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Mathematical Problems in Mechanics/Calculus of Variations

Derivation of nonlinear bending theory for shells from three-dimensional nonlinear elasticity by Gamma-convergence

Dérivation de la théorie non linéaire des coques en flexion à partir de l'élasticité non linéaire tridimensionelle par Gamma-convergence

Gero Friesecke [a], Richard D. James [b], Maria Giovanna Mora [c], Stefan Müller [c]

[a] *Mathematics Institute, University of Warwick, Coventry CV4 7AL, UK*
[b] *Department of Aerospace Engineering and Mechanics, 107 Akerman Hall, University of Minnesota, Minneapolis, MN 55455, USA*
[c] *Max Planck Institute for Mathematics in the Sciences, Inselstr. 22-26, 04103 Leipzig, Germany*

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Abstract

We show that the nonlinear bending theory of shells arises as a Γ-limit of three-dimensional nonlinear elasticity. ***To cite this article: G. Friesecke et al., C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

Résumé

Nous montrons que la théorie non linéaire des coques en flexion émerge comme Γ-limite de la théorie de l'élasticité tridimensionelle. ***Pour citer cet article : G. Friesecke et al., C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

Version française abrégée

Dans cette Note nous dérivons la théorie des coques non linéaires en flexion comme Γ-limite de la théorie d'élasticité tridimensionnelle non linéaire.

E-mail addresses: gf@maths.warwick.ac.uk (G. Friesecke), james@aem.umn.edu (R.D. James), mmora@mis.mpg.de (M. Giovanna Mora), sm@mis.mpg.de (S. Müller).

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Le problème tridimensionnel. Soit M une surface orientable dans $\mathbb{R}^3$ de classe C^2. Soit μ la normale de M et soit $M_h := \{x + s\mu(x)\colon\ x \in M,\ s \in (-h/2, h/2)\}$, pour $h > 0$ suffisamment petit. Nous supposons, pour simplifier, que M est donné par une seule application $\psi : \Omega' \subset \mathbb{R}^2 \to M$ où Ω' est un ouvert borné lipschitzien. Donc M_h est l'image de $\Omega := \Omega' \times (-1/2, 1/2)$ par l'application $\psi^{(h)}(z', z_3) = \psi(z') + hz_3\eta(z')$, où $\eta = \mu \circ \psi$.

L'énergie élastique d'une application $u : M_h \to \mathbb{R}^3$ est $E^{(h)}(u) = \int_{M_h} W(\nabla u)\,\mathrm{d}x$. Ici W est une fonction Borelienne, de classe C^2 dans un voisinage de SO(3), qui satisfait (4) et (5) ci-dessous.

Le problème bidimensionnel. Considérons la classe $\mathcal{A} = \{u \in W^{2,2}(M; \mathbb{R}^3)\colon\ (\nabla_{\tan} u)^{\mathrm{T}}(\nabla_{\tan} u) = I$ p.p. dans $M\}$ d'applications isométriques de M. Ici, la dérivée tangentielle $\nabla_{\tan} u(x)$ est une application de $T_x M$ à valeurs dans $\mathbb{R}^3$. La quantité importante est *l'application relative de Weingarten* $S_{M,u}$ qui mesure la différence entre les secondes formes fondamentales de M et de $N = u(M)$. Soit ν la normale de N et soit $Y = u \circ \psi : \Omega' \to N$. Nous définissons $S_{M,u}$ par

$$S_{M,u}(x)\nabla\psi = R^{\mathrm{T}}(x)\nabla'(\nu \circ Y) - \nabla'(\mu \circ \psi),$$

où $\nabla' = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2})$ et $R(x) = \nabla_{\tan} u(x)$. Des caractérisations équivalentes sont données par (7) ou (8) ci-dessous.

