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# Piecewise Rigid Body Mechanics

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**Summary.** We propose a new three-dimensional dynamic theory of transforming materials intended to make realistic simulations of the dynamic behavior of these materials accessible. The theory is appropriate for materials whose free energy function rises steeply from its energy wells. Essentially, the theory is the multiwell analog of ordinary rigid body mechanics with three additional features: the full stress is not treated as arbitrary (the average limiting tractions on each interface enter the theory as unknowns), a certain component of the local balance of linear momentum is used, and kinetic laws for interfacial motion are introduced based on ideas of Eshelby and Abeyaratne and Knowles. In an interesting special case of the resulting equations of motion, all material constants together with all information about the shape of the body collapse to a single dimensionless constant. We prove well-posedness up to the time of a collision between interfaces, and do a preliminary study of the problem of annihilation and nucleation of interfaces. Conservation laws and a dissipation inequality are identified. We also give generalizations of the theory to magnetic and thermodynamic piecewise rigid media. A probable application area for the theory is the assessment of the use of transforming materials at small scale as "motors" for propulsion or actuation.

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## 1. Introduction

The development of the theory presented here was motivated by the desire to write down realistic three-dimensional dynamic equations for the response of single crystal martensitic materials that are accessible to accurate simulation. These materials have free energy functions with multiple energy wells and moving interfaces between phases or between variants of the martensitic phase.

The difficulty with the study of the three-dimensional dynamics of transforming materials that undergo a shape change is well known. Even in the case of nonlinear elastic materials with a single energy well, the simplest equations are three-dimensional nonlinear hyperbolic conservation laws. After some time, these develop exceedingly complex arrays of shock waves that dissipate energy. In fact, even the structure of elementary shock waves in such materials (which are not genuinely nonlinear) is not known. These equations can be regularized by the addition of viscosity or viscoelasticity, but the relation between the energy dissipated by the shock waves and that dissipated by viscous or other mechanisms is completely unknown; moreover, most workers believe that viscosity or viscoelasticity are not the important dissipative mechanisms in these materials. The situation with multiple-energy-well materials is significantly less clear. These equations exhibit all the difficulties of three-dimensional hyperbolic conservation laws together with change-of-type from hyperbolic to elliptic, caused by the presence of energy wells. Not only is the structure of elementary shock and phase boundary interactions unknown, but the admissibility criterion appropriate for phase boundaries is not yet settled. (The search for the correct admissibility criterion has been an active area of research for the past 20 years with important contributions from Abeyaratne and Knowles [1] [2], Dafermos [11], Escobar and Clifton [14], Slemrod [37], Truskinovsky [39], and others.) Part of the problem with the assessment of these admissibility criteria is the near absence of three-dimensional dynamic solutions in experimentally accessible situations.

Fortunately, there is a class of martensitic materials with features that suggest a simplification of the dynamic equations. These are materials for which the free energy grows steeply away from its energy wells. This class includes many important examples, particularly some examples of recent interest. With such materials, it is common practice in the experimental literature to analyze the static microstructure or the conditions for transformation of a stressed specimen by evaluating various expressions *using the lattice parameters of the unstressed phases*. As an extreme example, we mention recent work of Vasko, Leo, and Shield [40] on the fracture of Cu-14.0wt.%Al-3.5wt.%Ni. Under load, they observe various austenite-martensite interfaces emanating from a sharp crack tip. In a variety of specimens with various orientations relative to the loads and various crack orientations, they measured the traces of these interfaces on the plane of observation. Then they determined theoretically an austenite-martensite interface (from among the 96 possible ones) using a maximum work criterion based on the crack tip stress field. They thereby select a unique austenite-martensite interface, and they calculate the angle its trace makes on the plane of observation relative to the plane of the crack; all of these calculations are done using the unstressed lattice parameters. They find that the selected interface agrees with the observed one in all cases, and the disagreement between the measured angle and predicted angle is less than a degree. Thus even with the unusually large stresses at the crack tip, there seems to be good reason to use lattice parameters of the unstressed

phases. In essence, they are constraining the deformation gradients to be on the energy wells (except for negligible transition layers) but allowing the stresses to be nonzero. As we quantify in Section 3.2, even under reasonably large stresses, the deviation away from an energy well is much less than the distance between the wells in this material.

Other interesting examples are the recently discovered ferromagnetic shape memory materials. These are martensitic but also ferromagnetic, and they can be made to undergo very large strains by applying a magnetic field. The main search criteria currently being used to seek new examples of these materials are mobile interfaces, high magnetic anisotropy, and high stiffness. The latter two criteria translate into steep growth away from energy wells, which in the ferromagnetic-martensitic case are defined by special pairs of (magnetization, deformation gradient).

In ordinary nonlinear elastic materials, steep growth away from energy wells means high stiffness, and the dynamic behavior can often be modeled using ordinary rigid body mechanics. Piecewise Rigid Body Mechanics (PRBM) is the multiwell analog of rigid body mechanics. The body is divided into a number of subbodies separated by interfaces. On each subbody the deformation gradient lies on an energy well. As interfaces move, the subbodies rotate and translate and exchange mass due to interfacial motion. There is also overall rotation and translation. These motions are subject to the balances of linear and rotational momentum for the whole body and for subregions of the body. The Cauchy stress is an arbitrary symmetric tensor (the reaction stress due to the constraint of rigidity) on the subbodies, except that the average of its normal component on the interfaces enters the theory in a nontrivial way. Also, a certain component of the local balance of linear momentum on subregions bounded by interfaces is retained. As expected, we observe that such a theory is incomplete. To remove the indeterminacy, we introduce a kinetic law for the interfaces, in the spirit of work of Eshelby [15] [16] and Abeyaratne and Knowles [1] [2], after deriving an expression for the configurational force on an interface in the framework on piecewise rigid body mechanics. All the dissipation in PRBM occurs at the interfaces. We show that the resulting theory is well posed (Section 8). We also have made a considerable effort to put the equations in their most useable form, and to reveal their basic structure in the sense of dynamical systems; in particular we identify conservation laws, a quadratic form representing the kinetic energy, and a dissipation inequality. An observation (Section 8.3) about a matrix that we term the mass distribution matrix is also expected to simplify numerical implementation considerably.

The special case corresponding to a cantilever, fixed at its root, and containing a parallel array of any number of interfaces (Figure 2) is worked out in Remark 8.1. The equations of motion are highly nonlinear; yet, when the cantilever is chosen to have constant reference density $\rho_o$, uniform cross-sectional area, and the kinetic law is chosen to be linear with mobility $\mu$, then the unforced equations are determined by the single dimensionless number

$$\frac{T}{a^2 L \mu \rho_o}, \tag{1}$$

where $T$ is a typical time, $L$ is the reference length of the cantilever, and $a$ is a certain dimensionless scalar that depends only on the lattice parameters of the two phases (see Section 4 for the formula for $a = |\mathbf{a}|$). This number embodies all the input to the equations in this case, including all information about the shape of the body and all material constants. This number compares inertial to configurational forces. With such

a complete collapse of material and geometric data, the case of the cantilever provides an attractive way to test the existence and forms of kinetic relations. (There is a similar collapse for nonlinear kinetic relations, Remark 8.1.) The latter comprises one of the most important problems in the study of phase transformations.

Interfaces separating phases can collide for certain choices of the time evolution of applied forces. Our existence/uniqueness theorem breaks down at an annihilation. In terms of the dynamical system, an annihilation represents a singular perturbation. Both the mathematical and physical treatment of an annihilation in PRBM are quite delicate, and we confine attention to a few theorems and remarks. The physical treatment is subtle because, as explained in Section 8.3, it is not expected that the conservation laws hold through an annihilation. It is thus an open issue what precisely should be the the new initial data with which to restart the initial value problem (with the pair of colliding interfaces removed).

We give generalizations (Section 9) of PRBM to include magnetism (Piecewise Rigid Magneto-Mechanics, PRMM) and thermodynamics (Piecewise Rigid Thermodynamics, PRT). In the former, we use the energy wells appropriate to a magnetoelastic (or ferromagnetic shape memory) material and we include magnetic forces and torques and magnetic configurational forces. We also require a version of Maxwell's equations to determine the field from the magnetization. In the latter, we allow a general form of (anisotropic) Fourier's law and we allow the "heights" of the energy wells to depend on temperature, which permits the introduction of latent and specific heats into the theory.

The use of high stiffness "at the wells" is not the only way to make dynamic phenomena in transforming materials accessible to simulation. Another method is to derive appropriate models of lower dimensional continua for these materials; this has been done recently by Purohit and Bhattacharya for strings [34] and beams [35] . These achieve simplification by reduction to a single spatial coordinate $x$, leading to partial differential equations in $(x, t)$.

The main appealing feature of PRBM and its generalizations is that, like rigid body mechanics, it is eminently useable. As indicated above, we hope it can be used to answer fundamental questions about the forms (or validity) of kinetic laws. For example, it is expected to be possible to calculate natural frequencies and damping vs. frequency of a shape memory cantilever containing a system of parallel interfaces, and to relate the behavior back to the form of the kinetic law. One could hope that its simplicity will allow the whole idea of the use of Eshelby-Abeyaratne-Knowles kinetics to be tested, though due respect has to be paid to its assumptions. Another appealing feature of the theory is that, except for the kinetic law, the shape of the body, and applied forces and fields, the input for the theory is only fundamental material constants, as listed below.

PRBM ... lattice parameters of (parent and) product phases, reference density;

PRMM ... lattice parameters of (parent and) product phases, reference density, easy axes of magnetization, saturation magnetization;

PRT ... lattice parameters of parent and product phases, reference density, thermal conductivity tensors of each phase, latent heat of transformation, specific heat of each phase. (2)

There are a number of interesting open questions posed by our theory. These mainly concern the precise formulation of the theory when complex microstructures are considered. Even in the case of a simple laminated microstructure, one could pose initial data modeling a sequence of finer and finer microstructures. Fine microstructures would actually be expected in some cases if the initial microstructure was created by the previous passage of an austenite/martensite interface through the body (Section 9.2). One could associate to this initial microstructure a Young measure of a special type (e.g., the weighted sum of two Dirac masses). Then, it would be interesting to know whether the dynamic equations posed here imply a dynamical system for the Young measure, e.g., whether the Young measure at any time is determined by its initial value. If not, which seems likely, it would be fascinating to understand what additional information about the microstructure is needed, or, what other mathematical objects associated with microstructure solve a dynamical system, that is, are determined by their own initial conditions. PRBM seems to be the simplest, realistic theory with which to study these phenomena.

Other open questions concern the treatment of the important case of noninvertible kinetic relations and the physically acceptable treatment of nucleation and annihilation. More generally, the development can be considered as a prototype for other "rigid body theories" in which mobile defects, treated with the kinetics based on ideas of Eshelby, are surrounded by stiff material.

While ordinary rigid body mechanics is applied routinely and with great success to the motion of large bodies like satellites and airplanes, the natural application area for PBRM is for MEMS (microelectromechanical systems) using transforming materials. Briefly, this is because at small scales, bodies of transforming material are expected to have only a few domains, and also because transforming materials (particularly hard ones) and dynamic phenomena are of particular interest. Also, shape memory materials having such multiwell energies exhibit the largest work output per cycle per volume of any actuator system (Krulevitch et al. [27]), and therefore are expected to be useful small-scale "motors." These issues are explained in more detail in [23] [25]. In fact PRBM arose as the generalization of a model for the motion of a MEMS cantilever made of ferromagnetic shape memory material [23]. The present version of the theory would be suitable for predicting the motion of the cantilever, detached from its base, so it could "swim" freely in, say, a Stokesian fluid, driven by a harmonically applied magnetic field.

## 2. Rigid Body Mechanics

Piecewise rigid body mechanics (PRBM) bears the same relation to the nonlinear elastic theory of phase transformations [6] as ordinary rigid body mechanics does to nonlinear elasticity theory. Therefore, to set the stage and notation for PRBM, and especially to motivate a strategy for proving that PRBM is well posed, we consider a formulation and the well-posedness of ordinary rigid body mechanics. Rigid body mechanics is based, at least intuitively, on the assumption that the body is sufficiently stiff that elastic deformations are negligible. To understand this better, consider the following formal argument. Consider a nonlinear elastic material governed by a frame-indifferent Helmholtz free-energy function $\varphi(\mathbf{F})$, defined for $3 \times 3$ matrices $\mathbf{F} \in \mathbb{R}^{3\times 3}$ with positive determinant. It

is crucial for ordinary rigid body mechanics that this underlying free energy has a single energy well which is conventionally taken to be at SO(3):

$$\begin{aligned} \varphi(\mathbf{F}) \ &\geq \ 0, \qquad \varphi(\mathbf{F}) = 0 \quad \text{if and only if} \\ \mathbf{F} \ &\in \ \mathrm{SO}(3) := \{\mathbf{Q} \in \mathbb{R}^{3\times 3} \colon \ \mathbf{Q}^T\mathbf{Q} = \mathbf{I}, \det \mathbf{Q} = +1\}. \end{aligned} \tag{3}$$

Let us assume for simplicity that

$$\varphi(\mathbf{F}) = \varphi_{\bar{\mu}}(\mathbf{F}) := \bar{\mu}\, \mathrm{dist}(\mathbf{F}, \mathrm{SO}(3))^2, \tag{4}$$

where dist(**F**,K) denotes the shortest Euclidean distance from the matrix $\mathbf{F} \in \mathbb{R}^{3\times 3}$ to the compact set $\mathrm{K} \subset \mathbb{R}^{3\times 3}$. In fact the right-hand side of (4) defines a frame-indifferent elastic material, and its linearized shear modulus is exactly $\bar{\mu}$. An open, bounded, connected reference configuration $\Omega \in \mathbb{R}^3$ is given together with a reference mass density $\rho_o \colon \Omega \to \mathbb{R}^+$. The dynamic initial value problem for the motion $\mathbf{y} \colon \Omega \times (0, T) \to \mathbb{R}^3$ is formally, in the absence of body forces,

$$\begin{aligned} \rho_o \mathbf{y}_{tt} \ &= \ \mathrm{div}\left(\frac{\partial \varphi_{\bar{\mu}}(\nabla \mathbf{y})}{\partial \mathbf{F}}\right), \\ \mathbf{y}(\mathbf{x}, 0) \ &= \ \mathbf{y}_o(\mathbf{x}), \\ \mathbf{y}_t(\mathbf{x}, 0) \ &= \ \mathbf{v}_o(\mathbf{x}), \quad \mathbf{x} \in \Omega; \end{aligned} \tag{5}$$

for smooth motions. If also there are no surface tractions applied to the body, then by the usual argument, (5) has an energy integral,

$$\int_\Omega \left\{ \frac{1}{2}\rho_o |\mathbf{y}_t(\mathbf{x}, t)|^2 + \varphi_{\bar{\mu}}(\nabla \mathbf{y}(\mathbf{x}, t)) \right\} d\mathbf{x} = const., \tag{6}$$

the constant being the value of the integral at $t = 0$. Since the system (5) is nonlinear hyperbolic, it is well known that the acceleration of a typical solution of (5) blows up in finite time and shocks form. This invalidates the passage from (5) to (6). Physically, the shocks dissipate energy and it is well accepted that, however we interpret a solution of (5), the energy integral (6) should hold with inequality:

$$\begin{aligned} \int_\Omega \left\{ \frac{1}{2}\rho_o |\mathbf{y}_t(\mathbf{x}, t)|^2 + \varphi_{\bar{\mu}}(\nabla \mathbf{y}(\mathbf{x}, t)) \right\} d\mathbf{x} \ &\leq \ const. \\ &= \ \int_\Omega \left\{ \frac{1}{2}\rho_o |\mathbf{v}_o(\mathbf{x})|^2 + \varphi_{\bar{\mu}}(\nabla \mathbf{y}_o(\mathbf{x})) \right\} d\mathbf{x}. \end{aligned} \tag{7}$$

To model a very stiff material, we do formal asymptotics by letting the shear modulus get large, $\bar{\mu} \to \infty$. To have a reasonable physical problem, we would then let the initial data (at least $\mathbf{y}_o$) also depend on $\bar{\mu}$ so as to preserve the initial total energy, which we label

$$\begin{aligned} (\mathrm{KE} + \mathrm{PE})|_{t=0} \ &= \ \int_\Omega \left\{ \frac{1}{2}\rho_o |\mathbf{v}_o^{(\bar{\mu})}(\mathbf{x})|^2 + \varphi_{\bar{\mu}}(\nabla \mathbf{y}_o^{(\bar{\mu})}(\mathbf{x})) \right\} d\mathbf{x} \\ &= \ const. \text{ (independent of } \bar{\mu}). \end{aligned} \tag{8}$$

The corresponding sequence of solutions $\mathbf{y}^{(\bar{\mu})}(\mathbf{x}, t)$ then satisfies

$$\int_\Omega \left\{ \frac{1}{2}\rho_o |\mathbf{y}_t^{(\bar{\mu})}(\mathbf{x}, t)|^2 + \varphi_{\bar{\mu}}(\nabla \mathbf{y}^{(\bar{\mu})}(\mathbf{x}, t)) \right\} d\mathbf{x} \leq const., \tag{9}$$

from which we deduce that for each $t \in (0, T)$,

$$\int_\Omega \operatorname{dist}(\nabla \mathbf{y}^{(\bar{\mu})}(\mathbf{x}, t), \mathrm{SO}(3))^2 \, d\mathbf{x} \to 0, \quad \text{as } \bar{\mu} \to \infty. \tag{10}$$

It is known (e.g., [6]) that, not only does this imply the strong convergence of $\nabla \mathbf{y}^{(\bar{\mu})}(\mathbf{x}, t)$ to the set SO(3), but in fact it implies that $\nabla \mathbf{y}^{(\bar{\mu})}(\mathbf{x}, t)$ converges strongly (i.e., pointwise) to a particular matrix $\mathbf{R}(t) \in \mathrm{SO}(3)$, i.e.,

$$\mathbf{y}^{(\bar{\mu})}(\mathbf{x}, t) \to \mathbf{R}(t)\mathbf{x} + \mathbf{c}(t) \quad \text{(strongly in } W^{1,2}(\Omega)), \tag{11}$$

for each fixed $t > 0$. The right-hand side is the starting kinematic assumption of rigid body mechanics. This convergence is strong enough to pass to the limit in the balances of linear and rotational momentum to get the standard equations of rigid body mechanics. The dimensionless parameter that emerges from this calculation is obtained by dividing through (8)–(9) by $(\mathrm{KE} + \mathrm{PE})|_{t=0}/\operatorname{vol}(\Omega)$:

$$\frac{\bar{\mu} \operatorname{vol} \Omega}{(\mathrm{KE} + \mathrm{PE})|_{t=0}}. \tag{12}$$

When the shear modulus is much larger than the initial total energy per unit volume and no forces are applied, one expects that rigid body mechanics is valid. This argument has many variants—for example, there is a corresponding argument in the adiabatic case with the Helmholtz free energy replaced by the internal energy—but they lead to similar conclusions. If applied loads were allowed, then these would contribute power terms to the total energy which would introduce more dimensionless numbers whose asymptotics would have to be respected. Nevertheless, even in its most hopeful form this argument has important implications for PRBM that we discuss below.

Now we return to a formulation and well-posedness of ordinary rigid body mechanics. Consider a rigid motion,

$$\mathbf{y}(\mathbf{x}, t) = \mathbf{R}(t)\mathbf{x} + \mathbf{c}(t), \tag{13}$$

where $\mathbf{R}$: $(0, \infty) \to \mathrm{SO}(3)$ and $\mathbf{c}$: $(0, \infty) \to \mathbb{R}^3$. Let $\Omega$ and $\rho_o$ be as above and define the following:

$$\begin{aligned} M &:= \int_\Omega \rho_o \, d\mathbf{x}, && \text{(total mass)}, \\ \mathbf{x}_c &:= \frac{1}{M} \int_\Omega \rho_o \mathbf{x} \, d\mathbf{x}, && \text{(center of mass of } \Omega), \\ \mathbf{y}_c(t) &:= \mathbf{y}(\mathbf{x}_c, t), && \text{(center of mass of } \mathbf{y}(\Omega, t)). \end{aligned} \tag{14}$$

It is useful to eliminate $\mathbf{c}(t)$ using these definitions to get the equivalent expression,

$$\mathbf{y}(\mathbf{x}, t) = \mathbf{R}(t)(\mathbf{x} - \mathbf{x}_c) + \mathbf{y}_c(t). \tag{15}$$

Let $\mathcal{F}$ denote the total applied force and $\mathcal{T}_o$ be the total applied torque calculated about $\mathbf{y}_o \in \mathbb{R}^3$. Assuming that (15) is expressed in an inertial frame, the balance of linear momentum is

$$\frac{d}{dt}\int_\Omega \rho_o \dot{\mathbf{y}}\, d\mathbf{x} = \mathcal{F}, \tag{16}$$

and the balance of rotational momentum is

$$\frac{d}{dt}\int_\Omega \rho_o (\mathbf{y}-\mathbf{y}_o)\wedge \dot{\mathbf{y}}\, d\mathbf{x} = \mathcal{T}_o. \tag{17}$$

In the usual way, substitution of (15) into the balance of linear momentum and use of the definitions (14) yields Newton's second law for the center of mass:

$$M\ddot{\mathbf{y}}_c = \mathcal{F}. \tag{18}$$

It remains to reformulate the balance of rotational momentum in a way that illuminates well-posedness. If $\mathbf{R}(t)$ is differentiable, then (by differentiation of the relation $\mathbf{R}^T\mathbf{R} = \mathbf{I}$),

$$\dot{\mathbf{R}} = \mathbf{R}\mathbf{W}, \qquad \mathbf{W}^T = -\mathbf{W}, \tag{19}$$

where the tensor $\mathbf{W}$ is skew. It has an associated axial vector $\boldsymbol{\omega} \in \mathbb{R}^3$ defined uniquely by

$$\mathbf{W}\mathbf{a} = \boldsymbol{\omega}\wedge\mathbf{a}, \quad \text{for all } \mathbf{a}\in\mathbb{R}^3. \tag{20}$$

The vector $\boldsymbol{\omega} \in \mathbb{R}^3$ might be termed the referential angular velocity; the conventional angular velocity is $\mathbf{R}\boldsymbol{\omega}$. We also recall the identities $\mathbf{R}(\mathbf{a}\wedge\mathbf{b}) = \mathbf{R}\mathbf{a}\wedge\mathbf{R}\mathbf{b}$ and $\mathbf{a}\wedge(\boldsymbol{\omega}\wedge\mathbf{b}) = [(\mathbf{a}\cdot\mathbf{b})\mathbf{I} - \mathbf{b}\otimes\mathbf{a}]\boldsymbol{\omega}$, which hold for all $\mathbf{R}\in\mathrm{SO}(3)$, $\mathbf{a}, \mathbf{b}, \boldsymbol{\omega}\in\mathbb{R}^3$. Substitute (15) into the balance of rotational momentum (17) and use the definition of the center of mass in $\Omega$:

$$\frac{d}{dt}\left[\mathbf{R}(t)\left\{\int_\Omega (\mathbf{x}-\mathbf{x}_c)\cdot(\mathbf{x}-\mathbf{x}_c)\mathbf{I} - (\mathbf{x}-\mathbf{x}_c)\otimes(\mathbf{x}-\mathbf{x}_c)\, d\mathbf{x}\right\}\boldsymbol{\omega} + M(\mathbf{y}_c(t)-\mathbf{y}_o)\wedge\dot{\mathbf{y}}_c(t)\right] = \mathcal{T}_o. \tag{21}$$

