left\{ \sum_{j=\lambda(s,t)}^{N} \left(\sum_{k=1}^{j} (-1)^k \ddot{s}_k(t) \right) (s_{j+1}(t) - s_j(t)) + \sum_{j=\lambda(s,t)}^{N} \left(\sum_{k=1}^{j} (-1)^k \dot{s}_k(t) \right) (\dot{s}_{j+1}(t) - \dot{s}_j(t)) \right. \\ & \left. - \left(\sum_{k=1}^{\lambda(s,t)} (-1)^k \ddot{s}_k(t) \right) (s - s_{\lambda(s,t)}(t)) - \left(\sum_{k=1}^{\lambda(s,t)} (-1)^k \dot{s}_k(t) \right) (-\dot{s}_{\lambda(s,t)}(t)) \right\} \\ = \quad & \int_{\mathscr{A}} \mathbf{t}_o(\mathbf{x}, \mathbf{n}_o)|_{\mathbf{x} \cdot \mathbf{n}_o = s} \, da_o + \int_{\partial \mathbf{y}(\mathscr{A} \times (s,L),t)} \frac{\gamma}{2} (\mathbf{m} \cdot \mathbf{n})^2 \mathbf{n} \, da \\ & - \int_{\mathbf{y}(\mathscr{A} \times (s,L),t)} (\mathbf{m} \cdot \nabla^2 \varphi) \, d\mathbf{y} + \int_{\mathbf{y}(\mathscr{A} \times (s,L),t)} (\mathbf{m} \cdot \nabla \mathbf{h}_0) \, d\mathbf{y} \end{aligned} \tag{136}$$

Between discontinuities it plays no role beyond determining the integrated traction there. However, evaluated on each side of a discontinuity, it couples to the other equations. But its jump is already included in the jump condition (128), which, due to its simplicity, we keep (temporarily) as an independent equation. Thus, the only useful information contributed by the balance of linear momentum is its value on one side of the discontinuity, which for definiteness we take to be the right, where it takes the simpler form,

$$\begin{aligned}\rho_o bh\,\mathbf{a} \quad & \left\{\sum_{j=i}^{N}\left(\sum_{k=1}^{j}(-1)^k\ddot{s}_k(t)\right)(s_{j+1}(t)-s_j(t))+\sum_{j=i}^{N}\left(\sum_{k=1}^{j}(-1)^k\dot{s}_k(t)\right)(\dot{s}_{j+1}(t)-\dot{s}_j(t))\right.\\ & \qquad \left.-\left(\sum_{k=1}^{i}(-1)^k\dot{s}_k(t)\right)(-\dot{s}_i(t))\right\},\\ = \quad & \int_{\mathscr{A}}\mathbf{t}_o(\mathbf{x},\mathbf{n}_o)|_{\mathbf{x}\cdot\mathbf{n}_o=s_i(t)+}\,da_o+\int_{\partial\mathbf{y}(\mathscr{A}\times(s_i(t)+,L),t)}\frac{\gamma}{2}(\mathbf{m}\cdot\mathbf{n})^2\mathbf{n}\,da\\ & \qquad -\int_{\mathbf{y}(\mathscr{A}\times(s_i(t)+,L),t)}(\mathbf{m}\cdot\nabla^2\varphi)\,d\mathbf{y}+\int_{\mathbf{y}(\mathscr{A}\times(s_i(t)+,L),t)}(\mathbf{m}\cdot\nabla\mathbf{h}_0)\,d\mathbf{y}. \qquad (137)\end{aligned}$$

This is to be combined with jump condition (128) and the kinetic law (135):

$$\begin{aligned}(-1)^{i+1}\rho_o\,\mathbf{a}(\dot{s}_i)^2 &= [\![\int_{\mathscr{A}}\mathbf{t}_o\,da_o]\!], \quad \text{at } \mathbf{x}\cdot\mathbf{n}_o=s_i(t)\\ \dot{s}_i(t) &= \mathscr{F}\left((\eta_1^2\eta_2\fint_{\mathscr{A}}\mathbf{h}\,da_o)^+\cdot\mathbf{R}\mathbf{m}_2-(\eta_1^2\eta_2\fint_{\mathscr{A}}\mathbf{h}\,da_o)^-\cdot\pm\mathbf{m}_1+\left\langle\fint_{\mathscr{A}}\mathbf{t}_o\,da_o\right\rangle\cdot\mathbf{a}\right). \qquad (138)\end{aligned}$$

As long as we avoid issues of nucleation and annihilation, which also could be included with suitable criteria, these equations are evidently well-posed. The jump condition can be used to eliminate the integrated traction behind each interface $\int_{\mathscr{A}}\mathbf{t}_o(\mathbf{x},\mathbf{n}_o)|_{\mathbf{x}\cdot\mathbf{n}_o=s_i(t)-}$ from the kinetic law (138). Then the unknowns can be taken to be the $N+1$ quantities,

$$\int_{\mathscr{A}}\mathbf{t}_o(\mathbf{x},\mathbf{n}_o)|_{\mathbf{x}\cdot\mathbf{n}_o=s_i(t)+},\quad s_1(t),\ldots,s_N(t), \qquad (139)$$

which are then to be determined from the balance of linear momentum (137) and the kinetic laws. Of course, the field has to be calculated using the magnetostatic equation and applied field at each time step.

Simulations will be presented in a forthcoming paper (Tang and James [43]).

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A Appendix: A bound on the demagnetization energy for the cantilever

The bound (119) is evaluated for two families of test functions.