Nous considérons les formes quadratiques $Q_3(G) := \partial^2 W/\partial F^2(I)(G, G)$, où I est l'identité, et $Q_2(x, G) := \min_{a \in \mathbb{R}^3}(G + a \otimes \mu(x))$. Évidemment, $Q_2(x, G)$ dépend seulement de la restriction de G à $T_x M$. L'énergie bidimensionnelle est donnée par $E(u) = (1/24)\int_M Q_2(x, S_{M,u}(x))\,\mathrm{d}\mathcal{H}^2$, si $u \in \mathcal{A}$ et $E(u) = \infty$ sinon.

Le résultat principal est la Γ-convergence des fonctionnelles $h^{-3}E^{(h)}$ vers E au sens suivant.

Théorème 0.1. (i) *Soit* $u^{(h)} : M_h \to \mathbb{R}^3$ *une suite telle que* $\limsup_{h\to 0} h^{-3}E^{(h)}(u^{(h)}) < \infty$. *Alors il existe* $u \in \mathcal{A}$, *une application* $R : \Omega \to \mathbb{R}^{3\times 3}$ *et des constantes* $c^{(h)}$ *tels que pour une sous-suite*

$$u^{(h)} \circ \psi^{(h)} - c^{(h)} \to u \circ \psi^{(0)} \quad \textit{dans } W^{1,2}\big(\Omega; \mathbb{R}^3\big), \tag{1}$$

$$\nabla u^{(h)} \circ \psi^{(h)} \to R \quad \textit{dans } L^2\big(\Omega; \mathbb{R}^{3\times 3}\big), \tag{2}$$

$$R_{,3} = 0, \quad R(z) \in \mathrm{SO}(3)\ \textit{p.p.}, \quad R \in W^{1,2}\big(\Omega; \mathbb{R}^{3\times 3}\big), \tag{3}$$

$$\liminf_{h\to 0} h^{-3}E^{(h)}\big(u^{(h)}\big) \geqslant E(u).$$

(ii) *Si* $u \in \mathcal{A}$ *il existe une suite* $u^{(h)} : M_h \to \mathbb{R}^3$ *telle que* (1)–(3) *soient vérifiées et* $\lim_{h\to 0} h^{-3}E^{(h)}(u^{(h)}) = E(u)$.

On pourrait ajouter aux fonctionnelles des termes linéaires en u et des conditions au bord. Par les arguments classiques de la Γ-convergence, on obtient la convergence des applications (presque) minimisantes.

Un outil essentiel de la preuve est un résultat de rigidité pour des applications proches d'un mouvement rigide (voir Theorem 4.1 ci-dessous ou [5], Théorème 2).

1. Introduction

The derivation of plate and shell theories is a problem having a long history with major contributions from Euler, D. Bernoulli, Kirchhoff, Love, E. and F. Cosserat, von Karman and a great many modern authors. The question which theory, if any, is predicted by three-dimensional nonlinear elasticity of thin objects has been open for a long time (formal results go back at least as far as Kirchhoff [8]; for three recent formal treatments see [3,2,4]). The first rigorous result for finite deformations was the derivation of nonlinear membrane theory by Le Dret and Raoult [9–11] through the use of Γ-convergence (for elastic strings a Γ-convergence result was obtained earlier by Acerbi, Buttazzo and Percivale [1]). The more delicate case of bending theory for plates was settled recently in [5,6], see also [12]. Here we extend this result to the nonlinear bending theory of shells. A key ingredient is an optimal oscillation estimate for deformations whose gradient is close to $\mathrm{SO}(n)$ (see Theorem 4.1 below), first derived in [5] and generalizing earlier work of John [7].

2. The setting

The three-dimensional problem. Let M be an oriented surface in $\mathbb{R}^3$ of class C^2. For $x \in M$ let $\mu(x)$ denote the normal at x and consider, for sufficiently small h, the thickened set M_h of thickness h

$$M_h := \left\{ x + s\mu(x):\ x \in M,\ s \in \left(-\frac{h}{2}, \frac{h}{2}\right)\right\}.$$

In the following we will assume for convenience that M is given by a single C^2 chart $\psi : \Omega' \subset \mathbb{R}^2 \to M$, where Ω' is a bounded Lipschitz domain. Then M_h is parametrized by the map

$$\psi^{(h)} : \Omega = \Omega' \times \left(-\frac{1}{2}, \frac{1}{2}\right) \to M_h, \quad \psi^{(h)}(z', z_3) = \psi(z') + hz_3\eta(z'), \quad \eta(z') = (\mu \circ \psi)(z').$$

The case of general surfaces can be handled by standard localization arguments.