Define the inertia tensor of $\Omega$ by

$$\mathcal{I} = \int_\Omega \{(\mathbf{x}-\mathbf{x}_c)\cdot(\mathbf{x}-\mathbf{x}_c)\mathbf{I} - (\mathbf{x}-\mathbf{x}_c)\otimes(\mathbf{x}-\mathbf{x}_c)\}\, d\mathbf{x}. \tag{22}$$

Since $\Omega$ is given, we assume $\mathcal{I}$ is known. Carrying out the differentiation in (21) and substituting from the balance of linear momentum, we get the useful form

$$\mathcal{I}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega}\wedge\mathcal{I}\boldsymbol{\omega} + \mathbf{R}^T((\mathbf{y}_c-\mathbf{y}_o)\wedge\mathcal{F}) = \mathbf{R}^T\mathcal{T}_o. \tag{23}$$

This equation links $\mathbf{R}$ and $\boldsymbol{\omega}$ which are also linked by the kinematic relation (19); to account for this fact, it is useful (when considering well-posedness) to treat $\mathbf{R}$ and $\boldsymbol{\omega}$ as

independent functions subject to (19) and (20). Summarizing, the balances of linear and rotational momentum for a rigid motion are equivalent to the system of equations

$$\begin{aligned} \dot{\mathbf{y}}_c &= \mathbf{v}_c, \\ M\dot{\mathbf{v}}_c &= \mathcal{F}, \\ \mathcal{I}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \wedge \mathcal{I}\boldsymbol{\omega} + \mathbf{R}^T((\mathbf{y}_c - \mathbf{y}_o) \wedge \mathcal{F}) &= \mathbf{R}^T\mathcal{T}_o, \\ \dot{\mathbf{R}} &= \mathbf{RW}, \quad \mathbf{W} = \boldsymbol{\omega} \wedge . \end{aligned} \tag{24}$$

Here, we have broken down (18) into two first-order equations. These are to be augmented by the initial conditions for the position and velocity of the center of mass, the referential angular velocity, and the initial rotation:

$$\begin{aligned} \mathbf{y}_c(0) &= \mathbf{y}_c^o, \\ \mathbf{v}_c(0) &= \mathbf{v}_c^o, \\ \boldsymbol{\omega}(0) &= \boldsymbol{\omega}_o, \\ \mathbf{R}(0) &= \mathbf{R}_o \in \mathrm{SO}(3). \end{aligned} \tag{25}$$

Let us assume that the force $\mathcal{F}$ and torque $\mathcal{T}_o$ are assigned continuous, bounded functions of $(\mathbf{y}, \mathbf{v}, \boldsymbol{\omega}, \mathbf{R}, t)$ defined on all of $\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^{3\times 3} \times [0, \infty)$, satisfying a Lipschitz condition in $(\mathbf{y}, \mathbf{v}, \boldsymbol{\omega}, \mathbf{R})$. Then, observing that the inertia tensor is invertible under our assumptions on $\Omega$, we see that the initial value problem is in the standard form $\dot{\boldsymbol{\Theta}} = f(\boldsymbol{\Theta}, t)$, $\boldsymbol{\Theta}(0) = \boldsymbol{\Theta}_o$, with

$$\boldsymbol{\Theta} = (\mathbf{y}_c, \mathbf{v}_c, \boldsymbol{\omega}, \mathbf{R}) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^{3\times 3}. \tag{26}$$

The standard existence-uniqueness-extension theorem for ODEs (e.g., Hartman [18]) then states that the initial value problem (24)–(25) has a unique continuously differentiable solution for all $t \geq 0$. The only issue that remains is the status of $\mathbf{R}(t) \in \mathbb{R}^{3\times 3}$ delivered by this theorem. Using $(24)_4$ and $(25)_4$, we have

$$\begin{aligned} \frac{d}{dt}(\mathbf{RR}^T) &= \mathbf{RWR}^T + \mathbf{RW}^T\mathbf{R}^T = 0, \\ (\mathbf{RR}^T)(0) &= \mathbf{I}, \end{aligned} \tag{27}$$

which shows (also using continuity, or Liouville's theorem for $\det \mathbf{R}$) that $\mathbf{R}(t) \in \mathrm{SO}(3)$, $t \geq 0$. As is well known by the experts, the above is not the basis for computing solutions numerically. Naive time-discretization of $(24)_4$ leads to departures from the strictly convex manifold SO(3), which is easily remedied by using manifold coordinates, like Euler angles or quarternions. If the force and torque also satisfy certain rather strong conditions, the system (24) is Hamiltonian, in which case it is desirable to use a numerical method that exactly preserves the Hamiltonian structure at the discrete level, or else preserves one or more conserved quantities, if they occur. These issues are discussed for example by Kane, Marsden, and Ortiz [26], and related numerical issues arise[1] in PRBM, but we do not address them in this paper.

[1] More specifically, the system of equations of PRBM has conservation laws, but it is not Hamiltonian.

## 3. Piecewise Rigid Body Mechanics

### *3.1. Free Energy of Materials That Undergo Structural Phase Transformations*

The simplest geometrically exact free-energy for a material that undergoes a structural phase transformation has the form $\varphi(\mathbf{F}, \theta)$, where $\mathbf{F} \in \mathbb{R}^{3\times 3}$, $\det \mathbf{F} > 0$ is the deformation gradient and $\theta > 0$ is the absolute temperature. As above, $\varphi$ is frame-indifferent:

$$\varphi(\mathbf{RF}, \theta) = \varphi(\mathbf{F}, \theta), \quad \text{for all } \mathbf{R} \in \text{SO(3)}, \tag{28}$$

and all $\mathbf{F} \in \{\mathbb{R}^{3\times 3}\colon \det \mathbf{F} > 0\}$. The key feature of $\varphi$ that sets it apart from free energies of ordinary nonlinear elastic materials is that $\varphi$ has energy wells.

A theory for the structure of the energy wells for martensitic materials that undergo first-order transformations is available [6]. It rests on an important crystallographic construction of Ericksen [12] and Pitteri [32]. The main results from that theory are the following. There are two point groups $\mathcal{P}_m \subset \mathcal{P}_a$ associated, respectively, with (low temperature) martensite and (high temperature) austenite phases and a set of positive-definite, symmetric $3 \times 3$ matrices, $\mathbf{U}_1, \ldots, \mathbf{U}_n$ where $n = \frac{\text{order of } \mathcal{P}_a}{\text{order of } \mathcal{P}_m}$. The matrices $\mathbf{U}_1, \ldots, \mathbf{U}_n$ satisfy

$$\begin{aligned} \mathbf{QU}_1\mathbf{Q}^T &= \mathbf{U}_1, \qquad\qquad \text{for all } \mathbf{Q} \in \mathcal{P}_m, \\ \{\mathbf{U}_1, \ldots, \mathbf{U}_n\} &= \{\mathbf{QU}_1\mathbf{Q}^T\colon \mathbf{Q} \in \mathcal{P}_a \backslash \mathcal{P}_m\}. \end{aligned} \tag{29}$$

Given the two groups, (29) is a strong restriction of the forms of these matrices; various special cases have been worked out (Ball and James [6], Bělík and Luskin [7]) and a comprehensive study for all pairs of groups will appear in a forthcoming book (Pitteri and Zanzotto [33]). Finally there is assumed another positive-definite symmetric matrix $\mathbf{U}_a$ associated with the austenite phase. Though it is not reflected by the notation, all of the matrices $\mathbf{U}_1, \ldots, \mathbf{U}_n$ and $\mathbf{U}_a$ depend weakly on temperature $\theta$ due to ordinary thermal expansion, and in all cases $\mathbf{U}_a|_{\theta=\theta_c} = \mathbf{I}$, where $\theta_c$ is the transformation temperature. The key assumption about $\varphi$ is that it is minimized at $\mathbf{U}_a$ for $\theta > \theta_c$, and it is equi-minimized at $\mathbf{U}_1, \ldots, \mathbf{U}_n$ for $\theta < \theta_c$. Accounting for the fact that $\varphi$ is also frame-indifferent, we shall assume that $\varphi$ is smooth and all minimizers of $\varphi$ are given as follows:

$$\begin{aligned} &\text{For } \theta > \theta_c, \quad \varphi(\cdot, \theta) \text{ is minimized on SO(3)}\mathbf{U}_a, \\ &\text{At } \theta = \theta_c, \quad \varphi(\cdot, \theta) \text{ is minimized on SO(3)}\mathbf{I} \cup \text{SO(3)}\mathbf{U}_1 \cup \cdots \cup \text{SO(3)}\mathbf{U}_n, \\ &\text{For } \theta < \theta_c, \quad \varphi(\cdot, \theta) \text{ is minimized on SO(3)}\mathbf{U}_1 \cup \cdots \cup \text{SO(3)}\mathbf{U}_n. \end{aligned} \tag{30}$$

Here we have used the notation $\text{SO(3)}\mathbf{A} = \{\mathbf{RA}\colon \mathbf{R} \in \text{SO(3)}\}$. A classical special case is the cubic to tetragonal transformation ($n = 24/8 = 3$), with

$$\begin{aligned} \mathbf{U}_1 &= \begin{pmatrix} \eta_2 & 0 & 0 \\ 0 & \eta_1 & 0 \\ 0 & 0 & \eta_1 \end{pmatrix}, \qquad \mathbf{U}_2 = \begin{pmatrix} \eta_1 & 0 & 0 \\ 0 & \eta_2 & 0 \\ 0 & 0 & \eta_1 \end{pmatrix}, \\ \mathbf{U}_3 &= \begin{pmatrix} \eta_1 & 0 & 0 \\ 0 & \eta_1 & 0 \\ 0 & 0 & \eta_2 \end{pmatrix}. \end{aligned} \tag{31}$$

This example will be studied below in the case for (the thermal martensite in) the ferromagnetic shape memory material $Ni_2MnGa$, which has measured values $\eta_1 = 1.013$, $\eta_2 = 0.952$ at $\theta = \theta_c$ ([38]).

### *3.2. Motivation and Formal Statement of the Theory*

In piecewise rigid body mechanics (i.e., the purely mechanical version of the theory), we neglect the temperature dependence of the free-energy density. That is, we neglect the small temperature dependence of the matrices $\mathbf{U}_1, \ldots, \mathbf{U}_n$ or $\mathbf{U}_a$, and we assume that we are sufficiently below or above the transformation temperature that no phase transformation takes place.[2] The interesting case is below the phase transformation temperature where there exist multiple wells, in which case the theory governs the rearrangement of martensite variants. We assume this for definiteness in the formulation below. The motivation for PRBM is similar to that of rigid body mechanics. Consider, for example, the Cu-14.0 wt.%Al-3.9 wt.%Ni alloy mentioned in the introduction, for which a great deal is known. This undergoes a cubic to orthorhombic transformation ($n = 24/4 = 6$) with six variants of martensite. The structure of the matrices is

$$\mathbf{U}_1 = \begin{pmatrix} \frac{\alpha+\gamma}{2} & \frac{\alpha-\gamma}{2} & 0 \\ \frac{\alpha-\gamma}{2} & \frac{\alpha+\gamma}{2} & 0 \\ 0 & 0 & \beta \end{pmatrix}, \qquad \mathbf{U}_2 = \begin{pmatrix} \frac{\alpha+\gamma}{2} & \frac{\gamma-\alpha}{2} & 0 \\ \frac{\gamma-\alpha}{2} & \frac{\alpha+\gamma}{2} & 0 \\ 0 & 0 & \beta \end{pmatrix},$$

$$\mathbf{U}_3 = \begin{pmatrix} \frac{\alpha+\gamma}{2} & 0 & \frac{\alpha-\gamma}{2} \\ 0 & \beta & 0 \\ \frac{\alpha-\gamma}{2} & 0 & \frac{\alpha+\gamma}{2} \end{pmatrix}, \qquad \mathbf{U}_4 = \begin{pmatrix} \frac{\alpha+\gamma}{2} & 0 & \frac{\gamma-\alpha}{2} \\ 0 & \beta & 0 \\ \frac{\gamma-\alpha}{2} & 0 & \frac{\alpha+\gamma}{2} \end{pmatrix},$$

$$\mathbf{U}_5 = \begin{pmatrix} \beta & 0 & 0 \\ 0 & \frac{\alpha+\gamma}{2} & \frac{\alpha-\gamma}{2} \\ 0 & \frac{\alpha-\gamma}{2} & \frac{\alpha+\gamma}{2} \end{pmatrix}, \qquad \mathbf{U}_6 = \begin{pmatrix} \beta & 0 & 0 \\ 0 & \frac{\alpha+\gamma}{2} & \frac{\gamma-\alpha}{2} \\ 0 & \frac{\gamma-\alpha}{2} & \frac{\alpha+\gamma}{2} \end{pmatrix}, \tag{32}$$

with the measured values $\alpha = 1.0619$, $\beta = 0.9178$, and $\gamma = 1.0230$ (Otsuka and Shimizu [31]). A simple calculation shows that the minimum Euclidean distance between wells[3] is

$$\min_{i,j}\{\text{dist}(\text{SO}(3)\mathbf{U}_i, \text{SO}(3)\mathbf{U}_j)\} = 0.055. \tag{33}$$

All of the linearly elastic moduli of the orthorhombic phase of this alloy have been measured (Yasunaga et al. [42] [43]). Near transformation temperature, where they are

[2] Because of the release of latent heat upon transformation, and its important side effects, the case of phase transformation is more reasonably treated under piecewise rigid thermomechanics in Section 9.2.

[3] The distance between each pair of wells is either 0.055 or 0.181, corresponding to approximately 5.5% and 18% strain. The smaller strain corresponds to wells related as in $\mathbf{U}_1$ and $\mathbf{U}_2$.

the softest,[4] they are (Voigt notation, GPa):

$$\begin{aligned} C_{11} &= 189, \qquad C_{22} = 141, \qquad C_{33} = 205, \qquad C_{44} = 54.9, \\ C_{55} &= 19.7, \qquad C_{66} = 62.6, \\ C_{12} &= 124, \qquad C_{13} = 45.5, \qquad C_{23} = 115. \end{aligned} \tag{34}$$

Using this data, we find that the minimum eigenvalue of the linear elasticity tensor is 14.6 GPa. Also, from work of Shield [36] we know that this alloy has little ductility and fracture occurs at stresses of the order of several hundred MPa in specimens that have been carefully prepared to avoid flaws (there is a strong orientation dependence). For certain oriented crystals, transformation to a monoclinic phase occurs at similar stresses. Even in these somewhat extreme situations, the maximum deviation from the energy wells over most of the body is expected to be less than 100 MPa/15 GPa $= 6.6 \times 10^{-3}$. Hence, in rather extreme situations,

$$\frac{\text{maximum expected deviation from the wells}}{\text{minimum distance between wells}} < \frac{0.0066}{0.055} \approx 0.1. \tag{35}$$

In cases in which these materials are used as actuators, this ratio is expected to be much lower. For example, in the ferromagnetic shape memory material $Ni_2MnGa$, the blocking stress (i.e., the stress at which the field-induced strain goes to zero) is about 8MPa and the typical operating range for $Ni_2MnGa$ actuators is 1–2 MPa. Motivated by the asymptotic argument that gave rigid body mechanics, we summarize piecewise rigid body mechanics schematically as follows:

I. Kinematics $\nabla \mathbf{y}(\mathbf{x}, t) \in \mathrm{SO}(3)\mathbf{U}_1 \cup \cdots \cup \mathrm{SO}(3)\mathbf{U}_n$,
II. Dynamics
   a. Balance of linear momentum

$$\frac{d}{dt}\int_{\mathcal{P}} \rho_o \dot{\mathbf{y}}\, d\mathbf{x} = \mathcal{F}(\mathcal{P}) \quad \text{for all smooth } \mathcal{P} \subset \Omega,$$

   b. Balance of rotational momentum

$$\frac{d}{dt}\int_{\mathcal{P}} \rho_o (\mathbf{y} - \mathbf{y}_o) \wedge \dot{\mathbf{y}}\, d\mathbf{x} = \mathcal{T}_o(\mathcal{P}) \quad \text{for all smooth } \mathcal{P} \subset \Omega. \tag{36}$$

The main reason that these assumptions are schematic is that the specification of forces $\mathcal{F}(\mathcal{P})$ and torques $\mathcal{T}_o(\mathcal{P})$ is left vague. In classical rigid body mechanics the Cauchy stress $\sigma(\mathbf{x}, t)$ is assumed to be an arbitrary symmetric $3 \times 3$ matrix-valued function, the "constraint stress," available as part of the solution. In this case, one contribution to the forces and moments on $\mathcal{P}$ is then, explicitly,

$$\begin{aligned} \mathcal{F}(\mathcal{P}) &= \int_{\partial \mathbf{y}(\mathcal{P},t)} \sigma \hat{\mathbf{n}}\, dA + \text{body forces}, \\ \mathcal{T}_o(\mathcal{P}) &= \int_{\partial \mathbf{y}(\mathcal{P},t)} (\mathbf{y} - \mathbf{y}_o) \wedge \sigma \hat{\mathbf{n}}\, dA + \text{body torques}. \end{aligned} \tag{37}$$

[4] This is true among those whose temperature dependence has been measured, and it is expected to be generally true.

A classical formal argument in turn shows that (using the arbitrariness of $\sigma$) the balances of linear and rotational momentum[5] place no restriction on the motion except when $\mathcal{P} = \Omega$, which gives the classical equations discussed in Section 2. In piecewise rigid body mechanics we also acknowledge the existence of the arbitrary symmetric constraint stress $\boldsymbol{\sigma}$ on, say, open regions where $\nabla \mathbf{y}$ belongs to a single well. However, as we shall illustrate below in special cases, this will only have the effect of eliminating the local balance rotational momentum; the balance of linear momentum will give an important restriction on the motion that arises at interfaces. These considerations granted, it will be clear below that the balances of linear and rotational momentum are not sufficient to determine the motion. The nonuniqueness is associated with the motion of interfaces across which the deformation gradient jumps and which contribute to energy dissipation: Briefly, these interfaces are "defects" in the sense of Eshelby [15]. The typical resolution is to write a kinetic law that relates the velocity of the interface to the configurational force on it. Thus, schematically, piecewise rigid body mechanics is completed by prescribing

$$\text{III. Kinetics:} \quad \text{interfacial velocity} = f(\text{configurational force}). \tag{38}$$

We note that in martensitic materials that undergo a reversible phase transformation, e.g., typical shape memory materials, it is well accepted that most of the dissipation of energy occurs at interfaces (and arises from interfacial motion), consistent with the above formulation. We now work toward making the kinetic law precise. For a perspective on kinetic laws that is intended to fit better with the overall structure of continuum mechanics, see Gurtin [17].

## 4. Rank-One Connections between Energy Wells

For typical choices of point groups $\mathcal{P}_m \subset \mathcal{P}_a$, the kinematic assumption $\nabla \mathbf{y}(\mathbf{x}, t) \in \mathrm{SO}(3)\mathbf{U}_1 \cup \cdots \cup \mathrm{SO}(3)\mathbf{U}_n$ permits continuous deformations with gradient supported on two or more wells, this being the case of interest. We briefly summarize the situation here. Consider two wells, which for definiteness we take to be $\mathrm{SO}(3)\mathbf{U}_1 \cup \mathrm{SO}(3)\mathbf{U}_2$. The classical Hadamard jump condition for a continuous deformation $\mathbf{y}$ whose gradient suffers a simple jump discontinuity across an interface in $\Omega$ with normal $\mathbf{n} \in \mathbb{R}^3$ is

$$(\nabla \mathbf{y})^+ - (\nabla \mathbf{y})^- = \mathbf{a} \otimes \mathbf{n}, \quad \mathbf{a} \in \mathbb{R}^3. \tag{39}$$

Here, the superscripts $+$ and $-$ refer to limiting values from opposite sides of the interface (i.e., if the interface $\mathcal{S}$ is designated as the level set $f = 0$ of a smooth function $f\colon \Omega \times (0, T) \to \mathbb{R}$, $|\nabla f| > 0$, then $+$ or $-$ refers to limiting values from the regions $\{f > \text{or} < 0\} \cap \Omega$, respectively). Thus, the condition of compatibility for continuous deformations that assume gradients on the two wells is

$$\mathbf{R}_2\mathbf{U}_2 - \mathbf{R}_1\mathbf{U}_1 = \hat{\mathbf{a}} \otimes \mathbf{n}, \tag{40}$$

[5] This may not be instantly obvious for the second equation. The key point is the symmetry of $\sigma$.

which is to be solved for $\mathbf{R}_{1,2} \in \mathrm{SO}(3)$, $\hat{\mathbf{a}}, \mathbf{n} \in \mathbb{R}^3$. Premultiply (40) by $\mathbf{R}_1^T$ and let $\hat{\mathbf{R}} = \mathbf{R}_1^T \mathbf{R}_2$, $\mathbf{a} = \mathbf{R}_1^T \hat{\mathbf{a}}$; (40) then becomes

$$\hat{\mathbf{R}}\mathbf{U}_2 - \mathbf{U}_1 = \mathbf{a} \otimes \mathbf{n}. \tag{41}$$

It is known that, given the restriction $\det \mathbf{U}_1 = \det \mathbf{U}_2$ (which is true by (29)) on the positive-definite symmetric matrices $\mathbf{U}_1$ and $\mathbf{U}_2$, a necessary and sufficient condition that (41) has a solution ($\hat{\mathbf{R}} \in \mathrm{SO}(3), \mathbf{a}, \mathbf{n} \in \mathbb{R}^3$) is that $\det(\mathbf{U}_1^2 - \mathbf{U}_2^2) = 0$ (Ericksen [13]). In that case there are two solutions, $(\mathbf{R}^{\mathrm{I}}, \mathbf{a}^{\mathrm{I}} \otimes \mathbf{n}^{\mathrm{I}})$ and $(\mathbf{R}^{\mathrm{II}}, \mathbf{a}^{\mathrm{II}} \otimes \mathbf{n}^{\mathrm{II}})$. Formulas for these solutions are given, for example, in [24], where further interpretative information can be found. In the common case that there is a $180^\circ$ rotation matrix $\mathbf{Q} \in \mathcal{P}_a$ (cf., (29)) such that $\mathbf{U}_2 = \mathbf{Q}\mathbf{U}_1\mathbf{Q}^T$, these two solutions can be interpreted as so-called Type I/Type II twins. In the special case that $\mathbf{U}_1$ and $\mathbf{U}_2$ are given by (31), the solutions are

$$\begin{aligned} \mathbf{a}^{\mathrm{I}} \otimes \mathbf{n}^{\mathrm{I}} &= \frac{\eta_1^2 - \eta_2^2}{\eta_1^2 + \eta_2^2}(\eta_2, -\eta_1, 0) \otimes (1, 1, 0) \qquad \text{and} \\ \mathbf{a}^{\mathrm{II}} \otimes \mathbf{n}^{\mathrm{II}} &= \frac{\eta_1^2 - \eta_2^2}{\eta_1^2 + \eta_2^2}(\eta_2, \eta_1, 0) \otimes (1, -1, 0). \end{aligned} \tag{42}$$

(Here, $\mathbf{R}^{\mathrm{I}}$ and $\mathbf{R}^{\mathrm{II}}$ follow from (41)). Note that in the general case there are only two reference normals $\mathbf{n}^{\mathrm{I}}$ and $\mathbf{n}^{\mathrm{II}}$. Thus, it may seem problematic to construct anything beyond a simple layered structure using these solutions. In fact, if $\mathbf{U}_1$ and $\mathbf{U}_2$ satisfy certain additional conditions (Bhattacharya [8], James and Kinderlehrer [21]), then a four-fold intersection of the two variants is possible, using these two normals. More generally, recent work of Müller and Šverák [28] [29] shows that there are generically exceedingly complicated solutions of $\nabla \mathbf{y}(\mathbf{x}, t) \in \mathrm{SO}(3)\mathbf{U}_1 \cup \cdots \cup \mathrm{SO}(3)\mathbf{U}_n$ which are not obtained in the obvious way by assembling the simple rank-one connections found above. This again suggests the need for a precise, general statement of the theory.