1. The first is simply $\tilde{\mathbf{m}}=\mathbf{m}$. Recalling that the number of interfaces is N, Γ is the interfacial energy per unit area, b is the depth of the cantilever and h is its thickness, we have the bound,

$$\begin{aligned}\left\{\text{interfacial energy}+\frac{\gamma}{2}\int_{\mathbb{R}^3}|\tilde{\mathbf{m}}|^2\,d\mathbf{y}\right\} &= N\Gamma hb|\mathbf{U}_1\mathbf{e}_2\times\mathbf{U}_1\mathbf{e}_3|+\frac{\gamma}{2}\det\mathbf{U}_1 hbLm_s^2,\\ &= N\Gamma hb|\mathrm{cof}\mathbf{U}_1(\mathbf{e}_2\times\mathbf{e}_3)|+\frac{\gamma}{2}\det\mathbf{U}_1 hbLm_s^2,\\ &= N\Gamma hb\left(\frac{\eta_1^4+\eta_1^2\eta_2^2}{2}\right)^{\frac{1}{2}}+\frac{\gamma}{2}\eta_1^2\eta_2 hbLm_s^2. \qquad (140)\end{aligned}$$

This energy is minimized at $N=0$ and is proportional to the thickness h.

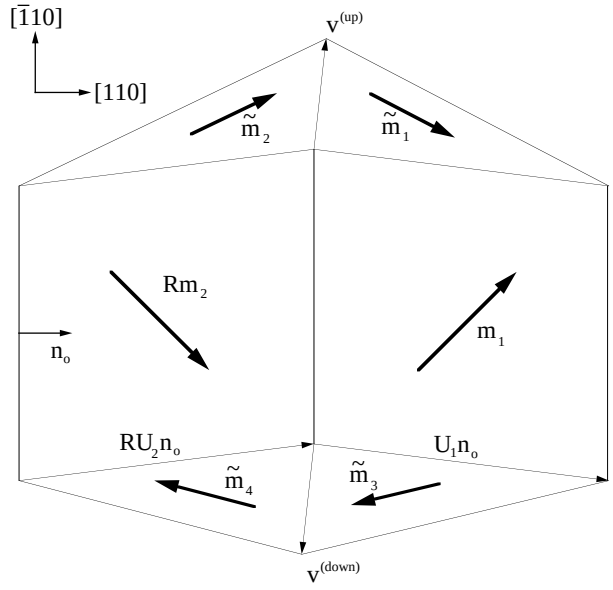

Fig. 3. Illustration of the second test function

2. The second family of test functions is illustrated in Fig. 3 for one period of the cantilever and for equi-spaced interfaces, the distance between interfaces being s in the reference configuration.
The function $\tilde{\mathbf{m}}$ is supported on the two triangular regions above and below the deformed cantilever and has the values indicated; it vanishes elsewhere including on the cantilever. The vectors $\mathbf{Rm}_2$ and $\mathbf{m}_1$ are shown for convenience. The vectors $\mathbf{v}^{(up)}$ and $\mathbf{v}^{(down)}$ define the triangles. To give $\operatorname{div}\tilde{\mathbf{m}} = \operatorname{div}\mathbf{m}$ the vectors $(\tilde{\mathbf{m}}_1, \tilde{\mathbf{m}}_2, \tilde{\mathbf{m}}_3, \tilde{\mathbf{m}}_4)$ must be parallel to their respective nearby edges and must give the same divergence as $\mathbf{m}$, i.e.,

$$\begin{aligned}
(\tilde{\mathbf{m}}_2 - \tilde{\mathbf{m}}_1)\cdot\mathbf{v}^{(up)} &= 0,\\
(\tilde{\mathbf{m}}_4 - \tilde{\mathbf{m}}_3)\cdot\mathbf{v}^{(down)} &= 0,\\
(\tilde{\mathbf{m}}_2 + \mathbf{Rm}_2)\cdot(\mathbf{RU}_2\mathbf{n}_o)^{\perp} &= 0,\\
(\tilde{\mathbf{m}}_1 + \mathbf{m}_1)\cdot(\mathbf{U}_1\mathbf{n}_o)^{\perp} &= 0,\\
(\tilde{\mathbf{m}}_4 + \mathbf{Rm}_2)\cdot(\mathbf{RU}_2\mathbf{n}_o)^{\perp} &= 0,\\
(\tilde{\mathbf{m}}_3 + \mathbf{m}_1)\cdot(\mathbf{U}_1\mathbf{n}_o)^{\perp} &= 0. \qquad (141)
\end{aligned}$$

These are solved for $(\tilde{\mathbf{m}}_1, \ldots, \tilde{\mathbf{m}}_4)$; there remain four free parameters (This is not obvious from the equation/unknown count but is known to the experts. Two of the equations are redundant because the average magnetization satisfies the jump condition based on the average interface normal). We calculate the bound $\int_{\mathbb{R}^3} |\tilde{\mathbf{m}}|^2 d\mathbf{y}$, substitute in the given parameters η_1, η_2 and m_s, and then optimize over the four free parameters. We then multiply by the number of periods $L/(2s)$. We get the total energy:

$$\frac{2bhL\Gamma}{s} + (4.7\times 10^6)bLs \quad \text{(c.g.s. units)}. \qquad (142)$$

Optimizing over s and comparing with the first bound, we get the results given in the text. The first bound is good for small h and the second is good for large h.

We should remark that it appears that there is no advantage to making unequal spacing between walls. Within the context of the two families of test functions given above, the first does not depend on the spacing. In the second the walls must be equally spaced to permit a solution of (141); in fact, any similar kind of construction would also require this. If the walls are not equi-spaced, then the average magnetization of a period would be nonzero. In that case one could then subtract off the average magnetization, then apply the construction above. Then terms involving the average magnetization would have to be estimated by another method, or else the triangles would have to bridge several periods.

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