For a map $u : M_h \to \mathbb{R}^3$ its elastic energy is

$$E^{(h)}(u) = \int_{M_h} W(\nabla u)\,\mathrm{d}x,$$

where the stored-energy density $W : \mathbb{R}^{3\times 3} \to [0, \infty]$ is Borel measurable, C^2 in a neighbourhood of SO(3) and satisfies

$$W \text{ is frame indifferent: } W(F) = W(RF) \text{ for all } R \in \mathrm{SO}(3), \tag{4}$$

$$W(F) \geqslant C\,\mathrm{dist}^2\big(F, \mathrm{SO}(3)\big), \quad C > 0, \qquad W(F) = 0 \quad \text{if } F \in \mathrm{SO}(3). \tag{5}$$

The two-dimensional problem. Let the surface M be as above and consider the following set of (infinitesimal) isometries of M

$$\mathcal{A} = \big\{ u \in W^{2,2}\big(M; \mathbb{R}^3\big):\ (\nabla_{\tan} u)^{\mathrm{T}}(\nabla_{\tan} u) = I \text{ a.e. on } M\big\}.$$

Here the tangential derivative $\nabla_{\tan} u(x)$ is viewed as linear map from T_xM to $\mathbb{R}^3$. Sometimes it will be convenient to extend this map to a proper rotation in $\mathbb{R}^3$. The two-dimensional energy will depend on a map $S_{M,u}(x) : T_xM \to T_xM$ which measures the difference between the second fundamental form of $N = u(M)$ and of M. Let $\psi : \Omega' \to M$ be the chart considered above, let $Y = u \circ \psi : \Omega' \to N$ and define $S_{M,u}(x)$ by

$$S_{M,u}(x)\nabla'\psi = R^{\mathrm{T}}(x)\nabla'(\nu \circ Y) - \nabla'(\mu \circ \psi), \tag{6}$$

where ν is the normal of N, $\nabla' = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2})$ and $R(x) = \nabla_{\tan} u(x)$. The closely related quadratic form

$$(\nabla'\psi)^{\mathrm{T}} S_{M,u} \nabla'\psi = (\nabla' Y)^{\mathrm{T}} \nabla'(\nu \circ Y) - (\nabla'\psi)^{\mathrm{T}} \nabla'(\mu \circ \psi) \tag{7}$$

is exactly the difference of the fundamental forms of N and M. In terms of the Weingarten maps S_M and S_N given by $S_M \nabla'\psi = \nabla'(\mu \circ \psi)$ etc. we have

$$S_{M,u}(x) = R^{\mathrm{T}}(x) S_N\big(u(x)\big) R(x) - S_M(x). \tag{8}$$

We thus call $S_{M,u}$ the *relative Weingarten map*.

Consider the quadratic form

$$Q_3(G) = \frac{\partial^2 W}{\partial F^2}(I)(G, G)$$

related to the linearization of the energy at the identity. It follows from (4), (5) that Q_3 is positive semidefinite and its kernel consists of the skew symmetric matrices. For each $x \in M$, with normal $\mu(x)$ we define the form

$$Q_2(x, G) = \min_{a\in\mathbb{R}^3} Q_3\big(G + a \otimes \mu(x)\big).$$

Since Q_3 vanishes on skew-symmetric matrices we have $Q_2(x, G) = Q_2(x, G') = Q_2(x, G'')$ where $G' = G(I - \mu \otimes \mu)$, $G'' = (I - \mu \otimes \mu)G'$. In particular Q_2 depends only on the restriction of G to $T_x M$.