## 5. Dissipation in Piecewise Rigid Bodies

In preparation for the formulation of the kinetic laws we consider the simplest possible situation of a single interface separating regions where $\nabla \mathbf{y}$ assumes two values $\mathbf{R}_1(t)\mathbf{U}_1$ and $\mathbf{R}_2(t)\mathbf{U}_2$, satisfying the compatibility condition (40). A general expression for such a deformation can be written in the following way. Let $s\colon (0, \infty) \to \mathbb{R}$ be smooth and define a characteristic function $\chi\colon \mathbb{R} \times (0, \infty) \to \{0, 1\}$ by

$$\chi(\xi, t) = \begin{cases} 0 & : \quad \xi < s(t), \\ 1 & : \quad \xi \geq s(t). \end{cases} \tag{43}$$

Also, let $\mathbf{R}\colon (0, \infty) \to \mathrm{SO}(3)$, $\mathbf{c}\colon (0, \infty) \to \mathbb{R}^3$, and let $\mathbf{a}, \mathbf{n} \in \mathbb{R}^3$ solve (41). Define a motion

$$\mathbf{y}(\mathbf{x}, t) = \mathbf{R}(t)\left(\mathbf{U}_1\mathbf{x} + \mathbf{a}\int_0^{\mathbf{x}\cdot\mathbf{n}} \chi(s, t)\, ds\right) + \mathbf{c}(t). \tag{44}$$

By differentiating this expression with respect to $\mathbf{x}$, one sees that it satisfies the hypotheses and it is also globally invertible. If we omit the assumptions of smoothness in time of $\mathbf{R}(t)$ and $s(t)$, this is the most general expression for a motion with an isolated interface, whose gradient assumes two values [5]. The referential normal velocity of propagation of the interface is $\dot{s}(t)$. Let the Piola-Kirchhoff (constraint) stress $\mathbf{T}$: $\Omega \times (0, \infty) \to \mathbb{R}^{3\times 3}$ be defined in terms of the symmetric Cauchy stress (37) by

$$\mathbf{T} = (\det \nabla \mathbf{y})\sigma(\nabla \mathbf{y})^{-T}. \tag{45}$$

Let us omit body forces (for simplicity) and write the balance of linear momentum for any open subregion $\mathcal{P} \subset \Omega$ (cf., (16), (37)):

$$\frac{d}{dt}\int_{\mathcal{P}} \rho_o \dot{\mathbf{y}}\, d\mathbf{x} = \int_{\partial \mathbf{y}(\mathcal{P},t)} \sigma \hat{\mathbf{n}}\, dA = \int_{\partial \mathcal{P}} \mathbf{T}\hat{\mathbf{n}}\, dA. \tag{46}$$

Here, the Piola-Kirchhoff stress $\mathbf{T}(\mathbf{x}, t)$ is assumed to be smooth separately on the two regions $\Omega \cap \{\mathbf{x}\cdot\mathbf{n} > s(t)\}$ and $\Omega \cap \{\mathbf{x}\cdot\mathbf{n} < s(t)\}$ with continuous extensions to the closures of these regions. Let $\Omega_t^+ = \Omega \cap \{\mathbf{x}\cdot\mathbf{n} > s(t)\}$, $\Omega_t^- = \Omega \cap \{\mathbf{x}\cdot\mathbf{n} < s(t)\}$, and $\mathcal{S}_t = \Omega \cap \{\mathbf{x}\cdot\mathbf{n} = s(t)\}$, and use the notation for the jump and average, $[\![a]\!] = a^+ - a^-$ and $\langle a\rangle = \frac{1}{2}(a^+ + a^-)$, based on the limiting values on $\mathcal{S}_t$ from from $\Omega_t^+$ and $\Omega_t^-$. Localizing the balance of linear momentum using the arbitrariness of $\mathcal{P}$, we get

$$\rho_o \ddot{\mathbf{y}} = \mathrm{div}_{\mathbf{x}}\, \mathbf{T} \quad \text{on } \Omega_t^+ \cup \Omega_t^-, \tag{47}$$

and we also get the jump condition,

$$-\rho_o [\![\dot{\mathbf{y}}]\!]\dot{s} = [\![\mathbf{t}]\!] \quad \text{on } \mathcal{S}_t, \tag{48}$$

where $\mathbf{t} = \mathbf{T}\mathbf{n}$ is the Piola-Kirchhoff traction.

The *dissipation* $\mathcal{D}$ is defined to be the rate of change of the total energy, which in this case is only the kinetic energy, minus the rate of work done at the boundary:

$$\mathcal{D} = \frac{d}{dt}\int_{\Omega} \frac{1}{2}\rho_o |\dot{\mathbf{y}}|^2\, d\mathbf{x} - \int_{\partial\Omega} \dot{\mathbf{y}}\cdot \mathbf{T}\hat{\mathbf{n}}\, da. \tag{49}$$

A straightforward calculation using Reynold's transport theorem and (47) and (48) yields

$$\begin{aligned}
\mathcal{D} &= \frac{d}{dt}\left\{\int_{\Omega_t^+} \frac{\rho_o}{2}|\dot{\mathbf{y}}|^2\, d\mathbf{x} + \int_{\Omega_t^-} \frac{\rho_o}{2}|\dot{\mathbf{y}}|^2\, d\mathbf{x}\right\} - \int_{\partial\Omega} \dot{\mathbf{y}}\cdot\mathbf{T}\hat{\mathbf{n}}\, da,\\
&= \int_{\Omega_t^+} \rho_o \dot{\mathbf{y}}\cdot\ddot{\mathbf{y}}\, d\mathbf{x} + \int_{\Omega_t^-} \rho_o \dot{\mathbf{y}}\cdot\ddot{\mathbf{y}}\, d\mathbf{x} - \int_{\partial\Omega_t^+} \dot{\mathbf{y}}\cdot\mathbf{T}\hat{\mathbf{n}}\, da\\
&\quad + \int_{\partial\Omega_t^-} \dot{\mathbf{y}}\cdot\mathbf{T}\hat{\mathbf{n}}\, da - \int_{\mathcal{S}_t} [\![\dot{\mathbf{y}}\cdot\mathbf{T}\mathbf{n}]\!]\, da,\\
&= -\left\{\int_{\mathcal{S}_t} \frac{\rho_o}{2}[\![|\dot{\mathbf{y}}|^2]\!]\dot{s} + [\![\dot{\mathbf{y}}\cdot\mathbf{t}]\!]\, da\right\},
\end{aligned} \tag{50}$$

which shows that in a piecewise rigid body, all dissipation occurs at the interface. In (50) we have used that the stress-power vanishes,

$$\int_\Omega \mathbf{T} \cdot \nabla \dot{\mathbf{y}} \, d\mathbf{x} = 0, \tag{51}$$

which follows from our kinematic assumption and the symmetry of the Cauchy stress. A simplified form of the dissipation results from using the kinematic jump conditions (which follow directly from (44)),

$$[\![\dot{\mathbf{y}}]\!] = -\dot{s}\mathbf{R}(t)\mathbf{a}, \qquad [\![\nabla \mathbf{y}]\!] = \mathbf{R}(t)\mathbf{a} \otimes \mathbf{n}, \tag{52}$$

together with the identities and $[\![a^2]\!] = 2[\![a]\!]\langle a\rangle$ and $[\![ab]\!] = [\![a]\!]\langle b\rangle + [\![b]\!]\langle a\rangle$. Use of these identities, the kinematic jump conditions (52), and the jump condition (48) in the last of (50) gives

$$\mathcal{D} = \dot{s}(t)\mathbf{R}(t)\mathbf{a} \cdot \left\langle \int_{\mathcal{S}_t} \mathbf{t} \, da \right\rangle. \tag{53}$$

The *configurational force* on the interface is the (normalized) multiplier of $\dot{s}$:

$$\mathbf{R}(t)\mathbf{a} \cdot \left\langle \fint_{\mathcal{S}_t} \mathbf{t} \, da \right\rangle. \tag{54}$$

Here and below, the slashed integral notation $\fint_{\mathcal{S}_t}$ denotes the area average $\frac{1}{\text{Area}(\mathcal{S}_t)} \int_{\mathcal{S}_t}$. In PRBM interfaces are "driven" by the component of the average traction on the amplitude of the discontinuity. This is the same formula for the configurational force that would be obtained by specializing the corresponding formula of the deformable case to the piecewise rigid ansatz.

## 6. Kinetic Laws for Piecewise Rigid Bodies

Now we can make explicit the form of the kinetic law, which is stated schematically in (38), for PRBM. Following the ideas of Eshelby and Abeyaratne and Knowles, we assume the kinetic law takes the form

$$\dot{s} = -f\left(\mathbf{R}(t)\mathbf{a} \cdot \left\langle \fint_{\mathcal{S}_t} \mathbf{t} \, da \right\rangle\right) \quad \text{on } \mathcal{S}_t, \tag{55}$$

where $f\colon \mathbb{R} \to \mathbb{R}$ satisfies $\xi f(\xi) \geq 0$, to respect the nonpositivity of the dissipation $\mathcal{D}$. Two typical forms of kinetic laws that have been discussed in the literature (Abeyaratne and Knowles [1] [2], Abeyaratne, Chu, and James [4], Bhattacharya [9]) are

$$\begin{aligned} f(\xi) &= \mu\xi, && \text{classical linear kinetics,} \\ f(\xi) &= \mu \begin{cases} -\sqrt{\xi^2 - b^2}, & \xi < -c, \\ 0, & -c \leq \xi \leq c, \\ \sqrt{\xi^2 - b^2}, & \xi > c, \end{cases} && \text{"pinning" kinetics.} \end{aligned} \tag{56}$$

Here, $\mu > 0$ is the *mobility* and $b, c$ are positive constants with $c^2 - b^2 = 0$.

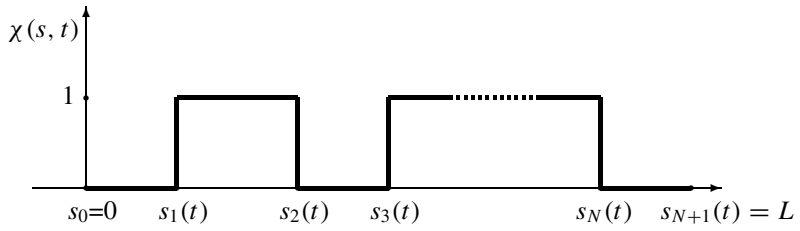


**Fig. 1.** The function $\chi(s,t)$ used in the ansatz (44).

## 7. Formulation of the Theory for a Single Laminate

The motion $\mathbf{y}(\mathbf{x},t)$ of a single laminate is still given by the formula (44), but the characteristic function $\chi$ has any number of jumps that propagate at independent velocities. So, we assume there are $N+2$ smooth functions $s_i\colon (0,\infty) \to [0,L]$ satisfying

$$0 = s_0(t) < s_1(t) < s_2(t) < \cdots < s_N(t) < s_{N+1}(t) = L. \tag{57}$$

Here, without loss of generality, $\Omega$ is assumed to satisfy $0 = \min_{\mathbf{x}\in\bar{\Omega}}(\mathbf{x}\cdot\mathbf{n})$ and $L = \max_{\mathbf{x}\in\bar{\Omega}}(\mathbf{x}\cdot\mathbf{n})$, and we have introduced the two stationary "fake" interfaces $s_0$ and $s_{N+1}$ so as to streamline subsequent formulas. We have also disallowed nucleation or annihilation of interfaces; this is discussed in Section 8. Now let $\mathbf{y}\colon \Omega\times(0,\infty)\to\mathbb{R}^3$ be given by (44) with the characteristic function $\chi\colon [0,L]\times(0,\infty)\to\{0,1\}$ defined by

$$\chi(\xi,t) = \begin{cases} 0 & : \quad s_i(t) \le \xi < s_{i+1}(t), \quad i \text{ even}, \\ 1 & : \quad s_i(t) < \xi \le s_{i+1}(t), \quad i \text{ odd}, \end{cases} \tag{58}$$

and pictured in Figure 1. The unknown functions in the case of the single laminate are

$$\mathbf{R}(t), \qquad \mathbf{c}(t), \qquad s_1(t),\ldots,s_N(t). \tag{59}$$

Now we seek equations of motion that, together with initial conditions and the assignment of forces and torques, will determine these functions. The goal is to put these in a convenient form for discussing well-posedness and for simulation. Our strategy is the following: Just as in rigid body mechanics we associated an unknown function with a law of mechanics according to the appearance of its highest derivative ($\mathbf{R}(t)$: balance of rotational momentum, $\mathbf{y}_c(t)$: balance of linear momentum), we here try to formulate the equations with the following associations:

$$\begin{aligned} \mathbf{R}(t)\colon &\quad \text{Balance of rotational momentum,} \\ \mathbf{c}(t)\colon &\quad \text{Balance of linear momentum,} \\ s_1(t),\ldots,s_N(t)\colon &\quad N \text{ kinetic laws.} \end{aligned} \tag{60}$$

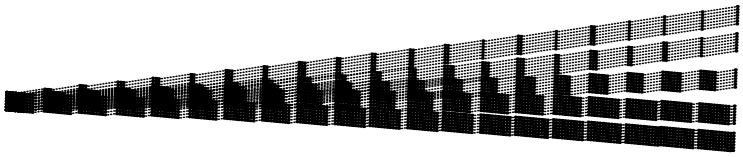

**Fig. 2.** A family of snapshots of shear motions of a cantilever in PRBM, corresponding to a particular choice of the functions $s_1(t), \ldots, s_N(t)$, $t = t_1, t_2, t_3, t_4, t_5$. Drawn using energy wells obtained from the measured lattice parameters of $Ni_2MnGa$.

As in rigid body mechanics it is convenient here also to eliminate $\mathbf{c}(t)$ in favor of the deformed center of mass. Define

$$\begin{aligned} M &= \int_\Omega \rho_o \, d\mathbf{x}, \\ \mathbf{x}_c &= \frac{1}{M} \int_\Omega \rho_o \mathbf{x} \, d\mathbf{x}, \\ \gamma(s,t) &= \int_0^s \chi(r,t) \, dr, \\ \gamma_c(t) &= \frac{1}{M} \int_\Omega \rho_o \gamma(\mathbf{x}\cdot\mathbf{n}, t) \, d\mathbf{x}, \\ \mathbf{y}_c(t) &= \frac{1}{M} \int_{\mathbf{y}(\Omega,t)} \left( \frac{\rho_o}{\det \mathbf{U}_1} \right) \mathbf{z} \, d\mathbf{z}. \end{aligned} \tag{61}$$

If we eliminate $\mathbf{c}(t)$ from (44) using these definitions, we get

$$\mathbf{y}(\mathbf{x}, t) = \mathbf{R}(t)[\mathbf{U}_1(\mathbf{x} - \mathbf{x}_c) + (\gamma(\mathbf{x}\cdot\mathbf{n}, t) - \gamma_c(t))\mathbf{a}] + \mathbf{y}_c(t). \tag{62}$$

Keep in mind that $\gamma$ and $\gamma_c$ are not independent functions because of $(61)_4$.

There appear to be physically interesting cases of PRBM even for the simple laminate. Figure 2 shows several snapshots of a cantilever whose "bending" (actually shear) is produced by an antiphase periodic motion of its interfaces. Figure 2 was obtained by simply plotting the deformed positions of a square array of dots $\mathbf{x} = \mathbf{x}_1, \mathbf{x}_2, \ldots$ according to the formula (44), using five particular sets of interfacial positions $s_1(t), \ldots, s_N(t)$, $t = t_1, t_2, t_3, t_4, t_5$. It does not represent the solution of a particular dynamical problem, though something similar is expected in realistic situations. This example is particularly interesting in the case of ferromagnetic shape memory materials in which the motion can be induced by an applied field. This cantilever is currently under construction guided by the results based on the theory given here and elsewhere [23]. We now formulate systematically the equations of motion.

### *7.1. Mass, Density, Sectional Properties, and Their Integrals*

To evaluate various integrals below, we will need expressions for certain sectional properties, designated by hats. Let $\{\mathbf{n}, \mathbf{n}_1^\perp, \mathbf{n}_2^\perp\}$ be an orthonormal basis with associated

coordinates $\{s, \xi_1, \xi_2\}$. Define the *cross-section* by

$$\mathcal{A}(s) = \{(\xi_1, \xi_2)\colon\ s\mathbf{n} + \xi_1\mathbf{n}_1^\perp + \xi_2\mathbf{n}_2^\perp \in \Omega\}, \quad s \in (0, L). \tag{63}$$

The *cross-sectional area* is denoted

$$|\mathcal{A}(s)| = \text{Area}(\mathcal{A}(s)). \tag{64}$$

The *mass per unit length (in the direction* $\mathbf{n}$*)* is

$$\hat{\rho}_o(s) = \int_{\mathcal{A}(s)} \rho_o \, d\xi_1 \, d\xi_2. \tag{65}$$

Note that $\hat{\rho}_o(s) > 0$, $s \in (0, L)$. The *sectional center of mass* is

$$\hat{\mathbf{x}}_c(s) = \frac{1}{\hat{\rho}_o(s)} \int_{\mathcal{A}(s)} \rho_o(\xi_1\mathbf{n}_1^\perp + \xi_2\mathbf{n}_2^\perp) \, d\xi_1 \, d\xi_2. \tag{66}$$

Note that $\hat{\mathbf{x}}_c \cdot \mathbf{n} = 0$. The *total mass of* $\{\mathbf{x} \in \Omega\colon\ \mathbf{x} \cdot \mathbf{n} < s\}$ is

$$\hat{M}(s) = \int_0^s \hat{\rho}_o(s) \, ds. \tag{67}$$

When $\rho_o$ is constant and $\Omega$ has a simple form, these quantities are given by simple analytical expressions. In order to bring out the essential structure of the equations of motion and the kinetic energy of the laminate, we introduce some auxiliary quantities. We shall denote with $\mathbf{A}$ the constant skew matrix whose axial vector is $\mathbf{a}$, $\mathbf{A} = \mathbf{a} \wedge$. Let

$$\mathbf{s}(t) \ = \ (s_1(t), -s_2(t), s_3(t), \ldots, -(-1)^N s_N(t)), \tag{68}$$

$$\mathbf{v}_s(t) \ = \ (\dot{s}_1(t), -\dot{s}_2(t), \dot{s}_3(t) \ldots, -(-1)^N \dot{s}_N(t)), \qquad \text{and} \tag{69}$$

$$\alpha(t) \ = \ (|\mathcal{A}(s_1(t))|, |\mathcal{A}(s_2(t))|, \ldots, |\mathcal{A}(s_N(t))|) \tag{70}$$

denote the vectors of interface positions, velocities, and areas, respectively. Let $\mathcal{M}$ be the following $N \times N$ symmetric *mass-distribution matrix*[6] whose $ij^{\text{th}}$ element is given by

$$\mathcal{M}_{ij} = \begin{cases} \frac{1}{M}\hat{M}(s_j)\left(M - \hat{M}(s_i)\right), & i > j, \\ \frac{1}{M}\hat{M}(s_i)\left(M - \hat{M}(s_j)\right), & i \le j, \end{cases} \tag{71}$$

and let $\mathcal{C}$ denote the $N \times 3$ *coupling matrix*[7] whose $i^{\text{th}}$ row is the vector

$$\mathcal{C}_i = \int_0^{s_i} \hat{\rho}_o(r)[\mathbf{U}_1(r\mathbf{n} + \hat{\mathbf{x}}_c(r) - \mathbf{x}_c)] \, dr. \tag{72}$$

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[6] The mass-distribution matrix $\mathcal{M}$ is one contribution to the mass matrix of the dynamical system, as can be seen from the equations of motion (112)–(117). $\mathcal{M}$ takes into account the fact that, due to interface motion, the mass of the body is redistributed among the subbodies between the interfaces.

[7] $\mathcal{C}$ couples interface motion and overall rotation.

Finally, let $\boldsymbol{\zeta}$, $\boldsymbol{\eta}$, and $\boldsymbol{\mu}$ be $N$-dimensional vectors whose $i^{\text{th}}$ components are given respectively by

$$\begin{aligned}
\zeta_i(t) &= \frac{\hat{M}(s_i(t))}{M}\sum_{j=0}^{N}\frac{1}{2}(1-(-1)^j)\int_{s_j(t)}^{s_{j+1}(t)} r\hat{\rho}_o(r)\,dr \\
&\quad -\sum_{j=0}^{i-1}\frac{1}{2}(1-(-1)^j)\int_{s_j(t)}^{s_{j+1}(t)} r\hat{\rho}_o(r)\,dr, \\
\eta_i(t) &= \frac{\hat{M}(s_i(t))}{M}\sum_{j=0}^{N}(-1)^j\hat{\rho}_o(s_j(t))\dot{s}_j^2(t)-\sum_{j=0}^{i-1}(-1)^j\hat{\rho}_o(s_j(t))\dot{s}_j^2(t), \\
\varpi_i(t) &= \hat{M}(s_i(t)),
\end{aligned} \tag{73}$$

for $i = 1, 2, \ldots, N$. Note that $\mathcal{M}, \mathcal{C}, \boldsymbol{\zeta}$, and $\varpi$ depend upon time only through the positions of the interfaces, while $\boldsymbol{\eta}$ depends also on their velocities. The script quantities $\mathcal{M}, \mathcal{C}$ will appear on the left-hand side of the equations of motion in our formulation, forming the mass matrix of the system, while the vectors $\boldsymbol{\zeta}$, $\boldsymbol{\eta}$, and $\varpi$ will appear as lower-order terms on the right-hand side. The following proposition gives explicit formulas for all the quantities that will appear in the equations of motion.