The two-dimensional energy is now defined by

$$E(u) = \frac{1}{24}\int_M Q_2\big(x, S_{M,u}(x)\big)\,\mathrm{d}\mathcal{H}^2 \quad \text{if } u \in \mathcal{A}, \qquad E(u) = \infty \quad \text{else.}$$

3. Convergence results

We essentially show that the functionals $h^{-3}E^{(h)}$ Γ-converge to E. This implies that (almost) minimizers of $h^{-3}E^{(h)}$ converge to minimizers of E, also when force terms and boundary conditions are added. Here we focus on the simplest case since the inclusion of body forces and boundary conditions is very similar to the situation for plates [6].

Theorem 3.1. (i) (compactness and ansatz-free lower bound) *Suppose that* $u^{(h)}: M_h \to \mathbb{R}^3$ *satisfy* $\limsup_{h\to 0} h^{-3}E^{(h)}(u^{(h)}) < \infty$. *Then there exists a* $u \in \mathcal{A}$, *a map* $R: \Omega \to \mathbb{R}^{3\times 3}$ *and constants* $c^{(h)}$ *such that for a subsequence*

$$u^{(h)} \circ \psi^{(h)} - c^{(h)} \to u \circ \psi^{(0)} \quad \text{in } W^{1,2}\big(\Omega; \mathbb{R}^3\big), \tag{9}$$

$$\nabla u^{(h)} \circ \psi^{(h)} \to R \quad \text{in } L^2\big(\Omega; \mathbb{R}^{3\times 3}\big), \tag{10}$$

$$R_{,3} = 0, \quad R(z) \in \mathrm{SO}(3) \text{ a.e.}, \quad R \in W^{1,2}\big(\Omega; \mathbb{R}^{3\times 3}\big), \tag{11}$$

and for the subsequence under consideration,

$$\liminf_{h\to 0} \frac{1}{h^3} E^{(h)}\big(u^{(h)}\big) \geqslant E(u). \tag{12}$$

(ii) (attainment of lower bound) *If* $u \in \mathcal{A}$ *then there exists a sequence* $u^{(h)}: M_h \to \mathbb{R}^3$ *such that* (9)–(11) *hold and*

$$\lim_{h\to 0} \frac{1}{h^3} E^{(h)}\big(u^{(h)}\big) = E(u).$$

4. Proofs

Compactness. The key ingredient is the following quantitative rigidity estimate (see [5,6] for a proof and a discussion of related results).

Theorem 4.1. *Let* U *be a bounded Lipschitz domain in* $\mathbb{R}^n$, $n \geqslant 2$. *Then there exists a constant* $C(U)$ *with the following property. For each* $v \in W^{1,2}(U;\ \mathbb{R}^n)$ *there exists an associated rotation* $R \in \mathrm{SO}(n)$ *such that*

$$\|\nabla v - R\|_{L^2(U)} \leqslant C(U)\big\|\mathrm{dist}\big(\nabla v, \mathrm{SO}(n)\big)\big\|_{L^2(U)}.$$

The proof of Theorem 4.1 shows that the constant $C(U)$ can be chosen independent of U for a family of sets that are Bilipschitz images of a cube (with uniform Lipschitz constants).

Consider a lattice of squares

$$S_{a,h} = a + \left(-\frac{h}{2}, \frac{h}{2}\right)^2, \quad a \in h\mathbb{Z}^2,$$

and let Ω'_h be the union of those squares with $S_{a,3h} \subset \Omega'$. For such a square we can apply Theorem 4.1 to the deformed cube $C_{a,h} = \psi^{(h)}(S_{a,h} \times (-\frac{1}{2}, \frac{1}{2})) \subset M_h$ since $C_{a,h}$ is a Bilipschitz image of $(-\frac{h}{2}, \frac{h}{2})^3$ under the map $\Psi^{(h)}(z) = \psi(z') + z_3\eta(z')$. We thus obtain a map $R^{(h)} : \Omega'_h \to \mathrm{SO}(3)$ which is constant on each $S_{a,h}$ and satisfies

$$\int_{-1/2}^{1/2} \int_{S_{a,h}} \left|\left(\nabla u^{(h)}\right) \circ \psi^{(h)} - R^{(h)}\right|^2 \det \nabla\psi^{(h)} \, \mathrm{d}z' \, \mathrm{d}z_3 \leqslant C \int_{C_{a,h}} W\left(\nabla u^{(h)}\right) \mathrm{d}x. \tag{13}$$