**Proposition 7.1.** *Assume that* $\Omega$ *and the reference normal* $\mathbf{n}$ *to the layers are such that* $\operatorname{meas}\mathcal{A}(s)$ *and* $\hat{\rho}(s)$ *are continuous for* $s \in (0, L)$. *Suppose* $(0 =)\ s_0 < s_1(\cdot) < \cdots < s_N(\cdot) < s_{N+1}(= L) \in C^2(0, T)$. *Let* $p$: $(0, L) \times (0, T) \to \{0, \ldots, N\}$ *satisfy*

$$s_{p(s,t)}(t) < s < s_{p(s,t)+1}(t), \tag{74}$$

*i.e.,* $p(s, t)$ *is the index of the interface just to the left of* $s$ *at time* $t$. *Then,*

$$\gamma(s,t) = \sum_{i=0}^{p(s,t)}(-1)^i s_i(t) + \frac{1}{2}(1-(-1)^{p(s,t)})s, \tag{75}$$

$$\dot{\gamma}(s,t) = \sum_{i=0}^{p(s,t)}(-1)^i \dot{s}_i, \tag{76}$$

$$\ddot{\gamma}(s,t) = \sum_{i=0}^{p(s,t)}(-1)^i \ddot{s}_i, \tag{77}$$

$$\gamma_c(t) = \frac{1}{M}\sum_{i=0}^{N}\left\{(-1)^i\left[M-\hat{M}(s_i)\right]s_i+\frac{1}{2}(1-(-1)^i)\int_{s_i(t)}^{s_{i+1}(t)} s\hat{\rho}_o(s)\,ds\right\}, \tag{78}$$

$$\dot{\gamma}_c(t) = \frac{1}{M}\sum_{i=0}^{N}(-1)^i\left[M-\hat{M}(s_i)\right]\dot{s}_i, \tag{79}$$

$$\ddot{\gamma}_c(t) = \frac{1}{M}\sum_{i=0}^{N}(-1)^i\left\{\left[M-\hat{M}(s_i)\right]\ddot{s}_i-\hat{\rho}_o(s_i)\dot{s}_i^2\right\}. \tag{80}$$

*Furthermore,*

$$\int_{\Omega\cap\{\mathbf{x}\cdot\mathbf{n}<s_i(t)\}} \rho_o(\dot\gamma(\mathbf{x}\cdot\mathbf{n},t)-\dot\gamma_c(t))\,d\mathbf{x} = (\mathcal{M}\mathbf{v}_s)_i, \tag{81}$$

$$\left(\frac{d}{dt}\int_0^s \hat\rho_o(\dot\gamma(r,t)-\dot\gamma_c(t))\,dr\right)\Bigg|_{s=s_i-0} = (\mathcal{M}\dot{\mathbf{v}}_s+\boldsymbol{\eta})_i, \tag{82}$$

$$\mathbf{a}\cdot\dot{\mathbf{W}}\left(\int_{\Omega\cap\{\mathbf{x}\cdot\mathbf{n}<s_i\}} \rho_o\mathbf{U}_1(\mathbf{x}-\mathbf{x}_c)\,d\mathbf{x}\right) = (\mathcal{C}\mathbf{A}\dot{\boldsymbol{\omega}})_i, \tag{83}$$

$$\mathbf{a}\cdot\mathbf{W}^2\left(\int_{\Omega\cap\{\mathbf{x}\cdot\mathbf{n}<s_i\}} \rho_o[\mathbf{U}_1(\mathbf{x}-\mathbf{x}_c)+(\gamma(\mathbf{x}\cdot\mathbf{n})-\gamma_c)\mathbf{a}]\,d\mathbf{x}\right) = (\mathcal{C}(\mathbf{A}\boldsymbol{\omega}\wedge\boldsymbol{\omega}))_i - |\mathbf{A}\boldsymbol{\omega}|^2(\mathcal{M}\mathbf{s}-\boldsymbol{\zeta})_i, \tag{84}$$

*for $i = 1, 2, 3, \ldots, N$, the subscript $i$ indicating the $i^{\text{th}}$ component of the vectors to which $i$ is attached, and*

$$\int_\Omega \rho_o\dot\gamma(\mathbf{x}\cdot\mathbf{n},t)((\mathbf{U}_1(\mathbf{x}-\mathbf{x}_c))\wedge\mathbf{a})\,d\mathbf{x} = (\mathcal{C}\mathbf{A})^T\mathbf{v}_s, \tag{85}$$

$$\int_\Omega \rho_o(\dot\gamma^2-\dot\gamma_c^2)\,d\mathbf{x} = \mathbf{v}_s\cdot\mathcal{M}\mathbf{v}_s. \tag{86}$$

*Proof.* The expressions for $\gamma(s,t)$ and $\gamma_c(t)$ and their time derivatives follow directly from the definition as the integral of $\chi$, and from the identity

$$\sum_{i=0}^N b_i\sum_{k=0}^i a_k = \sum_{i=0}^N a_i\sum_{k=i}^N b_k \tag{87}$$

applied with $b_i = \int_{s_i}^{s_{i+1}} g(r)\,dr$, $g(r) = \hat\rho_o(r)$, or $g(r) = r\hat\rho_o(r)$ and $a_k = (-1)^k s_k(t)$. For the sectional integrals notice that for any integrable $f\colon (0,L)\times(0,\infty)\to\mathbb{R}^n$ we have

$$\begin{aligned}\int_{\Omega\cap\{\mathbf{x}\cdot\mathbf{n}<s\}} \rho_o(\mathbf{x})f(\mathbf{x}\cdot\mathbf{n},t)\,d\mathbf{x} &= \int_0^s \hat\rho_o(s)f(s,t)\,ds,\\ &= \left(\sum_{i=0}^{p(s,t)-1}\int_{s_i(t)}^{s_{i+1}(t)} \hat\rho_o(s)f(s,t)\,ds\right)\\ &\quad+\int_{p(s,t)}^s \hat\rho_o(s)f(s,t)\,ds.\end{aligned} \tag{88}$$

To get (81), (83), (84), and (85), apply this equation with $f(s,t) = \{\gamma(s,t)-\gamma_c,\ \dot\gamma(s,t)-\dot\gamma_c,\ s\mathbf{n}+\hat{\mathbf{x}}_c(s)-\mathbf{x}_c,\ \dot\gamma(s,t)(s\mathbf{n}+\hat{\mathbf{x}}_c(s)-\mathbf{x}_c)\}$, respectively, after using the equations (63)–(67), (79), (80), the sum rule (87), and the definitions of the matrices $\mathbf{A}$, $\mathcal{M}$, and $\mathcal{C}$. To get (82), begin with the transport theorem in the form,

$$\frac{d}{dt}\left(\int_0^s \hat\rho_o\dot\gamma(r,t)\,dr\right) = \int_0^s \hat\rho_o\ddot\gamma(r,t)\,dr - \sum_{j=0}^{p(s,t)}\hat\rho_o(s_j)[\![\dot\gamma(r,t)]\!]\Big|_{r=s_j}\dot s_j. \tag{89}$$

Notice that from (76) we have $[\![\dot\gamma(r,t)]\!]|_{r=s_j} = (-1)^j \dot s_j$ and thus, using also the fact that $\dot\gamma_c$ depends only on t,

$$\frac{d}{dt}\left(\int_0^s \hat\rho_o(\dot\gamma(r,t)-\dot\gamma_c(t))\,dr\right) = \int_0^s \hat\rho_o(\ddot\gamma(r,t)-\ddot\gamma_c(t))\,dr - \sum_{j=0}^{p(s,t)} \hat\rho_o(s_j)(-1)^j \dot s_j^2. \tag{90}$$

Evaluate this at $s = s_i - 0$ and use (77) together with the sum rule (87) and the definition of $\mathbf{v}_s$ and $\boldsymbol\eta$ to get (82). To obtain (86), notice that, in view of (81) and (76), we have

$$\mathbf{v}_s \cdot \mathcal{M}\mathbf{v}_s = -\sum_{i=1}^{N}(-1)^i \dot s_i \int_0^{s_i} \hat\rho_o\left(\sum_{j=0}^{p(s,t)}(-1)^j \dot s_j\right) ds + \sum_{i=1}^{N}(-1)^i \hat M(s_i)\dot s_i \dot\gamma_c. \tag{91}$$

Express the integral in the first term as the sum of integrals each on each single domain $(s_{k-1}, s_k)$, and for the second term use (79) multiplied by $\dot\gamma_c$ to get

$$\mathbf{v}_s \cdot \mathcal{M}\mathbf{v}_s = -\sum_{i=1}^{N}(-1)^i \dot s_i \sum_{k=1}^{i}\int_{s_{k-1}}^{s_k} \hat\rho_o\left(\sum_{j=0}^{p(s,t)}(-1)^j \dot s_j\right) ds + M\dot\gamma_c \sum_{i=1}^{N}(-1)^i \dot s_i - M\dot\gamma_c^2. \tag{92}$$

Now use the expression (79) for $\dot\gamma_c$, and observe that we can sum up to the $N+1$ term in (79) as well as in the sums with index $i$ in (92), because $\dot s_{N+1} = 0$. Then, using the rule (87), switch the sums in $i$ and $k$ in the first term at the right-hand side of (92), and apply the definition of $\gamma_c$ in the second term:

$$\mathbf{v}_s \cdot \mathcal{M}\mathbf{v}_s = -\sum_{i=1}^{N+1}\int_{s_{i-1}}^{s_i} \hat\rho_o \sum_{j=0}^{p(s,t)}(-1)^j \dot s_j\, ds \left(\sum_{k=i}^{N+1}(-1)^k \dot s_k\right) + \sum_{k=1}^{N+1}\int_{s_{k-1}}^{s_k} \hat\rho_o \sum_{j=0}^{p(s,t)}(-1)^j \dot s_j\, ds \left(\sum_{i=1}^{N+1}(-1)^i \dot s_i\right) - M\dot\gamma_c^2. \tag{93}$$

After rearranging the terms and using (76), we get (86). □

### *7.2. Balance of Linear Momentum*

The introduction of the definitions (61) is designed to make the overall balance of linear momentum simple:

$$\frac{d}{dt}\int_\Omega \rho_o \dot{\mathbf{y}}\, d\mathbf{x} = \mathcal{F} \iff M\ddot{\mathbf{y}}_c = \mathcal{F}, \tag{94}$$

where $\mathcal{F}$ is the total applied force on $\mathbf{y}(\Omega, t)$.

The local balance of linear momentum determines div $\mathbf{T}$ whose only appearance in the other laws occurs in the kinetic law as the resultant traction on an interface. We now make this explicit. There are various ways to do this, e.g., by writing the balance of linear momentum for each interval $\{\mathbf{x} \in \Omega\colon\ s_i(t) < \mathbf{x}\cdot\mathbf{n} < s_{i+1}(t)\}$, and then combining these with the jump conditions across each interface. We find it easier to consider regions of the form $\{\mathbf{x} \in \Omega\colon\ \mathbf{x}\cdot\mathbf{n} < s_i,\ i \in 1, \ldots, N\}$. First we write the balance of linear momentum for the material region $\{\mathbf{x} \in \Omega\colon\ \mathbf{x}\cdot\mathbf{n} < s\}$:

$$\frac{d}{dt}\int_{\Omega\cap\{\mathbf{x}\cdot\mathbf{n}<s\}} \rho_o\dot{\mathbf{y}}\,d\mathbf{x} = \mathcal{F}(0,s), \tag{95}$$

where $\mathcal{F}(0,s)$ is the total external force from all sources (traction, surface and body forces) on the region $\{\mathbf{y}(\mathbf{x},t)\colon\ \mathbf{x}\in\Omega,\ \mathbf{x}\cdot\mathbf{n} < s\}$. For example, $\mathcal{F}(0,s)$ includes, in addition to the force due to traction at the cross-section $\mathcal{A}(s)$, forces that arise from moving fluids surrounding the body, and gravitational or electromagnetic forces on this region. The force arising from the traction $\mathbf{t}(s)\colon\ \mathcal{A}(s) \to \mathbb{R}^3$ on $\mathcal{A}(s)$, regarded as having normal $\mathbf{n}$, is assumed to have the form

$$\int_{\mathcal{A}(s)} \mathbf{t}\,da. \tag{96}$$

The remaining contribution to $\mathcal{F}(0,s)$, from distributed surface and body forces, is given by

$$\mathcal{F}_{\text{ext}}(0,s). \tag{97}$$

We have $\mathcal{F} = \mathcal{F}_{\text{ext}}(0,L)$, and, in summary, we write

$$\mathcal{F}(0,s) = \int_{\mathcal{A}(s)} \mathbf{t}\,da + \mathcal{F}_{\text{ext}}(0,s). \tag{98}$$

Substitute the expression $\dot{\mathbf{y}}(\mathbf{x},t) = \mathbf{RW}[\mathbf{U}_1(\mathbf{x}-\mathbf{x}_c) + (\gamma-\gamma_c)\mathbf{a}] + (\dot\gamma-\dot\gamma_c)\mathbf{Ra} + \dot{\mathbf{y}}_c$ into (95):

$$\begin{aligned}\frac{d}{dt}\Bigg\{\mathbf{RW}\left[\mathbf{U}_1\int_{\Omega\cap\{\mathbf{x}\cdot\mathbf{n}<s\}}\rho_o(\mathbf{x}-\mathbf{x}_c)\,d\mathbf{x} + \mathbf{a}\int_{\Omega\cap\{\mathbf{x}\cdot\mathbf{n}<s\}}\rho_o(\gamma-\gamma_c)\,d\mathbf{x}\right]\\ +\ \mathbf{Ra}\int_{\Omega\cap\{\mathbf{x}\cdot\mathbf{n}<s\}}\rho_o(\dot\gamma-\dot\gamma_c)\,d\mathbf{x} + \hat{M}(s)\dot{\mathbf{y}}_c\Bigg\} = \int_{\mathcal{A}(s)}\mathbf{t}\,da + \mathcal{F}_{\text{ext}}(0,s).\end{aligned} \tag{99}$$

That is,

$$\begin{aligned}&(\mathbf{RW}^2 + \mathbf{R}\dot{\mathbf{W}})\left[\mathbf{U}_1\int_{\Omega\cap\{\mathbf{x}\cdot\mathbf{n}<s\}}\rho_o(\mathbf{x}-\mathbf{x}_c)\,d\mathbf{x} + \mathbf{a}\int_{\Omega\cap\{\mathbf{x}\cdot\mathbf{n}<s\}}\rho_o(\gamma-\gamma_c)\,d\mathbf{x}\right]\\ &+ 2\mathbf{RWa}\left(\int_{\Omega\cap\{\mathbf{x}\cdot\mathbf{n}<s\}}\rho_o(\dot\gamma-\dot\gamma_c)\,d\mathbf{x}\right) + \mathbf{Ra}\left(\frac{d}{dt}\int_{\Omega\cap\{\mathbf{x}\cdot\mathbf{n}<s\}}\rho_o(\dot\gamma-\dot\gamma_c)\,d\mathbf{x}\right)\\ &+ \hat{M}(s)\ddot{\mathbf{y}}_c = \int_{\mathcal{A}(s)}\mathbf{t}\,da + \mathcal{F}_{\text{ext}}(0,s).\end{aligned} \tag{100}$$

Now notice that, actually, only the quantity $\mathbf{Ra}\cdot \fint_{\mathcal{A}(s)} \mathbf{t}\, da$ occurs in the kinetic law. Thus, we dot (100) by $\mathbf{Ra}$ and get the simplified formula for the configurational force:

$$\begin{aligned}\mathbf{Ra}\cdot\int_{\mathcal{A}(s)}\mathbf{t}\,da \;=\; & \mathbf{a}\cdot\mathbf{W}^2\left(\mathbf{U}_1\int_{\Omega\cap\{\mathbf{x}\cdot\mathbf{n}<s\}}\rho_o(\mathbf{x}-\mathbf{x}_c)\,d\mathbf{x}+\mathbf{a}\int_{\Omega\cap\{\mathbf{x}\cdot\mathbf{n}<s\}}\rho_o(\gamma-\gamma_c)\,d\mathbf{x}\right)\\ & +\mathbf{a}\cdot\dot{\mathbf{W}}\left(\mathbf{U}_1\int_{\Omega\cap\{\mathbf{x}\cdot\mathbf{n}<s\}}\rho_o(\mathbf{x}-\mathbf{x}_c)\,d\mathbf{x}\right)\\ & +|\mathbf{a}|^2\left(\frac{d}{dt}\int_{\Omega\cap\{\mathbf{x}\cdot\mathbf{n}<s\}}\rho_o(\dot\gamma-\dot\gamma_c)\,d\mathbf{x}\right)\\ & +\hat M(s)\ddot{\mathbf{y}}_c\cdot\mathbf{Ra}-\mathcal{F}_{\text{ext}}(0,s)\cdot\mathbf{Ra}. \end{aligned}\tag{101}$$

We need (101) evaluated on each side of each interface in order to calculate the average $\langle \mathbf{Ra}\cdot \fint_{\mathcal{A}(s)}\mathbf{t}\,da\rangle$ that appears in the kinetic laws. For this purpose it is desirable to work with (101) evaluated on just one side of the interface, here chosen arbitrarily to be $s = s_i - 0$, together with the jump condition that comes from the balance of linear momentum.[8] The jump condition is obtained from (100) or else by specialization, then integration over the cross-section, of (48). Assuming that the forces $\mathcal{F}$ and $\mathcal{F}_{\text{ext}}$ are continuous functions of $(s,t)$, the jump condition is

$$-\frac{1}{|\mathcal{A}(s_i(t))|}(\hat\rho_o(s_i)(-1)^i\dot s_i^2)\mathbf{Ra}=\left[\!\!\left[\fint_{\mathcal{A}(s)}\mathbf{t}\,da\right]\!\!\right].\tag{102}$$

Dotting this with $\mathbf{Ra}$, we get

$$-\frac{1}{|\mathcal{A}(s_i(t))|}(\hat\rho_o(s_i)(-1)^i\dot s_i^2)|\mathbf{a}|^2=\left[\!\!\left[\mathbf{Ra}\cdot\fint_{\mathcal{A}(s)}\mathbf{t}\,da\right]\!\!\right].\tag{103}$$

### *7.3. Balance of Rotational Momentum*

Substitution of the ansatz (62) into the overall balance of rotational momentum (17) gives

$$\begin{aligned}&\frac{d}{dt}\int_\Omega\rho_o\,(\mathbf{R}[\mathbf{U}_1(\mathbf{x}-\mathbf{x}_c)+(\gamma-\gamma_c)\mathbf{a}]+(\mathbf{y}_c-\mathbf{y}_o))\\ &\qquad\wedge\left(\dot{\mathbf{R}}[\mathbf{U}_1(\mathbf{x}-\mathbf{x}_c)+(\gamma-\gamma_c)\mathbf{a}]+(\dot\gamma-\dot\gamma_c)\mathbf{Ra}+\dot{\mathbf{y}}_c\right)d\mathbf{x}=\mathcal{T}_o.\end{aligned}\tag{104}$$

Write $\dot{\mathbf{R}}=\mathbf{RW}$ and use the identity $\mathbf{Rb}\wedge\mathbf{Rc}=\mathbf{R}(\mathbf{b}\wedge\mathbf{c})$ and the definitions (61):

$$\begin{aligned}\frac{d}{dt}\Big\{\mathbf{R}\int_\Omega\rho_o[&(\mathbf{U}_1(\mathbf{x}-\mathbf{x}_c)+(\gamma-\gamma_c)\mathbf{a})\wedge\mathbf{W}(\mathbf{U}_1(\mathbf{x}-\mathbf{x}_c)+(\gamma-\gamma_c)\mathbf{a})\\ &+(\dot\gamma-\dot\gamma_c)\mathbf{U}_1(\mathbf{x}-\mathbf{x}_c)\wedge\mathbf{a}]\,d\mathbf{x}+M(\mathbf{y}_c-\mathbf{y}_o)\wedge\dot{\mathbf{y}}_c\Big\}=\mathcal{T}_o.\end{aligned}\tag{105}$$

[8] The advantage of this formulation is that the jump is extremely simple, whereas the average is not.

Note that, from Proposition 7.1, $\dot{\gamma}$ is not absolutely continuous and we cannot (without introducing Dirac masses) differentiate under the integral sign in (105). Following the lessons of rigid body mechanics, we introduce the axial vector $\boldsymbol{\omega}$ corresponding to $\mathbf{W}$, $\mathbf{W} = \boldsymbol{\omega} \wedge$. We also define the referential moment of inertia by

$$\mathcal{I}(t) = \int_\Omega \rho_o \{|(\mathbf{U}_1(\mathbf{x} - \mathbf{x}_c) + (\gamma - \gamma_c)\mathbf{a})|^2 \mathbf{I} - (\mathbf{U}_1(\mathbf{x} - \mathbf{x}_c) + (\gamma - \gamma_c)\mathbf{a}) \otimes (\mathbf{U}_1(\mathbf{x} - \mathbf{x}_c) + (\gamma - \gamma_c)\mathbf{a})\} \, d\mathbf{x}. \tag{106}$$

Recalling the passage from (22) to (24), we write (105) in the form

$$\begin{aligned} &\frac{d}{dt}\{\mathbf{R}\mathcal{I}\boldsymbol{\omega}\} + \mathbf{R}\left\{\boldsymbol{\omega} \wedge \int_\Omega \rho_o(\dot{\gamma} - \dot{\gamma}_c)((\mathbf{U}_1(\mathbf{x} - \mathbf{x}_c)) \wedge \mathbf{a}) \, d\mathbf{x}\right\}, \\ &\quad + \mathbf{R}\frac{d}{dt}\int_\Omega \rho_o(\dot{\gamma} - \dot{\gamma}_c)((\mathbf{U}_1(\mathbf{x} - \mathbf{x}_c)) \wedge \mathbf{a}) \, d\mathbf{x}, \\ &\quad + M(\mathbf{y}_c - \mathbf{y}_o) \wedge \ddot{\mathbf{y}}_c = \mathcal{T}_o, \end{aligned} \tag{107}$$

or, taking into account the definition of $\mathbf{x}_c$,

$$\begin{aligned} &\mathcal{I}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \wedge \mathcal{I}\omega + \dot{\mathcal{I}}\boldsymbol{\omega} \\ &\quad + \boldsymbol{\omega} \wedge \int_\Omega \rho_o \dot{\gamma}((\mathbf{U}_1(\mathbf{x} - \mathbf{x}_c)) \wedge \mathbf{a}) \, d\mathbf{x}, \\ &\quad + \frac{d}{dt}\int_\Omega \rho_o \dot{\gamma}((\mathbf{U}_1(\mathbf{x} - \mathbf{x}_c)) \wedge \mathbf{a}) \, d\mathbf{x}, \\ &\quad + M\mathbf{R}^T((\mathbf{y}_c - \mathbf{y}_o) \wedge \ddot{\mathbf{y}}_c) = \mathbf{R}^T\mathcal{T}_o. \end{aligned} \tag{108}$$

This is the equation for $\boldsymbol{\omega}$, which is supplemented by the relation $\dot{\mathbf{R}} = \mathbf{R}\mathbf{W}$, $\mathbf{W} = \boldsymbol{\omega} \wedge$. The two integrals in (108) are explicit functions of $s_1(t), \ldots, s_N(t)$ and their first and second time derivatives, as can be obtained from (85). The local balance of rotational momentum implies as usual that the Cauchy stress is symmetric. If the local balance of rotational momentum is evaluated for a region bounded by an interface, e.g., a region of the form $\Omega \cap \{\mathbf{x} \cdot \mathbf{n} < s_i(t)\}$, the moment of the traction on the interface appears on the right-hand side. This does not couple to any other equations because the traction and its moment on an interface can be assigned independently. (This makes the remarks of Section 3.2 explicit in this case of a single laminate.)

### *7.4. Kinetic Laws*

The general form of the kinetic law is given above in (55). Specialized to a laminate, this is

$$(-1)^i \dot{s}_i(t) = f\left(\mathbf{R}\mathbf{a} \cdot \left\langle \fint_{\mathcal{A}(s_i(t))} \mathbf{t} \, da \right\rangle\right), \quad i \in \{1, \ldots, N\}. \tag{109}$$

## 8. Summary and Analysis of the Equations of Motion

### *8.1. Summary*

The way the traction occurs in the various laws suggests that we introduce the new scalar variable $\tau_i^{\pm}$: $[0, \infty) \to \mathbb{R}$ by

$$\tau_i^{\pm}(t) = \mathbf{R}(t)\mathbf{a} \cdot \fint_{\mathcal{A}(s)} \mathbf{t}\, da|_{s=s_i(t)\pm 0}, \tag{110}$$

and the corresponding $N$-dimensional vectorial quantities $\boldsymbol{\tau}^{\pm}$. Similarly, for the external forces on the region $\mathbf{y}(\mathbf{x}, t)$, $\mathbf{x} \in \Omega \cap \{\mathbf{x} \cdot \mathbf{n} < s_i(t)\}$, we introduce the $N$-dimensional *external distributed force* vector,

$$\mathcal{F}_s = (\mathbf{Ra} \cdot \mathcal{F}_{\text{ext}}(0, s_1(t)),\ \mathbf{Ra} \cdot \mathcal{F}_{\text{ext}}(0, s_2(t)), \ldots, \mathbf{Ra} \cdot \mathcal{F}_{\text{ext}}(0, s_N(t))). \tag{111}$$

Note that $\mathcal{F}_s$ can be assigned independently of $\mathcal{F}$ (cf. the discussion of Section 7.2).

The equations that do not just determine a part of the stress now can be summarized. We use the definitions given in Section 7 and the results of Proposition 7.1 to recast the equations of motion into the following vectorial form:

$$\dot{\mathbf{y}}_c = \mathbf{v}_c, \tag{112}$$

$$M\dot{\mathbf{v}}_c = \mathcal{F}, \tag{113}$$

$$\dot{\mathbf{s}} = \mathbf{v}_s, \tag{114}$$

$$\begin{aligned} |\mathbf{a}|^2 \mathcal{M}\dot{\mathbf{v}}_s + \mathcal{C}\mathbf{A}\dot{\boldsymbol{\omega}} &= \alpha \cdot \boldsymbol{\tau}^- - |\mathbf{a}|^2 \boldsymbol{\eta} + \mathcal{F}_s - \frac{1}{M}(\mathbf{Ra} \cdot \mathcal{F})\boldsymbol{\varpi} \\ &\quad - \mathcal{C}(\mathbf{A}\boldsymbol{\omega} \wedge \boldsymbol{\omega}) + |\mathbf{A}\boldsymbol{\omega}|^2(\mathcal{M}\mathbf{s} - \boldsymbol{\zeta}), \end{aligned} \tag{115}$$

$$\begin{aligned} (\mathcal{C}\mathbf{A})^T \dot{\mathbf{v}}_s + \mathcal{I}\dot{\boldsymbol{\omega}} &= \mathbf{R}^T \mathcal{T}_o - \mathbf{R}^T((\mathbf{y}_c - \mathbf{y}_o) \wedge \mathcal{F}) - \boldsymbol{\omega} \wedge \mathcal{I}\boldsymbol{\omega} \\ &\quad - \dot{\mathcal{I}}\boldsymbol{\omega} - (\dot{\mathcal{C}}\mathbf{A})^T \mathbf{v}_s - \boldsymbol{\omega} \wedge (\mathcal{C}\mathbf{A})^T \mathbf{v}_s, \end{aligned} \tag{116}$$

$$\dot{\mathbf{R}} = \mathbf{RW}. \tag{117}$$

The tractions $\boldsymbol{\tau}^-$ are linked to the tractions $\boldsymbol{\tau}^+$ and to the interface velocities through the kinetic laws and the balance of linear momentum at interfaces

$$\begin{aligned} \dot{s}_i &= -f\left((-1)^{i+1}\frac{(\tau_i^+ + \tau_i^-)}{2}\right), \\ \tau_i^+ - \tau_i^- &= \frac{(-1)^{i+1}}{|\mathcal{A}(s_i)|}\hat{\rho}_o(s_i)\dot{s}_i^2|\mathbf{a}|^2. \end{aligned} \tag{118}$$

*Remark* 8.1 (Cantilever). The simplest interesting case of these equations of motion is perhaps the situation appropriate to the cantilever, Figure 2. Put $\mathbf{R} = \mathbf{I}$.[9] The cantilever is

[9] If one chooses $\mathbf{R} \in \text{SO}(3)$, not necessarily the identity, then $\mathbf{a}$ is everywhere replaced by $\mathbf{Ra}$ and otherwise the same equations of motion of the cantilever emerge.

fixed at the root, so we assume that $\mathbf{y}(0,t)=0$. Use the kinematics (62) and $\mathbf{y}(0,t)=0$ to get

$$\dot{\mathbf{y}}_c = \dot{\gamma}_c(t)\mathbf{a}. \tag{119}$$

Differentiate this with respect to $t$ and use the overall balance of linear momentum (113) to evaluate the reaction force $\mathcal{F}_0$ at the root of the cantilever:

$$\frac{1}{M}\mathbf{a}\cdot(\mathcal{F}_0+\mathcal{F}_L) = |\mathbf{a}|^2\ddot{\gamma}_c. \tag{120}$$

This allows us to eliminate the reaction force and gives the following for the forces that occur on the right-hand side of the balance of linear momentum (115):

$$\begin{aligned}\mathcal{F}_s - \frac{1}{M}(\mathbf{a}\cdot\mathcal{F})\varpi \;&=\; |\mathbf{a}|^2\ddot{\gamma}_c\left(M-\hat{M}(s_1),\ldots,M-\hat{M}(s_N)\right)\\ &\quad -(\mathbf{a}\cdot\mathcal{F}_L,\ldots,\mathbf{a}\cdot\mathcal{F}_L).\end{aligned} \tag{121}$$

We have chosen $\mathbf{R}=\mathbf{I}$ for all $t$. Thus, $\omega=0$ and the balance of rotational momentum is satisfied simply by choosing $\mathcal{T}_{\mathrm{o}}$ appropriately. $\mathcal{T}_{\mathrm{o}}$ is then the reaction torque at the root of the cantilever. We are left with the balance of linear momentum (115) in the form

$$|\mathbf{a}|^2\mathcal{M}\dot{\mathbf{v}}_s = \alpha\cdot\boldsymbol{\tau}^- - |\mathbf{a}|^2\boldsymbol{\eta} + \mathcal{F}_s - \frac{1}{M}(\mathbf{a}\cdot\mathcal{F})\varpi. \tag{122}$$

For simplicity we choose the case of a cantilever of constant reference density and uniform referential cross-section, and we also choose linear kinetics, that is,

$$\rho_{\mathrm{o}} = const, \qquad \mathcal{A}(s)=\mathcal{A}=const., \qquad f(\xi)=\mu\xi, \tag{123}$$

so that $\hat{M}(s)=\rho_{\mathrm{o}}\mathcal{A}s$ and $M=\rho_{\mathrm{o}}\mathcal{A}L$. Denote $a=|\mathbf{a}|$ and $\tau_L=\mathbf{a}\cdot\mathcal{F}_L$. Substitute all this into the balance of linear momentum. Finally, nondimensionalize the resulting system by defining dimensionless interfacial positions $r_i(t)\overset{\text{def}}{=}(1/L)s_i(tT)$, where $T$ is a typical time. Note that $0<r_1(t)<\cdots<r_N(t)<1$. A particularly simple form emerges:

$$\begin{aligned}
(r_2-r_1)[-\ddot{r}_1] \;&=\; \left(\frac{T}{a^2L\mu\rho_o}+\frac{(\dot{r}_2-\dot{r}_1)}{2}\right)(\dot{r}_2+\dot{r}_1),\\
(r_3-r_2)[-\ddot{r}_1+\ddot{r}_2] \;&=\; -\left(\frac{T}{a^2L\mu\rho_o}+\frac{(\dot{r}_3-\dot{r}_2)}{2}\right)(\dot{r}_3+\dot{r}_2),\\
(r_4-r_3)[-\ddot{r}_1+\ddot{r}_2-\ddot{r}_3] \;&=\; \left(\frac{T}{a^2L\mu\rho_o}+\frac{(\dot{r}_4-\dot{r}_3)}{2}\right)(\dot{r}_4+\dot{r}_3),\\
&\;\;\vdots \qquad\qquad\qquad\qquad \vdots\\
(1-r_N)[-\ddot{r}_1+\ddot{r}_2-\ddot{r}_3+\cdots+(-1)^N\ddot{r}_N] \;&=\; \frac{\tau_L T^2}{a^2L^2\mathcal{A}\rho_o}\\
&\quad +(-1)^N\dot{r}_N\left(\frac{-T}{a^2L\mu\rho_o}+\frac{1}{2}\dot{r}_N\right).
\end{aligned} \tag{124}$$

It is seen that the dynamics of the unloaded cantilever is governed by the single dimensionless number $T/(a^2 L\mu\rho_o)$, which by inspection can be interpreted as the ratio of inertial to configurational forces. If constant loads are allowed ($\tau_L \neq 0$), then we also require the dimensionless load amplitude $\tau_L T^2/(a^2 L^2 \mathcal{A}\rho_o)$; if in addition these loads are periodic in time, then we need moreover the dimensionless frequency $fT$ where $f$ is the dimensional frequency.

This complete collapse of material and geometric data is easily traced to the form of the equations of motion. It also extends to nonlinear kinetic relations, as long as the other conditions $\rho_o = const$, $\mathcal{A}(s) = \mathcal{A} = const.$, are respected. To see this, put the applied forces to zero and note that each term in the balance of linear momentum is proportional to $a^2\rho_o\mathcal{A}$ except for one term that arises from $\tau^-$, this being the contribution to $\tau^-$ from the kinetic relation (i.e., invert the kinetic relation and solve (118) for $\tau^-$). However, from the form (118) of $\tau^-$, one sees that this contribution is proportional to $\mathcal{A}f^{-1}$, and therefore, in the case of invertible, nonlinear kinetic relations all material and geometric data collapse to the single dimensionless function $(T/a^2 L\rho_o) f^{-1}(\frac{T}{L}\dot{r})$.

### *8.2. Well-posedness up to the Time of the First Interfacial Collision*

Introduce the vector of variables $\Theta = (\mathbf{y}_c, \mathbf{v}_c, \mathbf{s}, \mathbf{v}_s, \mathbf{R}, \boldsymbol{\omega})$. We begin by putting the system (112)–(117) into the standard form $\dot{\Theta} = f(\Theta; t)$. To do so, we need to check whether the matrix

$$\mathbb{M} = \begin{pmatrix} M\mathbf{I} & 0 & 0 \\ 0 & |\,\mathbf{a}\,|^2\,\mathcal{M} & \mathcal{C}\mathbf{A} \\ 0 & (\mathcal{C}\mathbf{A})^T & \mathcal{I} \end{pmatrix} \tag{125}$$

is invertible. This is done below.

**Proposition 8.1.** *Assume the hypotheses of* Proposition 7.1. *Then,* $\mathbb{M}$ *given by* (125) *is positive-definite on* $(0, T)$. *As* $t \to T$, *one of the following holds:* (i) *there is an* $\varepsilon > 0$ *such that* $\det \mathbb{M} > \varepsilon > 0$, *or* (ii) *there is an increasing sequence* $t_i \to T$ *such that* $\det \mathbb{M}(t_i) \to 0$ *and at least one pair of interfaces collides at T,* $s_{k+1}(t_i) - s_k(t_i) \to 0$ *holds for some* $k \in \{0, \ldots, N\}$. *If* (ii) *holds and only one pair* $s_{k+1}, s_k$ *of internal* $(k \neq 0, N)$ *interfaces collides at T, in the sense that* $\liminf(s_{j+1}(t_i) - s_j(t_i)) > 0$ *for* $j \neq k$, *then the corresponding limiting null-space of* $\mathbb{M}(t_i)$ *is one-dimensional and of the form* $(0, 0, 0; 0, \ldots, \alpha, -\alpha, \ldots, 0; , 0, 0, 0)$ *where* $\alpha = \frac{\pm 1}{\sqrt{2}}$ *lies in the* $3 + k^{\text{th}}$ *place.*

*Proof.* These assertions are anticipated to follow from the positiveness of the kinetic energy:

$$KE = \frac{1}{2}\int_\Omega \rho_o(\mathbf{x})|\dot{\mathbf{y}}(\mathbf{x})|^2\, d\mathbf{x}. \tag{126}$$

Substituting the time derivative of (62) into the expression for *KE* gives, after some simplification,

$$KE = \frac{1}{2}\mathcal{I}\boldsymbol{\omega}\cdot\boldsymbol{\omega} + \frac{1}{2}|\mathbf{a}|^2\int_\Omega \rho_o(\dot\gamma-\dot\gamma_c)^2\,d\mathbf{x} + \frac{1}{2}M|\dot{\mathbf{y}}_c|^2 - \mathbf{W}\mathbf{a}\cdot\int_\Omega \rho_o(\dot\gamma-\dot\gamma_c)\mathbf{U}_1(\mathbf{x}-\mathbf{x}_c)\,d\mathbf{x}. \tag{127}$$

Note that from (76), (79) we have

$$\int_\Omega \rho_o\dot\gamma\,d\mathbf{x} = M\dot\gamma_c, \tag{128}$$

and thus

$$\int_\Omega \rho_o(\dot\gamma-\dot\gamma_c)^2\,d\mathbf{x} = \int_\Omega \rho_o\dot\gamma^2\,d\mathbf{x} - M\dot\gamma_c^2. \tag{129}$$

Using this, (86), and the definition of the mass-distribution matrix, we find that

$$\int_\Omega \rho_o(\dot\gamma-\dot\gamma_c)^2\,d\mathbf{x} = \mathcal{M}\mathbf{v}_s\cdot\mathbf{v}_s. \tag{130}$$

Finally, from the definition of $\mathcal{B}$ we have

$$-\mathbf{W}\mathbf{a}\cdot\int_\Omega \rho_o(\dot\gamma-\dot\gamma_c)\mathbf{U}_1(\mathbf{x}-\mathbf{x}_c)\,d\mathbf{x} = -\boldsymbol{\omega}\cdot(\mathcal{C}\mathbf{A})^T\mathbf{v}_s. \tag{131}$$

Therefore, the kinetic energy takes the form

$$KE = \frac{1}{2}\begin{pmatrix}\mathbf{v}_c & \mathbf{v}_s & \boldsymbol{\omega}\end{pmatrix}\cdot\begin{pmatrix} M\mathbf{I} & 0 & 0\\ 0 & |\mathbf{a}|^2\mathcal{M} & \mathcal{C}\mathbf{A}\\ 0 & (\mathcal{C}\mathbf{A})^T & \mathcal{I}\end{pmatrix}\begin{pmatrix}\mathbf{v}_c\\ \mathbf{v}_s\\ \boldsymbol{\omega}\end{pmatrix}. \tag{132}$$

Because $KE = 0$ if and only if $\dot{\mathbf{y}} = 0$ almost everywhere in $\Omega$, to prove the thesis it is sufficient to show that given any vector $(\mathbf{v}_c, \mathbf{v}_s, \boldsymbol{\omega})$ whose components are not simultaneously all equal to zero, there exists an open set in $\Omega$ on which $\dot{\mathbf{y}} \neq 0$. We see this by contradiction. Assume the existence of vectors $\mathbf{v}_c$, $\mathbf{v}_s$, $\boldsymbol{\omega}$ not all identically zero for which $\dot{\mathbf{y}} = 0$ almost everywhere in $\Omega$, i.e.,

$$\mathbf{R}\mathbf{W}\,[\mathbf{U}_1(\mathbf{x}-\mathbf{x}_c) + (\gamma-\gamma_c)\mathbf{a}] + (\dot\gamma-\dot\gamma_c)\mathbf{R}\mathbf{a} + \mathbf{v}_c = 0, \qquad \text{a.e. } \mathbf{x}\in\Omega. \tag{133}$$

Integration on $\Omega$ and use of the definitions $(61)_{2,4}$ of center of mass $\mathbf{x}_c$ and of $\gamma_c$ yield

$$\mathbf{v}_c = 0. \tag{134}$$

After canceling the rotation $\mathbf{R}$, we get the condition

$$\mathbf{W}\,[\mathbf{U}_1(\mathbf{x}-\mathbf{x}_c) + (\gamma-\gamma_c)\mathbf{a}] + (\dot\gamma-\dot\gamma_c)\mathbf{a} = 0, \qquad \text{a.e. } \mathbf{x}\in\Omega. \tag{135}$$

Now choose

$$\mathbf{x} = c_1\mathbf{n} + \xi_1\mathbf{n}_1^\perp + \xi_2\mathbf{n}_2^\perp, \tag{136}$$

with $c_1 \in (s_k, s_{k+1})$ for some $k$, $0 \le k \le N$, and with $(\xi_1, \xi_2) \in \mathcal{A}(c_1)$. Substitute into (135) and use the arbitrariness of $\xi_1, \xi_2$ to get

$$\mathbf{W}\mathbf{U}_1\mathbf{n}_1^\perp = 0, \tag{137}$$

$$\mathbf{W}\mathbf{U}_1\mathbf{n}_2^\perp = 0. \tag{138}$$

Because $\mathbf{n}_1^\perp$, $\mathbf{n}_2^\perp$ are linearly independent vectors and $\mathbf{U}_1$ is a pure stretch, this proves that $\mathbf{W}$ and thus $\boldsymbol{\omega}$ vanish. Then (135) becomes

$$\dot{\gamma}(\mathbf{x}\cdot\mathbf{n}) = \dot{\gamma}_c, \tag{139}$$

or, equivalently,

$$\begin{aligned} 0 &= \dot{\gamma}_c, \\ -\dot{s}_1 &= \dot{\gamma}_c, \\ -\dot{s}_1 + \dot{s}_2 &= \dot{\gamma}_c, \\ &\cdots \\ -\dot{s}_1 + \dot{s}_2 + \cdots + (-1)^N \dot{s}_N &= \dot{\gamma}_c. \end{aligned} \tag{140}$$

This system of $N$ equations admits the unique solution $\dot{\gamma}_c = 0, \dot{s}_i = 0, i = 1, 2, \ldots, N$. Therefore $\mathbf{v}_s = 0$, which gives a contradiction, proving that $\mathbb{M}$ is positive-definite on $(0, T)$.

Thus, as $t$ nears $T$, $\det \mathbb{M}(t)$ remains bounded strictly away from zero, or there is an increasing sequence $t_i$ such that $\det \mathbb{M}(t_i) \to 0$. Let $\mathbf{e}^i = (\mathbf{v}_c^i, \mathbf{v}_s^i, \boldsymbol{\omega}^i) \in \mathbb{R}^3 \times \mathbb{R}^N \times \mathbb{R}^3$ be a normalized eigenvector corresponding to a vanishing eigenvalue,

$$\mathbb{M}(t_i)\mathbf{e}^i \to 0, \tag{141}$$

so that $KE = \frac{1}{2}\mathbf{e}^i \cdot \mathbb{M}(t_i)\mathbf{e}^i \to 0$. Assume that after extracting a subsequence, $\mathbf{e}^i$ converges. Here, $KE$ is still given by the formula (126), with $t = t_i$ and $(\mathbf{v}_c, \mathbf{v}_s, \boldsymbol{\omega}) = (\mathbf{v}_c^i, \mathbf{v}_s^i, \boldsymbol{\omega}^i)$. We get again, using the boundedness of $\dot{\mathbf{y}}$ on this sequence, that $\dot{\mathbf{y}}(\mathbf{x}, t_i) \to 0$ for a.e. $\mathbf{x} \in \Omega$, and we follow the steps (133)–(138) to derive that

$$\mathbf{v}_c^i \to 0, \qquad \boldsymbol{\omega}^i \to 0. \tag{142}$$

So, the approximate null-vector $\mathbf{e}^i$ tends to the subspace $(0, 0, 0; \mathbf{v}_s; 0, 0, 0)$. We also get that for a.e. $\mathbf{x} \in \Omega$,

$$\gamma^i(\mathbf{x}\cdot\mathbf{n}, t_i) - \gamma_c^i(t_i) \to 0, \tag{143}$$

where $\gamma^i$, $\gamma_c^i$ are calculated using $\mathbf{v}_s^i$. Suppose for a moment that the $s_j(t_i)$, $j \in \{0, \ldots, N+1\}$ satisfy $\liminf(s_{j+1}(t_i) - s_j(t_i)) > 0$, then each region $\Omega \cap \{s_j(t_i) < \mathbf{x}\cdot\mathbf{n} < s_{j+1}(t_i)\}$ has measure that is bounded (independent of $i$) away from zero and therefore the argument (139)–(140) also can be repeated. This gives

$$\mathbf{v}_s^i \to 0, \tag{144}$$

which contradicts the normalization $|\mathbf{e}^i| = 1$. Thus, for some $k \in \{0, \ldots, N\}$ we have, for a subsequence of $t_i$, that $s_{k+1}(t_i) - s_k(t_i) \to 0$.

If only one pair $s_k, s_{k+1}$ collides at T, in the sense explained in the Proposition, then we are unable to impose (143) for $s_k < \mathbf{x} \cdot \mathbf{n} < s_{k+1}$. More precisely, if we denote $v_s^i = (\dot{s}_1^i, -\dot{s}_2^i, \ldots, -(-1)^N \dot{s}_N^i)$, then (143) leads to a system like (140) but with the $k^{\text{th}}$ equation removed, i.e.,

$$\begin{aligned}
0 - \dot{\gamma}_c &\to 0,\\
-\dot{s}_1^i - \dot{\gamma}_c &\to 0,\\
-\dot{s}_1^i + \dot{s}_2^i - \dot{\gamma}_c &\to 0,\\
&\cdots\\
-\dot{s}_1^i + \dot{s}_2^i + \cdots + (-1)^{k-1}\dot{s}_{k-1}^i - \dot{\gamma}_c &\to 0,\\
-\dot{s}_1^i + \dot{s}_2^i + \cdots + (-1)^{k+1}\dot{s}_{k+1}^i - \dot{\gamma}_c &\to 0,\\
&\cdots\\
-\dot{s}_1^i + \dot{s}_2^i + \cdots + (-1)^N s_N^i - \dot{\gamma}_c &\to 0.
\end{aligned} \tag{145}$$

This yields that $\mathbf{e}^i \to (0, 0, 0; 0, \ldots, \alpha, -\alpha, \ldots, 0; , 0, 0, 0)$ with $\alpha = \frac{\pm 1}{\sqrt{2}}$ in the $3 + k^{\text{th}}$ place. □

*Remark* 8.2. For the case in which the $1^{\text{st}}$ or $N^{\text{th}}$ interface passes out of the body, the form of the limiting null-vector can be read off of (145).

As we approach the question of well-posedness, the delicate situation $\det \mathbb{M}(t_i) \to 0$ will require special care. However, even while $\det \mathbb{M}$ is bounded away from zero, the solution could blow up. That this does not occur, as a consequence of the boundedness of the applied force and torque, is proved using an explicit form of the dissipation inequality (cf. (50)), which is of independent interest.

**Proposition 8.2.** *(Dissipation) Suppose that the force $\mathcal{F}$ and torque $\mathcal{T}_{\text{o}}$ are assigned continuous functions of $(\Theta, t) \in \mathcal{D} = \mathbb{R}^3 \times \mathbb{R}^3 \times (0, L)^N \times \mathbb{R}^N \times \text{SO}(3) \times \mathbb{R}^3 \times [t_0 - \delta, t_0 + \delta)$, the kinetic relation satisfies $\xi f(\xi) \geq 0$, and let $\Theta \in C^1$ be a local solution near $t_0$ of* (112)–(117)*. Then,*

$$\begin{aligned}
\frac{d}{dt}(KE) &= \frac{d}{dt}\left[\frac{1}{2}(\mathbf{v}_c, \mathbf{v}_s, \boldsymbol{\omega}) \cdot \mathbb{M}(\mathbf{v}_c, \mathbf{v}_s, \boldsymbol{\omega})\right]\\
&\leq \mathbf{v}_c \cdot \mathcal{F} + \left[\mathcal{F}_s - \frac{1}{M}(\mathcal{F} \cdot \mathbf{Ra})\varpi\right] \cdot \mathbf{v}_s\\
&\quad + \mathbf{R}\boldsymbol{\omega} \cdot [\mathcal{T}_{\text{o}} - (\mathbf{y}_c - \mathbf{y}_{\text{o}}) \wedge \mathcal{F}]
\end{aligned} \tag{146}$$

*holds near $t_0$. The right-hand side of* (146) *is the rate of work of the applied forces and torques.*

*Proof.* The symmetry of $\mathbb{M}$ yields

$$\frac{d}{dt}(KE) = \frac{1}{2}\dot{\mathbb{M}}(\mathbf{v}_c, \mathbf{v}_s, \boldsymbol{\omega}) \cdot (\mathbf{v}_c, \mathbf{v}_s, \boldsymbol{\omega}) + \mathbb{M}(\dot{\mathbf{v}}_c, \dot{\mathbf{v}}_s, \dot{\boldsymbol{\omega}}) \cdot (\mathbf{v}_c, \mathbf{v}_s, \boldsymbol{\omega}). \tag{147}$$

Expanding the derivative of $\mathbb{M}$ in the first term and using the equations of motion in the second term, one gets, after a few simplifications,

$$\begin{aligned}\frac{d}{dt}(KE) \;=\; & |\mathbf{a}|^2 \left[\frac{1}{2}\dot{\mathcal{M}}\mathbf{v}_s \cdot \mathbf{v}_s - \boldsymbol{\eta} \cdot \mathbf{v}_s\right] \\ & + \left[-\frac{1}{2}\dot{\mathcal{I}}\boldsymbol{\omega} \cdot \boldsymbol{\omega} - \mathcal{C}(\mathbf{A}\boldsymbol{\omega} \wedge \boldsymbol{\omega}) \cdot \mathbf{v}_s + |\mathbf{A}\boldsymbol{\omega}|^2(\mathcal{M}\mathbf{s} - \boldsymbol{\zeta}) \cdot \mathbf{v}_s\right] \\ & + \mathbf{v}_c \cdot \mathcal{F} + \left[\mathcal{F}_s - \frac{1}{M}(\mathcal{F} \cdot \mathbf{Ra})\boldsymbol{\varpi}\right] \cdot \mathbf{v}_s \\ & + \mathbf{R}\boldsymbol{\omega} \cdot [\mathcal{T}_{\mathrm{o}} - (\mathbf{y}_c - \mathbf{y}_{\mathrm{o}}) \wedge \mathcal{F}].\end{aligned} \tag{148}$$

Making use of the sum rule (87), a straightforward calculation shows that

$$\frac{1}{2}\dot{\mathcal{M}}\mathbf{v}_s \cdot \mathbf{v}_s = \mathbf{v}_s \cdot \boldsymbol{\eta} + \frac{1}{2}\sum_{i=0}^{N} \hat{\rho}_o(s_i)\dot{s}_i{}^3. \tag{149}$$

Taking the derivative of the inertia tensor $\mathcal{I}$ as defined in (106), one finds

$$\begin{aligned}\dot{\mathcal{I}}\boldsymbol{\omega} \cdot \boldsymbol{\omega} \;=\; & 2[|\boldsymbol{\omega}|^2\mathbf{a} - (\boldsymbol{\omega} \cdot \mathbf{a})\boldsymbol{\omega}] \cdot \int_\Omega \rho_o(\dot{\gamma} - \dot{\gamma}_c)\mathbf{U}_1(\mathbf{x} - \mathbf{x}_c)\, d\mathbf{x} \\ & + 2[|\boldsymbol{\omega}|^2|\mathbf{a}|^2 - (\mathbf{a} \cdot \boldsymbol{\omega})^2] \int_0^L \hat{\rho}_o(\gamma - \gamma_c)(\dot{\gamma} - \dot{\gamma}_c)\, dr.\end{aligned} \tag{150}$$

Working with the definitions given in Section 7.1, we get

$$|\boldsymbol{\omega}|^2\mathbf{a} - (\boldsymbol{\omega} \cdot \mathbf{a})\boldsymbol{\omega} \;=\; -\mathbf{A}\boldsymbol{\omega} \wedge \boldsymbol{\omega}, \tag{151}$$

$$|\mathbf{a}|^2|\boldsymbol{\omega}|^2 - (\boldsymbol{\omega} \cdot \mathbf{a})^2 \;=\; |\mathbf{A}\boldsymbol{\omega}|^2, \tag{152}$$

$$\int_\Omega \rho_o(\dot{\gamma} - \dot{\gamma}_c)\mathbf{U}_1(\mathbf{x} - \mathbf{x}_c)\, d\mathbf{x} \;=\; \mathcal{C}^T\mathbf{v}_s, \tag{153}$$

$$\int_0^L \hat{\rho}_o\gamma(\dot{\gamma} - \dot{\gamma}_c) \;=\; \mathcal{M}\mathbf{s} \cdot \mathbf{v}_s - \boldsymbol{\zeta} \cdot \mathbf{v}_s, \tag{154}$$

and these, substituted into (150), give

$$-\frac{1}{2}\dot{\mathcal{I}}\boldsymbol{\omega} \cdot \boldsymbol{\omega} - \mathbf{v}_s \cdot \mathcal{C}(\mathbf{A}\boldsymbol{\omega} \wedge \boldsymbol{\omega}) + |A\boldsymbol{\omega}|^2(\mathcal{M}\mathbf{s} - \boldsymbol{\zeta}) \cdot \mathbf{v}_s = 0. \tag{155}$$

Thus,

$$\begin{aligned}\frac{d}{dt}(KE) \;=\; & |\mathbf{a}|^2 \left[\frac{1}{2}\sum_{i=0}^{N} \hat{\rho}_o(s_i)\dot{s}_i^3 + \alpha\boldsymbol{\tau} \cdot \mathbf{v}_s\right] \\ & + \mathbf{v}_c \cdot \mathcal{F} + \left[\mathcal{F}_s - \frac{1}{M}(\mathcal{F} \cdot \mathbf{Ra})\boldsymbol{\varpi}\right] \cdot \mathbf{v}_s \\ & + \mathbf{R}\boldsymbol{\omega} \cdot [\mathcal{T}_{\mathrm{o}} - (\mathbf{y}_c - \mathbf{y}_{\mathrm{o}}) \wedge \mathcal{F}].\end{aligned} \tag{156}$$

To complete the proof, we substitute the balances of linear momentum at interfaces (103) into the first term,

$$\begin{aligned}\frac{d}{dt}(KE) \;=\; & \sum_{i=0}^{N} \frac{1}{2}(-1)^{i+1}\dot{s}_i \mathcal{A}_i(\tau_i^+ + \tau_i^-) \\ & + \mathbf{v}_c \cdot \mathcal{F} + \left[\mathcal{F}_s - \frac{1}{M}(\mathcal{F}\cdot \mathbf{Ra})\varpi\right]\cdot \mathbf{v}_s \\ & + \mathbf{R}\boldsymbol{\omega}\cdot[\mathcal{T}_{\mathrm{o}} - (\mathbf{y}_c - \mathbf{y}_{\mathrm{o}})\wedge \mathcal{F}],\end{aligned} \tag{157}$$

the first term on the right of which is nonpositive by the condition $\xi f(\xi) \geq 0$. □

To exploit the dissipation inequality (146), we need the following inequality of Gronwall type. For this purpose, we recycle the notation above so the intended application is immediately clear; the notation is local to the lemma.

**Lemma 8.1.** *Let $\mathbb{M} \in C^0([0,T])$ be a positive-definite $N \times N$ matrix with minimum eigenvalue $\lambda_{\min} > \varepsilon > 0$ on $[0,T]$ and let $\mathbf{g} \in C^0([0,T],\mathbb{R}^N)$. Suppose $\mathbf{v} \in C^0([0,T],\mathbb{R}^N)$ satisfies*

$$\frac{1}{2}\mathbf{v}\cdot\mathbb{M}\mathbf{v} \leq \int_0^t \mathbf{g}(\tau)\cdot\mathbf{v}(\tau)\,d\tau, \tag{158}$$

*for $0 < t < T$. Then*

$$|\mathbf{v}(t)| \leq \frac{1}{\varepsilon}\int_0^t |\mathbf{g}(\tau)|\,d\tau. \tag{159}$$

*Proof.* We have $\frac{1}{2}\mathbf{v}\cdot\mathbb{M}\mathbf{v} \leq \int_0^t |\mathbf{g}(\tau)||\mathbf{v}(\tau)|\,d\tau$. Let

$$w(t) = \int_0^t |\mathbf{g}(\tau)||\mathbf{v}(\tau)|\,d\tau, \tag{160}$$

so that

$$\frac{1}{2}\varepsilon|\mathbf{v}|^2 \leq \frac{1}{2}\mathbf{v}\cdot\mathbb{M}\mathbf{v} \leq w. \tag{161}$$

Multiply this by $|\mathbf{g}|^2$ and use $\dot{w} = |\mathbf{g}||\mathbf{v}|$ to get

$$\frac{1}{2}\varepsilon|\dot{w}|^2 \leq |\mathbf{g}|^2 w. \tag{162}$$

Using that $\dot{w} \geq 0$ and $w(0) = 0$, take the square root, divide by $\sqrt{w}$, and integrate:

$$w(t) \leq \frac{1}{2\varepsilon}\left(\int_0^t |\mathbf{g}(\tau)|\,d\tau\right)^2. \tag{163}$$

Use (161) again and we get the result. □

Now we can prove well-posedness up to the time that a pair of interfaces collides.

**Theorem 8.1.** *Suppose* $\rho_o = const. > 0$ *and assume that* $\Omega$ *and* $\mathbf{n}$ *are such that* $|\mathcal{A}(s)|$ *is Lipschitz. Assume also that the forces* $\mathcal{F}$ *and* $\mathcal{F}_s$ *and torque* $\mathcal{T}_o$ *are assigned bounded functions of* $(\Theta, t) \in \mathcal{D} = \mathbb{R}^3 \times \mathbb{R}^3 \times (0, L)^N \times \mathbb{R}^N \times \mathrm{SO}(3) \times \mathbb{R}^3 \times [0, \infty)$ *satisfying a Lipschitz condition in* $\Theta = (\mathbf{y}_c, \mathbf{v}_c, \mathbf{s}, \mathbf{v}_s, \mathbf{R}, \boldsymbol{\omega})$*, and that the kinetic relation* $f(\tau)$ *with* $\tau f(\tau) \geq 0$ *is invertible with Lipschitz inverse* $f^{-1}$*. Then, there is a unique solution* $\Theta \in C^1(\mathcal{D})$ *of* (112)–(117) *on an interval* $[0, t^*)$ *corresponding to the initial condition* $\Theta(0) = \Theta_0 = (\mathbf{y}_c^0, \mathbf{v}_c^0, \mathbf{s}^0, \mathbf{v}_s^0, \mathbf{R}^0, \boldsymbol{\omega}^0) \in \mathcal{D}$ *with* $\mathbf{s}^0 = (s_1^0, \ldots, s_N^0)$ *satisfying* $0 < s_1^0 < s_2^0 < \cdots < s_N^0 < L$*. Either* $t^* = \infty$ *or a pair of interfaces collides at* $t^*$ *in the sense of* Proposition 8.1.

*Proof.* We use (118) and the invertibility of the kinetic relation to eliminate $\tau^-$ from the right-hand side of the equations of motion (112)–(117); this introduces $f^{-1}$ on the right-hand side of these equations. On bounded subsets of $\mathcal{D}$, the right-hand side of the equations of motion is Lipschitz under these hypotheses, so the standard existence/uniqueness/extension theorem of O.D.E. (see, e.g., Hartman [18]) gives the existence of a unique solution up to time $t^*$ with $\det \mathbb{M} > 0$ on $(0, t^*)$. There must be an increasing sequence $t_i \to t^*$ on which $\det \mathbb{M}(t_i) \to 0$; otherwise, $\det \mathbb{M}(t) > \varepsilon > 0$ for some $\varepsilon > 0$ on $(0, t^*)$. However, in the latter case we could invoke Proposition 8.2 and Lemma 8.1. Since SO(3) and the forces and torques are bounded, and $|\mathbf{y}_c - \mathbf{y}_o|$ grows no faster than $c_1 + c_2 t + \frac{1}{M} \max |\mathcal{F}| t^2$ from integration of the equations of motion for the center of mass, then (158) holds with $\mathbf{g}$ a quadratic polynomial. Hence, the solution $\Theta(t)$ would remain bounded on $[0, t^*]$. In this case an initial value problem with initial condition at a time $t_0$ that is sufficiently close to $t^*$ would have a unique extension (with $\det \mathbb{M} > 0$) to an interval $(t_0, t_1)$ with $t_1 > t^*$. Hence, only the first alternative is possible, which proves the theorem. □

*Remark* 8.3. The conditions on the domain $\Omega$ and on $\rho_o$ can be weakened. More importantly, the invertibility condition on the kinetic relation forbids some kinetic relations that embody a critical nonzero configurational force for motion of the interface.

We record below the form of equilibrium solutions that follows immediately from the definitions.

**Lemma 8.2.** *(Equilibrium solutions) Assume the hypotheses of Theorem 8.1, suppose that* $\mathcal{F} = \mathcal{F}_s = \mathcal{T}_o = 0$*, and assign that the initial conditions* $\Theta(0) = (\mathbf{y}_c^0, 0, \mathbf{s}^0, 0, \mathbf{R}^0, 0)$ *with* $\mathbf{s}^0 = (s_1^0, \ldots, s_N^0)$ *satisfying* $0 < s_1^0 < s_2^0 < \cdots < s_N^0 < L$*. Then* $\Theta(t) = \Theta(0)$ *is the unique solution to the equations of motion* (112)–(117) *(in* $C^0$*) for* $t \geq 0$*. This solution corresponds to a piecewise rigid motion* $\mathbf{y}(\mathbf{x}, t) = \mathbf{y}_0(\mathbf{x})$ *of the form* (44) *that meets the initial conditions*

$$\mathbf{y}(\mathbf{x}, 0) = \mathbf{y}_0(\mathbf{x}), \tag{164}$$

$$\dot{\mathbf{y}}(\mathbf{x}, 0) = 0. \tag{165}$$

*Proof.* This follows immediately when it is noticed that the continuity/invertibility of $f$ and $\xi f(\xi) \geq 0$ imply that $f(0) = 0$. □

As is common in mechanical systems, the system of equations (112)–(117) with no applied forces or torques is dissipative but also has conservation laws.

**Proposition 8.3.** *(Conservation laws) Assume the hypotheses of Theorem 8.1 and* $\mathcal{F} = \mathcal{F}_s = \mathcal{T}_o = 0$. *Then, on* $(0, t^*)$ *we have*

*(i) Conservation of linear momentum:*

$$M\dot{\mathbf{v}}_c = 0; \tag{166}$$

*(ii) Conservation of rotational momentum:*

$$\frac{d}{dt}[\mathbf{R}((\mathcal{C}\mathbf{A})^T\mathbf{v}_s + \mathcal{I}\boldsymbol{\omega})] = 0. \tag{167}$$

*Proof.* These are easy consequences of the equations of motion. □

### *8.3. Remarks on the Annihilation and Nucleation of Interfaces*

In this section we do a preliminary analysis of the annihilation and nucleation of interfaces. We discuss mainly annihilation, nucleation being completely analogous. We treat only the case in which a single pair of interfaces $(s_k, s_{k+1})$ collides at an internal point. For simplicity, we suppose throughout this section (and Appendix 1) that the following hypothesis holds.

**Hypothesis $\mathcal{H}$.** Assume the hypotheses of Theorem 8.1 and consider a resulting solution $\Theta(t) = (\mathbf{y}_c, \mathbf{v}_c, \mathbf{s}, \mathbf{v}_s, \mathbf{R}, \boldsymbol{\omega})$, $t \in [0, t^*)$ of (112)–(117) with $\det \mathbb{M}(t) > 0$ on $(0, t^*)$ and an increasing sequence $t_i \to t^*$ with $s_{k+1}(t_i) - s_k(t_i) \to 0$ for some $k \in \{1, \ldots, N-1\}$. Assume that for $j \neq k$, $j \in \{0, \ldots, N\}$, $\inf_{t\in(0,t^*)}(s_{j+1}(t) - s_j(t)) > 0$, $t \in [0, t^*)$. This implies in particular that there is an interval $(a, b)$ on which $0 < a < s_k(t) < s_{k+1}(t) < b < L$, and on this interval we assume that $|\mathcal{A}(s)| > c > 0$.

The last assumption is to ensure that $\hat{M}(s)$ also satisfies a lower Lipschitz condition $|\hat{M}(s_2) - \hat{M}(s_1) \geq \rho_o c|s_2 - s_1|$, $s_1, s_2 \in (a, b)$. Note that this would not typically be satisfied at a boundary annihilation, so these might call for a different treatment.

A possibility based on what we have shown so far is that $s_{k+1}(t_i) - s_k(t_i) \to 0$ on the particular subsequence $t_i$ but that $s_{k+1}(t) - s_k(t)$ oscillates near $t^*$. Thus, in order to establish that a collision actually occurs, the first issue is to clarify the limiting behavior of $s_{k+1}(t) - s_k(t)$ near $t^*$. Recall the form of $\mathbb{M}$ given by (125). We first observe that $\mathbb{M}^{-1}$ has a rather special structure. Recall the definition (71) of the mass-distribution matrix $\mathcal{M}$, which is typically the main submatrix of $\mathbb{M}$. First we note that, while $\mathcal{M}$ is a full

matrix, $\mathcal{M}^{-1}$ has the particular tridiagonal structure,

$$\mathcal{M}^{-1} = \begin{pmatrix} a_1 & b_2 & 0 & \dots & & \dots & 0 \\ b_2 & \ddots & & & & & \vdots \\ 0 & & a_{k-1} & b_k & 0 & & \\ \vdots & & b_k & a_k & b_{k+1} & & \vdots \\ & & 0 & b_{k+1} & a_{k+1} & & 0 \\ \vdots & & & & & \ddots & b_N \\ 0 & \dots & & \dots & 0 & b_N & a_N \end{pmatrix}, \tag{168}$$

where

$$a_k = \frac{\hat{M}(s_{k+1}) - \hat{M}(s_{k-1})}{(\hat{M}(s_{k-1}) - \hat{M}(s_k))(\hat{M}(s_k) - \hat{M}(s_{k+1}))}, \qquad b_k = \frac{1}{\hat{M}(s_{k-1}) - \hat{M}(s_k)}. \tag{169}$$

This can be verified by direct calculation; it simplifies greatly the implementation of these equations for large numbers of interfaces. Hence, as $s_{k+1}(t_i) - s_k(t_i) \to 0$, the $2 \times 2$ submatrix of $\mathcal{M}^{-1}$ given by

$$\begin{pmatrix} a_k & b_{k+1} \\ b_{k+1} & a_{k+1} \end{pmatrix} \tag{170}$$

is singular, but (under our hypotheses) the remaining entries in $\mathcal{M}^{-1}$ are bounded on $(0, t^*)$.

Our method will be based on the procedure of concealing the singularity inside a change of variables. Consider the change of variables $\mathbf{w}_s = \mathcal{M}\mathbf{v}_s$ in the kinetic energy. That is, define the positive-definite symmetric matrix $\tilde{\mathbb{M}}$ on $(0, t^*)$ by

$$(\mathbf{v}_c, \mathbf{w}_s, \boldsymbol{\omega}) \cdot \tilde{\mathbb{M}} \begin{pmatrix} \mathbf{v}_c \\ \mathbf{w}_s \\ \boldsymbol{\omega} \end{pmatrix} = (\mathbf{v}_c, \mathcal{M}^{-1}\mathbf{w}_s, \boldsymbol{\omega}) \cdot \mathbb{M} \begin{pmatrix} \mathbf{v}_c \\ \mathcal{M}^{-1}\mathbf{w}_s \\ \boldsymbol{\omega} \end{pmatrix}. \tag{171}$$

We want to apply the dissipation inequality (146), but using $\tilde{\mathbb{M}}$ and the new variables $\mathbf{w}_s$. For this, we need some properties of its terms.

**Proposition 8.4.** *Assume Hypotheses $\mathcal{H}$. The determinant of the positive-definite matrix $\tilde{\mathbb{M}}$ is bounded strictly away from* 0 *on* $(0, t^*)$. *Suppose in addition that the external distributed force $\mathcal{F}_s$ satisfies the componentwise Lipschitz condition, $|(\mathcal{F}_s)_{k+1} - (\mathcal{F}_s)_k| < C|s_{k+1} - s_k|$. The quantities $\mathcal{F}_s \cdot \mathcal{M}^{-1}\mathbf{w}_s$ and $\varpi \cdot \mathcal{M}^{-1}\mathbf{w}_s$ that appear on the right-hand side of* (146) *satisfy the bounds*

$$\begin{aligned} |\mathcal{F}_s \cdot \mathcal{M}^{-1}\mathbf{w}_s| &\le C|\mathbf{w}_s|, \\ |\varpi \cdot \mathcal{M}^{-1}\mathbf{w}_s| &\le C|\mathbf{w}_s|. \end{aligned} \tag{172}$$

*Proof.* $\tilde{\mathbb{M}}$ is positive-definite on $(0, t^*)$ because $\mathbb{M}$ is positive-definite and $\mathcal{M}$ is invertible on this interval. The proof that $\det \tilde{\mathbb{M}}$ is bounded away from zero is rather involved and is therefore presented separately in Appendix 1.

Now consider the quantities $|\mathcal{F}_s \cdot \mathcal{M}^{-1}\mathbf{w}_s|$ and $|\varpi \cdot \mathcal{M}^{-1}\mathbf{w}_s|$. By direct calculation using the form (168) of $\mathcal{M}^{-1}$,

$$\begin{aligned}
|\mathcal{F}_s \cdot \mathcal{M}^{-1}\mathbf{w}_s| \;\le\; & C_1|\mathbf{w}_s| + |(\mathbf{w}_s)_k||a_k(\mathcal{F}_s)_k + b_{k+1}(\mathcal{F}_s)_{k+1}| \\
& + |(\mathbf{w}_s)_{k+1}||b_{k+1}(\mathcal{F}_s)_k + a_{k+1}(\mathcal{F}_s)_{k+1}|, \\
\le\; & C_1|\mathbf{w}_s| + |(\mathbf{w}_s)_k|\,[|a_k + b_{k+1}||(\mathcal{F}_s)_k| + |b_{k+1}||(\mathcal{F}_s)_{k+1} - (\mathcal{F}_s)_k|]\,, \\
& + |(\mathbf{w}_s)_{k+1}|\,[|b_{k+1} + a_{k+1}||(\mathcal{F}_s)_k| + |a_{k+1}||(\mathcal{F}_s)_{k+1} - (\mathcal{F}_s)_k|]\,. \quad (173)
\end{aligned}$$

Focus on the two terms containing $|a_{k+1}|$ and $|b_{k+1}|$. Using the hypothesis on $\mathcal{F}_s$ and the lower Lipschitz condition on $\hat{M}(s)$, it follows that these are bounded. Therefore,

$$\begin{aligned}
|\mathcal{F}_s \cdot \mathcal{M}^{-1}\mathbf{w}_s| \;\le\; & C_2|\mathbf{w}_s| + C_3[|(\mathbf{w}_s)_k|(|a_k + b_{k+1}| + 1) \\
& + |(\mathbf{w}_s)_{k+1}|(||b_{k+1} + a_{k+1}| + 1)]. \quad (174)
\end{aligned}$$

However, while $a_k, b_{k+1}, a_{k+1}$ are singular, the sums $a_k + b_{k+1}$ and $b_{k+1} + a_{k+1}$ are bounded:

$$a_k + b_{k+1} = \frac{1}{\hat{M}(s_k) - \hat{M}(s_{k-1})}, \qquad b_{k+1} + a_{k+1} = \frac{1}{\hat{M}(s_{k+2}) - \hat{M}(s_{k+1})}. \quad (175)$$

Combining (174) and (175), we get the first of (172). The second of (172) is proved in exactly the same way, as $\varpi$ is bounded and satisfies the same kind of Lipschitz condition as $\mathcal{F}_s$. □

**Theorem 8.2.** *The limit $s_{k+1}(t) - s_k(t) \to 0$ as $t \to t^*$ from below. Similarly, the limits $s_i(t) \to s_i^*$, and $\mathbf{v}_c \to \mathbf{v}_c^*$ exist as $t \to t^*$.*

*Proof.* We change variables $\mathbf{w}_s = \mathcal{M}\mathbf{v}_s$ in the dissipation inequality (146) and use Proposition 8.4. This gives

$$\frac{d}{dt}(\mathbf{v}_c, \mathbf{w}_s, \boldsymbol{\omega}) \cdot \tilde{\mathbb{M}} \begin{pmatrix} \mathbf{v}_c \\ \mathbf{w}_s \\ \boldsymbol{\omega} \end{pmatrix} \le C|(\mathbf{v}_c, \mathbf{w}_s, \boldsymbol{\omega})|, \quad (176)$$

where $C$ depends on the constant appearing in (172) as well as on the bounds on forces and torques. By Proposition 8.4 $\det \tilde{\mathbb{M}}$ is bounded away from zero, and therefore we can use Lemma 8.1 to conclude that $|\mathbf{v}_c|$, $|\boldsymbol{\omega}|$, and $|\mathbf{w}_s|$ are uniformly bounded on $(0, t^*)$, in particular,

$$|\mathcal{M}\mathbf{v}_s| \le Ct^*, \quad \text{on } (0, t^*). \quad (177)$$

This bounds all components of $\mathbf{v}_s$ on basis vectors that are perpendicular to the limiting null-space of $\mathcal{M}$. In particular, it bounds $\mathbf{v}_s \cdot (0, \ldots, 1, 1, \ldots, 0)$, where the first 1 is in the

$k^{\text{th}}$ place. Hence $|\dot{s}_{k+1}(t) - \dot{s}_k(t)| \leq C$, and therefore, since we have one subsequence on which $s_{k+1}(t_i) - s_k(t_i) \to 0$, that $s_{k+1}(t) - s_k(t) \to 0$. (If not, then there would necessarily be two subsequences of $s_{k+1}(t) - s_k(t)$ that would tend to different limits, implying, by the mean value theorem, that the derivative blows up.) Similarly, $|\dot{s}_i(t)| \leq C$ for $i \neq k, k+1$, and therefore $s_i(t) \to s_i^*$ for $i \neq k, k+1$.

The existence of a limiting value of $\mathbf{v}_c$ is immediate from its equation of motion and boundedness of the force. □

As stated, these theorems do not provide enough information with which to restart an initial-value problem. It is not known if $s_k(t) + s_{k+1}(t)$ has a limit at $t^*$, which would be essential for posing a new initial-value problem. Also, the status of the limiting value of $\omega$, which appears to be intimately connected to the same question for $s_k(t) + s_{k+1}(t)$, is unknown. Furthermore, to restart an initial-value problem, we might expect to have to know the status of the derivatives $\dot{s}_i(t)$ as $t \to t^*$; that this may not be the case is discussed now.

To understand the conditions one should impose at an annihilation in PRBM, we considered the analogous problem for elastic bars with a double-well energy, in the framework of dynamic nonlinear elastic bar theory. These details are completely straightforward and are not recounted here. We assumed the wells were piecewise quadratic with equal elastic moduli $E$. Then we posed initial data for a Riemann-type problem with two interfaces (separating constant states of velocity $v$ and strain $\varepsilon$), leading to an annihilation. See the $x - t$ diagram in Figure 3. It is well known that such problems exhibit nonuniqueness [19], but we chose representative solutions that would satisfy any of the admissibility criteria that have been proposed (of which we are aware) for phase boundaries, and our conclusions are independent of which of these criteria are imposed. To understand what conditions should be imposed in PRBM, we took the natural limit $E \to \infty$ (cf. Section 2). Passage to this limit does not impose any conditions on the speeds of the incoming phase boundaries. At annihilation, two acoustic waves emerge and propagate in opposite directions toward the boundaries (with large velocity), as illustrated in Figure 3. What happens next depends on the boundary conditions. A simple case is that in which the right boundary is free of stress and the left is impedance matched, which is the case illustrated in Figure 3. In that case there is a reflection at the right boundary, and the reflected acoustic wave travels at high speed to the left boundary. This gives rise to a narrow sector bounded by these acoustic waves labelled by the strain-velocity pair $(\varepsilon_3, v_3)$ in Figure 3. We found the (at first) unexpected conclusion that, while velocity and strain remain bounded in the limit of $E \to \infty$, the stress in this sector tends to infinity. Moreover, the impulse at the left boundary arising from the large stress over the short period of time has a finite limit, that in fact exactly balances the momentum in the bar before annihilation.

It then becomes clear that, from the perspective of PRBM, whose initial data following an annihilation correspond to the times after these acoustic waves of infinite speed leave the body, there is a sudden disappearance of momentum, and therefore the *conservation laws for the equations of PRBM apparently should not be used to reset initial data following an annihilation*. Unfortunately, what precisely happens depends on the boundary conditions and the relative stiffnesses of the bodies adjoining the piecewise rigid one. For free boundary conditions at both ends, the lost momentum is trapped in

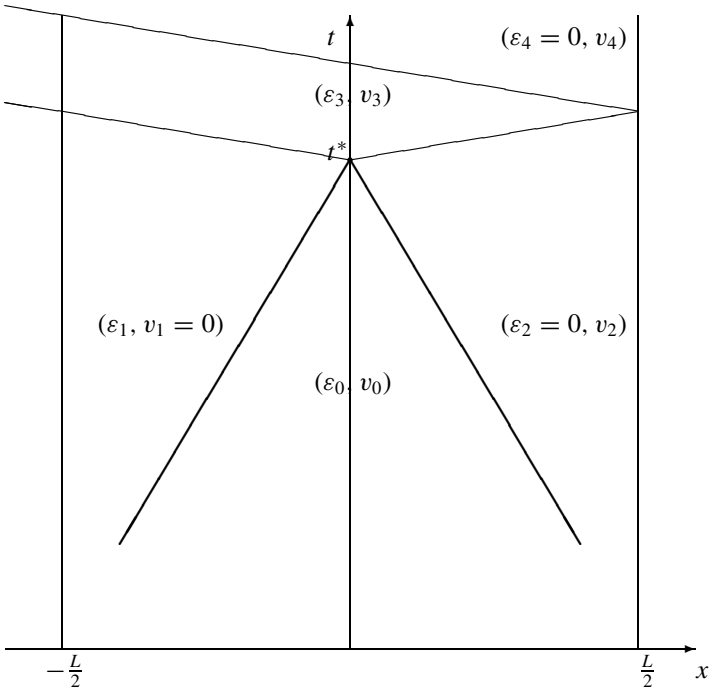


**Fig. 3.** An annihilation in bar theory; see text.

the bar as ringing waves, and in that case one might be led to restart the equations in a different way, consistent with the conservation laws (thanks to the weak continuity of momentum in this case, as $E \to \infty$). This needs further study.

These issues have been confronted by experts in the simpler case of the collision of ordinary rigid bodies, but we cannot seem to find a consensus in the literature of what is the correct criterion.

All of these issues are also of concern at a nucleation. Fortunately, in many small-scale applications, nucleation is rather difficult to achieve without special procedures, and thus is not such a concern.

## 9. Generalizations

### *9.1. Piecewise Rigid Magneto-Mechanics*

Piecewise rigid magneto-mechanics (PRMM) is the obvious generalization of PRBM to include ferromagnetism. A special case of PRBM has been discussed by James [23]. Here we briefly outline the theory in the case where the magnetism is treated by magnetostatics; this is the case when electrical conductivity is negligible and negligible changes of applied fields or loads occur during the time it takes for a light wave to traverse the system.

There are four modifications required in this case. The energy wells for a ferromagnetic material need to be used; the equations of magnetostatics that determine the magnetic

field from the magnetization distribution have to be put alongside the other equations; expressions for magnetic forces and torques have to be added on the right-hand sides of the balances of linear and rotational momentum; the formula for the configurational force has to be modified to include magnetic configurational forces. We discuss these modifications in order.

(i) *Energy wells of a ferromagnetic material and the PRMM ansatz.* In all ferromagnetic materials magnetization is accompanied by some amount of deformation. When this deformation is relatively large, the materials are termed magnetostrictive. The largest shape change that accompanies magnetization is found in materials that undergo a phase change; these are ferromagnetic shape memory materials. In the simplest case all these materials are governed by free-energy functions $\varphi(\mathbf{F}, \mathbf{m})$ of deformation gradient $\mathbf{F}$ and magnetization $\mathbf{m}$ that satisfy invariance restrictions of the form $\varphi(\mathbf{F}, \mathbf{m}) = \varphi(\mathbf{RFQ}, \mathbf{Rm})$ for all $\mathbf{R} \in \mathrm{SO}(3)$ and for all $\mathbf{Q} \in \mathcal{P}$, $\mathcal{P}$ being the symmetry group of the undistorted crystal (for the notation, see Section 3.1; see James and Kinderlehrer [21] for further background). PRMM is expected to be appropriate when $\varphi$ grows steeply away from its energy wells. It is worth noting that the current search for new ferromagnetic shape memory materials is based on steep growth (e.g., high magnetic anisotropy and high modulus) as a screening criterion for potential new alloys. Based on this invariance, the energy wells of PRMM are given by

$$\{\mathrm{SO}(3)(\mathbf{U}_1, \mathbf{m}_1), \ldots, \mathrm{SO}(3)(\mathbf{U}_n, \mathbf{m}_n)\}, \tag{178}$$

where $\mathrm{SO}(3)(\mathbf{U}, \mathbf{m})$ is shorthand for $(\mathbf{RU}, \mathbf{Rm})$, $\mathbf{R} \in \mathrm{SO}(3)$ (note that the *same* rotation matrix goes in front of $\mathbf{U}$ and $\mathbf{m}$), and

$$\{(\mathbf{U}_1, \mathbf{m}_1), \ldots, (\mathbf{U}_n, \mathbf{m}_n)\} = \{(\mathbf{QU}_1\mathbf{Q}^T, \mathbf{Qm}_1)\colon\ \mathbf{Q} \in \mathcal{P}\}. \tag{179}$$

In the ferromagnetic shape memory material $Ni_2MnGa$, the energy wells are given by (31) together with the corresponding magnetizations $\mathbf{m}_1 = m_s(1, 0, 0)$, $\mathbf{m}_2 = m_s(0, 1, 0)$, $\mathbf{m}_3 = m_s(0, 0, 1)$, respectively, expressed in the same basis as (31). A typical value of the saturation magnetization $m_s$ for $Ni_2MnGa$ is 600 emu/cm$^3$ about 10 degrees C below transformation temperature. Now we discuss the generalization of the ansatz (44) to PRMM. As formulated above (following James and Kinderlehrer [21]), the total free energy is

$$\int_\Omega \varphi\left(\nabla \mathbf{y}(\mathbf{x}), \mathbf{m}(\mathbf{y}(\mathbf{x}))\right)\, d\mathbf{x}. \tag{180}$$

That is, the magnetization is composed with the deformation, reflecting the fact that $\mathbf{m}$ is supported on the deformed configuration $\mathbf{y}(\Omega)$. Keeping this in mind, it is easy to write down the analog of the ansatz (44) for the simple laminate:

$$\begin{aligned} \mathbf{y}(\mathbf{x}, t) &= \mathbf{R}(t)\left(\mathbf{U}_1\mathbf{x} + \mathbf{a}\int_0^{\mathbf{x}\cdot\mathbf{n}} \chi(s, t)\, ds\right) + \mathbf{c}(t), \\ \mathbf{m}(\mathbf{y}(\mathbf{x}, t), t) &= \mathbf{R}(t)(\chi(\mathbf{x}\cdot\mathbf{n}, t)\hat{\mathbf{R}}\mathbf{m}_2 + (1 - \chi(\mathbf{x}\cdot\mathbf{n}, t))\mathbf{m}_1), \end{aligned} \tag{181}$$

where $\hat{\mathbf{R}} \in \mathrm{SO}(3)$, $\mathbf{a}$, $\mathbf{n}$ are related as in (41). Note that, since $\mathbf{y}$ is invertible on $\Omega$, (181) defines the magnetization uniquely. Also note that in PRMM under the

ansatz (181) *no new kinematic variables are introduced*. By differentiating the first of (181) with respect to $\mathbf{x}$, one can easily see that this ansatz implies that $(\nabla \mathbf{y}(\mathbf{x}, t), \mathbf{m}(\mathbf{y}(\mathbf{x}, t), t))$ lies on the two compatible energy wells $\mathrm{SO}(3)(\mathbf{U}_1, \mathbf{m}_1) \cup \mathrm{SO}(3)(\mathbf{U}_2, \mathbf{m}_2)$. For example, the extended ansatz (181) corresponds in the case ($\mathbf{R} = \mathbf{I}$) of Figure 2 to the superposition of a magnetization $\hat{\mathbf{R}}\mathbf{m}_2$ on the "light" variant and $\mathbf{m}_1$ on the "dark" variant; in physical terms, these stay on the easy axes of the crystal throughout the motion.

(ii) *Magnetostatic equations for the self-field.* For this subsection, we suppress the dependence on $t$. Both the magnetic field $\mathbf{h}$ and magnetization $\mathbf{m}$ enter formulas below for the force and configurational force. The magnetic field $\mathbf{h}$: $\mathbb{R}^3 \to \mathbb{R}^3$ is the sum of an assigned applied field $\mathbf{h}_a$ (interpreted as the field that would be present if the ferromagnetic body were removed) and the field produced by the magnetization, $\mathbf{h}_{\text{self}}$. From Maxwell's equations in the static case, $\operatorname{curl} \mathbf{h}_{\text{self}} = 0$ on all of space, so that

$$\mathbf{h}_{\text{self}} = -\nabla \psi \text{ on } \mathbb{R}^3, \tag{182}$$

and then the Maxwell equations $\operatorname{div} \mathbf{b} = 0$ and $\mathbf{b} = \mathbf{h} + 4\pi \mathbf{m}$ (cgs units) yield

$$\operatorname{div}(-\nabla \psi + 4\pi \mathbf{m}) = 0. \tag{183}$$

As discussed by e.g. James and Kinderlehrer [20], given $\mathbf{m} \in L^2(\mathbf{y}(\Omega))$, this equation has a unique solution $\mathbf{h}_{\text{self}} \in L^2(\mathbb{R}^3)$, so the magnetic field $\mathbf{h}$ is determined by the magnetization and the applied field.

(iii) *Magnetic forces and torques.* In PRMM expressions for the applied magnetic forces and torques,

$$\mathcal{F}, \qquad \mathcal{F}_s, \qquad \mathcal{T}_o \tag{184}$$

are needed. Recall from Section 8.1 the component of force $\mathcal{F}_{si}$ is the resultant force on the region $\Omega \cap \{\mathbf{x} \cdot \mathbf{n} < s_i(t)\}$. Recall also that $\mathcal{F}$ and $\mathcal{T}_o$ are the overall force and torque (about the fixed point $\mathbf{y}_o$), respectively, on $\mathbf{y}(\Omega, t)$. In the formulation of Brown [10] (which we adopt) the basic formulas for force and torque on all of $\mathbf{y}(\Omega, t)$ are

$$\mathcal{F} = \int_{\mathbf{y}(\Omega,t)} \mathbf{m} \cdot \nabla \mathbf{h}_a \, d\mathbf{x}, \tag{185}$$

$$\mathcal{T}_o = \int_{\mathbf{y}(\Omega,t)} (\mathbf{y} - \mathbf{y}_o) \wedge (\mathbf{m} \cdot \nabla \mathbf{h}_a) + \mathbf{m} \wedge \mathbf{h}_a \, d\mathbf{x}. \tag{186}$$

Note that these are easy to calculate because they only depend on the magnetization, which has been given by the ansatz (181), and the (assigned) applied field. The formula for the $\mathcal{F}_s$ follows the same pattern as (185) except that the field appearing in the formula is that produced by all "coils and magnetized matter" *outside* the region $\mathbf{y}(\mathbf{x}, t)$, $\mathbf{x} \in \Omega \cap \{\mathbf{x} \cdot \mathbf{n} < s_i(t)\}$; that is, if we let $\mathcal{V}_i(t) = \{\mathbf{y}(\mathbf{x}, t)\colon\ \mathbf{x} \in \Omega \cap \{\mathbf{x} \cdot \mathbf{n} < s_i(t)\}\}$, then

$$\begin{aligned} \mathcal{F}_{si} &= (\text{traction force}) + \int_{\mathcal{V}_i} \mathbf{m} \cdot \nabla \mathbf{h}_i \, d\mathbf{y}, \\ &\text{where} \quad \mathbf{h}_i = \mathbf{h}_a - \nabla \psi_i, \\ &\text{and} \quad \operatorname{div}(-\nabla \psi_i + \mathbf{m} \chi_{\mathbf{y}(\Omega,t) \setminus \mathcal{V}_i}) = 0. \end{aligned} \tag{187}$$

Here, $\chi_{\mathbf{y}(\Omega,t)\setminus\mathcal{V}_i}$ is the characteristic function of the indicated region. This is a rather nasty expression that requires us to recalculate $N$ fields at each time step. Fortunately, there is a simplification of this formula that is discussed by Brown [10], Section 5.4 (see also James [23] and Müller and Schlömerkemper [30]). The simplified formula requires only the field of the whole body and the magnetization. It is

$$\mathcal{F}_{si} = (\text{traction force}) + 2\pi \int_{\partial\mathcal{V}_i} (\mathbf{m}\cdot\mathbf{n})^2 \mathbf{n}\, da - \int_{\mathcal{V}_i} (\mathbf{m}\cdot\nabla^2\psi)\, d\mathbf{y} + \int_{\mathcal{V}_i} (\mathbf{m}\cdot\nabla\mathbf{h}_a)\, d\mathbf{y}. \tag{188}$$

Here, $\psi$ is obtained from (183). We note that this formula is to some extent disputed; calculations of Müller and Schlömerkemper [30] that are based directly on the atomic forces produced by a lattice of dipoles give a different boundary term than that in (188).

(iv) *Configurational force on a magnetoelastic interface.*
Finally, magnetic fields alone (without stress) clearly can drive a magnetoelastic interface, so the formula for the configurational force needs modification. This has been done under the present assumptions by James [23], so we simply record the formula for the configurational force specialized to PRMM:

$$\int_{\mathcal{S}_t} [\![\det\nabla\mathbf{y}\left(\mathbf{h}\cdot\mathbf{m} + 2\pi(\mathbf{m}\cdot\mathbf{n})^2\right) + \langle\mathbf{t}\rangle\cdot\nabla\mathbf{y}\mathbf{n}]\!]\, da. \tag{189}$$

Here, we have considered a single interface and we have used the notation of Section 5. The kinetic law is, therefore,

$$\dot{s}_i = f\left(\int_{\mathcal{S}_t} [\![\det\nabla\mathbf{y}(\mathbf{h}\cdot\mathbf{m} + 2\pi(\mathbf{m}\cdot\mathbf{n})^2) + \langle\mathbf{t}\rangle\cdot\nabla\mathbf{y}\mathbf{n}]\!]\, da\right). \tag{190}$$

We do not attempt a study of the well-posedness of PRMM. However, we observe that, since in PRMM no new kinematic variable are introduced—the unknowns are still $\mathbf{R}(t)$, $\mathbf{c}(t)$, $s_1(t), \ldots, s_N(t)$—and, since the magnetization determines the field uniquely, then the study of well-posedness of PRMM is expected to be similar to that of PRBM.

### *9.2. Piecewise Rigid Thermodynamics*

Piecewise rigid thermodynamics (PRT) is the natural thermomechanical extension of PRBM. We permit only deformations with gradient on the energy wells, but we allow the value of the free energy at each well to be an arbitrary smooth function of absolute temperature $\theta > 0$, as is appropriate when we consider both austenite and martensite phases. We also allow the heat conduction problem to be general. Here, we formulate the theory for a simple laminate.

Since the presence of thermodynamics does not alter the forms of the laws of mechanics, we carry over the equations of Sections 7 and 8 unchanged, except that now we admit a more general thermodynamic configurational force. The two wells will be

denoted $SO(3)\mathbf{U}_1$ and $SO(3)\mathbf{U}_2$ to keep the notation consistent with that of Section 7, even though in the thermodynamic case it is more natural, say, to associate $\mathbf{U}_1$ with austenite and $\mathbf{U}_2$ with martensite (We discuss at the end of this section the extension to the interesting case of austenite-twinned martensite).

Let $\varphi_1(\theta)$ and $\varphi_2(\theta)$ represent the values of the free energy at the two wells $SO(3)\mathbf{U}_1$ and $SO(3)\mathbf{U}_2$, respectively. By the frame-indifference of the free energy, these functions do not depend on the value of the deformation gradient on the respective wells. We adopt the following notation for thermodynamic quantities:

$$\begin{aligned} &\varepsilon \ \ldots \ \textit{internal energy per unit mass},\\ &\eta \ \ldots \ \textit{entropy per unit mass},\\ &\varphi \ \ldots \ \textit{free energy per unit mass},\\ &\mathbf{q} \ \ldots \ \textit{referential heat flux}, \end{aligned} \tag{191}$$

and we religiously use a Lagrangian description. The basic unknown functions in PRT are the motion (62), which is determined by the functions $\mathbf{y}_c(t), \mathbf{R}(t) \in SO(3), s_1(t), \ldots, s_N(t), t > 0$, and the temperature field $\theta(\mathbf{x}, t), \mathbf{x} \in \Omega, t > 0$. For the purpose of formulating the equations, we assume that $\theta \in C^\infty$ on sets of the form $\{\mathbf{x} \in \Omega\colon\ s_i(t) < \mathbf{x}\cdot\mathbf{n} < s_{i+1}(t), i = 0, \ldots, N\}$, and $\theta$ has traces from the right and left on the interfaces $\{\mathbf{x}\cdot\mathbf{n} = s_i(t)\}$. We assume the (anisotropic) Fourier's law as the constitutive equation for the referential heat flux:

$$\begin{aligned} \mathbf{q} \ &= \ -\mathbf{K}\nabla\theta, \qquad \mathbf{x} \in \Omega, \quad t > 0,\\ &\text{where} \quad \mathbf{K} = \chi\mathbf{K}_2 + (1-\chi)\mathbf{K}_1, \end{aligned} \tag{192}$$

and the matrices $\mathbf{K}_1$ and $\mathbf{K}_2$ are positive-definite *thermal conductivity tensors* associated with phases 1 and 2 and $\chi$ is the characteristic function introduced in (58).

*Remark* 9.1. Fourier's law is more commonly written in the spatial description. If we let *grad* denote the spatial gradient, $\mathbf{F} = \nabla\mathbf{y}$ the deformation gradient with polar decomposition $\mathbf{F} = \chi\mathbf{R}_2(t)\mathbf{U}_2 + (1-\chi)\mathbf{R}_1(t)\mathbf{U}_1$, $\tilde{\mathbf{q}} = \frac{1}{\det\mathbf{F}}\mathbf{F}\mathbf{q}$ the spatial heat flux, $\tilde{\theta}(\mathbf{y}, t)$ the Eulerian temperature field, $\tilde{\chi}$ the Eulerian phase marker ($\tilde{\chi}(\mathbf{y}(\mathbf{x}, t), t) = \chi(\mathbf{x}\cdot\mathbf{n}),\ \mathbf{x} \in \Omega$), then by straightforward calculations (192) is equivalent to

$$\tilde{\mathbf{q}} = -(\tilde{\chi}\mathbf{R}_2\tilde{\mathbf{K}}_2\mathbf{R}_2^T + (1-\tilde{\chi})\mathbf{R}_1\tilde{\mathbf{K}}_1\mathbf{R}_1^T)\,grad\,\theta, \tag{193}$$

where

$$\tilde{\mathbf{K}}_\alpha = \frac{1}{\det\mathbf{U}_\alpha}\mathbf{U}_\alpha^{-T}\mathbf{K}_\alpha\mathbf{U}_\alpha, \quad \alpha = 1, 2. \tag{194}$$

The measured heat conductivity tensors of the two phases would be identified with the tensors $\tilde{\mathbf{K}}_\alpha$.[10] The referential heat conductivity tensors $\mathbf{K}_\alpha$ are then obtained from (194), and in practice they differ little from their spatial counterparts.

[10] The presence of the rotation matrices $\mathbf{R}_\alpha$ in (193) is familiar to workers in continuum mechanics, but Fourier's law is not usually written this way. The presence of the rotation matrices accounts for the fact that the principal axes of the conductivity tensor rotate with the crystal during deformation.

By use of the thermodynamic relations $\varphi = \varepsilon - \theta\eta$, $\eta = -\partial\varphi/\partial\theta$, and the assumption on the free energy, we get the following formulas for the (Helmholtz) free energy, entropy, and internal energy:

$$\begin{aligned} \varphi(\theta) &= \chi\varphi_2(\theta) + (1-\chi)\varphi_1(\theta), \\ \eta(\theta) &= -(\chi\varphi_2'(\theta) + (1-\chi)\varphi_1'(\theta)), \\ \varepsilon(\theta) &= \chi(\varphi_2(\theta) - \theta\varphi_2'(\theta)) + (1-\chi)(\varphi_1(\theta) - \theta\varphi_1'(\theta)). \end{aligned} \tag{195}$$

Here, we have suppressed the dependence on $(\mathbf{x}, t)$. We also get a formula for the *specific heat at constant deformation gradient* ($C_F = -\theta\partial^2\varphi/\partial\theta^2$),

$$C_F = -\theta(\chi\varphi_2''(\theta) + (1-\chi)\varphi_1''(\theta)). \tag{196}$$

There are now two main tasks: to formulate and specialize the energy equation and to calculate the thermodynamic configurational force. We first consider the energy equation. Omitting radiation, the general form of the energy equation is

$$\frac{d}{dt}\int_{\mathcal{P}} \rho_0 \left(\epsilon + \frac{1}{2}|\dot{\mathbf{y}}|^2\right) d\mathbf{x} = \int_{\partial\mathcal{P}} (\dot{\mathbf{y}} \cdot \mathbf{T}\hat{\mathbf{n}} - \mathbf{q}\cdot\hat{\mathbf{n}})\, da, \tag{197}$$

for all regular $\mathcal{P} \subset \Omega$. By standard arguments[11] in continuum thermodynamics—the use of Reynold's transport theorem, the power theorem, and the thermodynamic relations between internal energy and entropy—this is equivalent to the following local form:

$$\rho_0\theta\dot{\eta} = -div\,\mathbf{q} \text{ on the sets } \{\mathbf{x}\in\Omega:\ s_i(t) < \mathbf{x}\cdot\mathbf{n} < s_{i+1}(t), i = 0, \ldots, N\}, \tag{198}$$

and the jump conditions,

$$[\![\rho_0\epsilon]\!]\dot{s}_i = [\![\mathbf{q}\cdot\mathbf{n}]\!] - [\![\dot{\mathbf{y}}]\!]\langle\mathbf{T}\mathbf{n}\rangle, \quad i \in \{1, \ldots, N\}. \tag{199}$$

If we now insert the constitutive equations (192), (195), and (196) into (198), we get

$$\rho_0 C_F\dot{\theta} = div(\mathbf{K}\nabla\theta), \tag{200}$$

or, written out in full,

$$-\rho_0\theta(\chi\varphi_2''(\theta) + (1-\chi)\varphi_1''(\theta))\dot{\theta} = div((\chi\mathbf{K}_2 + (1-\chi)\mathbf{K}_1)\nabla\theta). \tag{201}$$

Hence, PRT also achieves a substantial simplification of the heat conduction problem: On each region between interfaces *in the reference configuration*, the energy equation is the classical heat equation. This is supplemented by the jump condition (199) which expresses that the moving interfaces are sources of energy.

Now we turn to the configurational force. One consequence of the work of Abeyaratne and Knowles [1] [3] on kinetics of phase boundaries in the thermodynamic setting is

[11] The only slight difference from the standard line of argument is that in PRT we do not have the constitutive relation for the Piola-Kirchhoff stress, $\mathbf{T} = \rho_o\partial\varphi/\partial\mathbf{F}$; however, the identity $\mathbf{T}\cdot\nabla\dot{\mathbf{y}} = 0$ (the vanishing of the stress power) of PRT allows the same steps to go through.

an assessment of continuity conditions of temperature across the phase boundary. In the nonadiabatic case this can be argued on the basis of the well-posedness of the heat conduction problem (199), (200); that is, if for a moment we regard $[\![\dot{\mathbf{y}}]\!]\langle \mathbf{Tn}\rangle$ as assigned and pose typical initial and boundary conditions, then this is a heat equation with a moving energy source, which can be solved via Green's functions to yield a continuous temperature field. Thus we assume the temperature is continuous. In the thermodynamic setting, the formula for the configurational force is obtained from the entropy production, given by

$$\mathcal{E} = \frac{d}{dt}\int_\Omega \rho_0 \eta \, d\mathbf{x} + \int_{\partial\Omega} \frac{\mathbf{q}\cdot\hat{\mathbf{n}}}{\theta}\, da. \tag{202}$$

The second law of thermodynamics is $\mathcal{E} \geq 0$. Use of Reynold's transport theorem and the constitutive relations for entropy and heat flux gives

$$\mathcal{E} = \sum_{i=0}^{N} \int_{\Omega\cap\{s_i<\mathbf{x}\cdot\mathbf{n}<s_{i+1}\}} -\frac{1}{\theta^2}\mathbf{q}\cdot\nabla\theta \, d\mathbf{x} + \int_{\Omega\cap\{\mathbf{x}\cdot\mathbf{n}=s_i(t)\}} \left[\!\!\left[ -\rho_0\eta\dot{s}_i + \frac{1}{\theta}\mathbf{q}\cdot\mathbf{n}\right]\!\!\right] da. \tag{203}$$

Using the continuity of the temperature, we substitute for the $\frac{1}{\theta}[\![\mathbf{q}\cdot\mathbf{n}]\!]$ using the jump condition (199) to give

$$\begin{aligned}\mathcal{E} \;=\; & \sum_{i=0}^{N} \int_{\Omega\cap\{s_i<\mathbf{x}\cdot\mathbf{n}<s_{i+1}\}} -\frac{1}{\theta^2}\mathbf{q}\cdot\nabla\theta \, d\mathbf{x} \\ & + \int_{\Omega\cap\{\mathbf{x}\cdot\mathbf{n}=s_i(t)\}} \frac{1}{\theta}[\![\rho_0\varphi - (\nabla\mathbf{y}\mathbf{n})\cdot\langle\mathbf{Tn}\rangle]\!]\dot{s}_i \, da. \end{aligned} \tag{204}$$

Dissipation therefore arises from heat condition in bulk and from dissipation of energy at interfaces. Motivated by this formula (and following previous works), the configurational force is

$$[\![\rho_0\varphi - (\nabla\mathbf{y}\mathbf{n})\cdot\langle\mathbf{Tn}\rangle]\!]. \tag{205}$$

The kinetic law is, therefore,

$$\dot{s}_i = f([\![\rho_0\varphi - (\nabla\mathbf{y}\mathbf{n})\cdot\langle\mathbf{Tn}\rangle]\!]), \tag{206}$$

with $\xi f(\xi) \geq 0$. Note that the positive-definiteness of the conductivity tensors already implies that the first term of (204) is nonnegative, so that the second law of thermodynamics holds.

While we do not do a complete analysis of the well-posedness of PRT here, we make some remarks on this issue. The main concern is the quantity $[\![\dot{\mathbf{y}}]\!]\langle\mathbf{Tn}\rangle = -\dot{s}_i[\![(\nabla\mathbf{y}\mathbf{n})\cdot\langle\mathbf{Tn}\rangle]\!]$, which in PRBM was integrated over the interface. Here, the jump condition for the energy equation seems to require the local value of the average traction at interfaces, which does not couple to other equations. To discuss this concern, let us assume for simplicity linear kinetics,

$$\dot{s}_i = \mu([\![\rho_0\varphi - (\nabla\mathbf{y}\mathbf{n})\cdot\langle\mathbf{Tn}\rangle]\!]). \tag{207}$$

We first eliminate the term $[\![\dot{\mathbf{y}}]\!]\langle\mathbf{Tn}\rangle$ from the jump condition (199) using the kinetic law (207). Then, the energy equation and jump condition involve only the temperature

field and the unknown velocities of the interfaces. Apparently, they can be solved using Green's functions to yield the temperature field as a functional of the unknown interfacial positions. Now we integrate the kinetic law (207) over each interface. This integrated kinetic law will couple the heat conduction equation with the balances of linear and rotational momentum as formulated above. This formulation has the promise of being well-posed.

We now make some final remarks about the extension of PRT to perhaps its most interesting application: an array of austenite-martensite interfaces. The kinematics of the austenite-martensite interface has been studied by Wechsler, Lieberman, and Read [41], and the connection between their calculations and "energy wells" was given by Ball and James [5]. Except for a small transition layer, the deformation gradient associated with this structure is supported on three energy wells, $\mathrm{SO}(3)\mathbf{U}_a$, $\mathrm{SO}(3)\mathbf{U}_1$, and $\mathrm{SO}(3)\mathbf{U}_2$, the first associated with austenite and the others with two variants of martensite. The presence of the austenite places a strong kinematic restriction on the martensite bands that meet it, with the following consequences: the bands of martensite are very fine and the volume fraction of one martensite variant relative to the other is fixed. If the austenite-martensite interfaces are sufficiently far apart, then some bands of martensite will not impinge on the austenite-martensite interface, and these bands will not be subject to the strong kinematic restrictions. In practice, depending on the applied loads, these might tend to detwin, which would lead to a more complicated situation (for example, the kinetic law for martensite-martensite interfaces would surely be different from that of austenite-martensite interfaces). Assuming that the austenite-martensite interfaces are sufficiently close together so that each martensite band meets an austenite-martensite interface, then it is reasonable to model this situation as a simple laminate. The two deformation gradients would then be given by $\mathbf{R}_a(t)\mathbf{U}_a$ and $\mathbf{R}_m(t)(\lambda\hat{\mathbf{R}}\mathbf{U}_2 + (1-\lambda)\mathbf{U}_1)$, where $\lambda$ is the relative volume fraction of martensite variants and we have the compatibility conditions,

$$\begin{aligned} \mathbf{R}_m(\lambda\hat{\mathbf{R}}\mathbf{U}_2 + (1-\lambda)\mathbf{U}_1) - \mathbf{R}_a\mathbf{U}_a &= \mathbf{a}\otimes\mathbf{n}, \\ \hat{\mathbf{R}}\mathbf{U}_2 - \mathbf{U}_1 &= \mathbf{c}\otimes\mathbf{m}. \end{aligned} \tag{208}$$

In fact, these equations are exactly the equations of the crystallographic theory of martensite. For the energy equation, it is of course natural to use the thermal conductivity $\mathbf{K}_a$ in the austenite region; in the twinned martensite region it is natural to use the formula for the effective conductivity of a laminate $\mathbf{K}_{\mathrm{eff}}$. The expression for the latter should be quite simple because of the twinning relations that follow from the second of (208) and the structure of the martensite wells (29). (Finding the measured conductivity tensor for single crystal martensite is likely to present a greater problem.) A simple estimate for the entropy of austenite and martensite (needed for the energy equation) can be obtained from the measure latent heat of transformation and the specific heats of the individual phases. With these assumptions, this problem falls into the framework of PRT given above.

## Appendix 1

Here we show that the determinant of the positive-definite matrix $\tilde{\mathbb{M}}$ defined by (171) is bounded away from zero on $(0, t^*)$. This was used in Proposition 8.4.

**Lemma 9.1.** *Assume Hypotheses $\mathcal{H}$ of Section 8.3. Then* $\det \tilde{\mathbb{M}} > \varepsilon > 0$ *on* $(0, t^*)$.

*Proof.* Clearly the determinant of $\tilde{\mathbb{M}}$ could only possibly sneak to zero at $t^*$. If that were true, there would be an increasing sequence $t_i \to t^*$ and a family of unit vectors $\tilde{\mathbf{e}}^i = (\tilde{\mathbf{v}}_c^i, \tilde{\mathbf{w}}_s^i, \tilde{\boldsymbol{\omega}}^i)$ such that $\tilde{\mathbb{M}}(t_i)\tilde{\mathbf{e}}^i \to 0$. After taking another subsequence, we can assume that $\tilde{\mathbf{e}}^i \to \bar{\mathbf{e}} = (\bar{\mathbf{v}}_c, \bar{\mathbf{w}}_s, \bar{\boldsymbol{\omega}})$, $|\bar{\mathbf{e}}| = 1$. Let $\mathbf{e}^i = (\tilde{\mathbf{v}}_c^i, \mathcal{M}^{-1}(t_i)\tilde{\mathbf{w}}_s^i, \tilde{\boldsymbol{\omega}}^i)$ and assume a further subsequence is taken so that $\mathbf{e}^i/|\mathbf{e}^i| \to \mathbf{e}$, $|\mathbf{e}| = 1$. By direct calculation,

$$\det \mathcal{M}^{-1} = M \prod_{j=0}^{N} \frac{1}{\hat{M}(s_j) - \hat{M}(s_{j+1})}, \tag{209}$$

and so, on the present sequence it is bounded from below. Thus, using also the condition $|\tilde{\mathbf{e}}^i| = 1$, we have $\liminf |\mathbf{e}^i| > 0$. Therefore,

$$\tilde{\mathbf{e}}^i \cdot \tilde{\mathbb{M}}(t_i)\tilde{\mathbf{e}}^i = \mathbf{e}^i \cdot \mathbb{M}(t_i)\mathbf{e}^i \to 0, \tag{210}$$

which shows immediately that $\mathbf{e} = \pm(0, 0, 0; \ldots, \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, \ldots, 0; 0, 0, 0)$ and also that

$$\tilde{\mathbf{v}}_c^i \to 0, \qquad \tilde{\boldsymbol{\omega}}^i \to 0, \tag{211}$$

and so $|\tilde{\mathbf{w}}_s^i| \to 1$. Let $\mathbf{v}_s^i = \mathcal{M}^{-1}(t_i)\tilde{\mathbf{w}}_s^i$ and denote the middle part of the null vector $\mathbf{e}$, i.e., the $N$-dimensional vector between the semicolons, by $\mathbf{e}_N$. We can write

$$\frac{\mathbf{v}_s^i}{|\mathbf{v}_s^i|} = \mathbf{e}_N + \delta^i \mathbf{p}_i, \qquad |\mathbf{p}_i| = 1, \qquad \delta^i \geq 0, \qquad \delta^i \to 0. \tag{212}$$

Using the form of $\tilde{\mathbb{M}}$ (cf. (171) and (125)) and the conditions (210) and $|\tilde{\mathbf{w}}_s^i| \to 1$, we have the following restrictions on $\mathbf{v}_s^i$:

$$\begin{aligned} |\mathbf{a}|^2|\mathbf{v}_s^i|^2 \left(\frac{\mathbf{v}_s^i}{|\mathbf{v}_s^i|} \cdot \mathcal{M}(t_i)\frac{\mathbf{v}_s^i}{|\mathbf{v}_s^i|}\right) + 2|\mathbf{v}_s^i| \left(\mathcal{C}(t_i)\frac{\mathbf{v}_s^i}{|\mathbf{v}_s^i|}\right)(\mathbf{A}\boldsymbol{\omega}^i) \;&\to\; 0, \\ \mathcal{M}\mathbf{v}_s^i \;&\to\; \bar{\mathbf{w}}_s, \end{aligned} \tag{213}$$

$|\bar{\mathbf{w}}_s| = 1$. We will show that these are contradictory. We have from $\mathbf{v}_s^i = \mathcal{M}^{-1}(t_i)\tilde{\mathbf{w}}_s^i$ and $\frac{\mathbf{v}_s^i}{|\mathbf{v}_s^i|} \to \mathbf{e}_N$ that $|\mathbf{v}_s^i| \to \infty$. For sufficiently large $i$ the smallest eigenvalue of $\mathcal{M}(t_i)$ corresponds to the largest eigenvalue of $\mathcal{M}^{-1}(t_i)$, which is the singular one. A short calculation based on (168) and (169) gives it as $c_i(\hat{M}(s_{k+1}(t_i)) - \hat{M}(s_k(t_i)))$, where $\liminf c_i > 0$. Hence, we divide the first of (213) by $|\mathbf{v}_s^i|$ and pass to the limit $i \to \infty$, using the boundedness of $\mathcal{C}(t_i)$; we get

$$(\hat{M}(s_{k+1}(t_i)) - \hat{M}(s_k(t_i)))|\mathbf{v}_s^i| \to 0. \tag{214}$$

Hence, $|\mathbf{v}_s^i|\mathcal{M}(t_i)\mathbf{e}_N \to 0$, and the the second of (213), together with the decomposition (212), gives

$$\delta^i|\mathbf{v}_s^i|\mathcal{M}(t_i)\mathbf{p}_i \to \bar{\mathbf{w}}_s, \qquad |\bar{\mathbf{w}}_s| = 1. \tag{215}$$

**Case 1.** $\delta^i|\mathbf{v}_s^i|$ is bounded. In this case, by taking further subsequences, we can assume that

$$\delta^i|\mathbf{v}_s^i| \to \nu, \qquad \mathbf{p}_i \to \mathbf{p}, \tag{216}$$

with $\mathbf{p}$ not in the limiting null-space of $\mathcal{M}$ and $\nu \neq 0$ (cf. (215)). After substitution of (212), the first of (213) becomes

$$\begin{aligned}|\mathbf{a}|^2|\mathbf{v}_s^i|^2(\mathbf{e}_N \cdot \mathcal{M}(t_i)\mathbf{e}_N + 2\delta_i\mathbf{e}_N \cdot \mathcal{M}(t_i)\mathbf{p}_i + (\delta^i)^2\mathbf{p}_i \cdot \mathcal{M}(t_i)\mathbf{p}_i)\\ + 2|\mathbf{v}_s^i|(\mathcal{C}(t_i)(\mathbf{e}_N + \delta^i\mathbf{p}_i))(\mathbf{A}\boldsymbol{\omega}^i) \to 0.\end{aligned} \tag{217}$$

The term $|\mathbf{v}_s^i|\mathcal{C}(t_i)\mathbf{e}_N \to 0$ by the special form of $\mathcal{C}$ and the condition (214). (Here, we use $\inf|\mathcal{A}(s_k(t_i)| > 0$ to estimate $s_{k+1}(t_i) - s_k(t_i)$ in terms of $\hat{M}(s_{k+1}(t_i)) - \hat{M}(s_k(t_i))$ as explained just after Hypothesis $\mathcal{H}$.) Therefore, the last term of (217) tends to zero, because $\delta^i|\mathbf{v}_s^i|$ is bounded and $\boldsymbol{\omega}^i \to 0$, which leaves

$$|\mathbf{v}_s^i|^2(\mathbf{e}_N \cdot \mathcal{M}(t_i)\mathbf{e}_N + 2\delta_i\mathbf{e}_N \cdot \mathcal{M}(t_i)\mathbf{p}_i + (\delta^i)^2\mathbf{p}_i \cdot \mathcal{M}(t_i)\mathbf{p}_i) \to 0. \tag{218}$$

We see, using the condition $|\mathbf{v}_s^i|\mathcal{M}(t_i)\mathbf{e}_N \to 0$, that the middle term vanishes. Thus the positive first and third terms must separately tend to zero, which in the latter case yields

$$\limsup_{i\to\infty} \nu^2\mathbf{p} \cdot \mathcal{M}(t_i)\mathbf{p} = 0. \tag{219}$$

It follows that $\mathbf{p} = \pm\mathbf{e}_N$, which contradicts the condition (215).

**Case 2.** $\delta^i|\mathbf{v}_s^i| \to \infty$. In this case we have by (215) that, for a suitable subsequence and choice of $\pm$,

$$\mathbf{p}_i \to \pm\mathbf{e}_N. \tag{220}$$

However, this contradicts the fact, which follows from (212), that $\mathbf{p}_i$ is approximately perpendicular to $\mathbf{e}_N$; that is, taking the inner product of the first of (212) by itself,

$$1 = 1 + 2\delta^i\mathbf{p}_i \cdot \mathbf{e}_N + (\delta^i)^2, \tag{221}$$

giving that

$$\delta^i = 0, \quad i \geq i_0, \tag{222}$$

for some positive integer $i_0$. This contradicts (215). □

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