Applying the same estimate to a neighbouring cell $S_{b,h}$ and to $S_{a,3h} \supset S_{b,h}$ and taking into account that $\det \nabla\psi^{(h)} \sim h$ we easily deduce (see [6]) for $|\zeta|_\infty := \max(|\zeta_1|, |\zeta_2|) \leqslant h$

$$\int_{\Omega'_h} \left|R^{(h)}(z' + \zeta) - R^{(h)}(z')\right|^2 \leqslant \frac{C}{h} \int_{M_h} W\left(\nabla u^{(h)}\right) \mathrm{d}x.$$

If $\zeta \in \mathbb{R}^2$ is a general translation vector and $\omega' \subset \Omega'$ with dist $(\omega', \partial\Omega') \geqslant C|\zeta|$ then iterative application of this estimate yields

$$\int_{\omega'} \left|R^{(h)}(z' + \zeta) - R^{(h)}(z')\right|^2 \leqslant \frac{C}{h^3}|\zeta|^2 \int_{M_h} W\left(\nabla u^{(h)}\right) \leqslant C|\zeta|^2. \tag{14}$$

From this one deduces (9)–(11) by standard means (see [6] for the details, including convergence and regularity up to the boundary). In particular one deduces from (13) after summation over the relevant lattice points a

$$\int_{\Omega'_h \times (-1/2, 1/2)} \left|\left(\nabla u^{(h)}\right) \circ \psi^{(h)} - R^{(h)}\right|^2 \mathrm{d}z \leqslant Ch^2. \tag{15}$$

Lower bound. Let $F^{(h)} := (\nabla u^{(h)}) \circ \psi^{(h)}$. In view of (15) it is natural to introduce the scaled deviation from SO(3)

$$G^{(h)} = \frac{(R^{(h)})^{\mathrm{T}} F^{(h)} - I}{h} \chi_{\Omega'_h}.$$

Then, for a subsequence, $G^{(h)} \rightharpoonup G$ in $L^2(\Omega)$. Let $E_h = \{|G^{(h)}| > h^{-1/2}\}$. Then meas $E_h \to 0$ and using positivity of W, Taylor expansion for $z \in \Omega'_h \setminus E_h$ and the fact that $h^{-1} \det \nabla\psi^{(h)}(z) \to \det((\nabla\psi)^{\mathrm{T}}(\nabla\psi))^{1/2}(z') =: J_\psi(z')$ uniformly one deduces (see [6])

$$\liminf_{h \to 0} \frac{1}{h^3} \int_{M_h} W\left(\nabla u^{(h)}\right) \mathrm{d}x \geqslant \frac{1}{2} \int_{\Omega' \times (-1/2, 1/2)} Q_3(G) J_\psi \, \mathrm{d}x \geqslant \frac{1}{2} \int_{\Omega' \times (-1/2, 1/2)} Q_2\left(\psi(z'), G'\right) J_\psi \, \mathrm{d}z, \tag{16}$$

where $G'(z) = G(z)(I - \eta(z) \otimes \eta(z))$ and $\eta = \mu(\psi(z))$.

The main point is now to identify G'. To show that G' is affine in z_3 we consider the difference quotient in vertical direction

$$H^{(h)} = \Delta_3^s G^{(h)} := \frac{1}{s}\left(G^{(h)}(\cdot + se_3) - G^{(h)}\right).$$

Let $Y^{(h)} = u^{(h)} \circ \psi^{(h)}$. Then $\nabla Y^{(h)} = F^{(h)} \nabla \psi^{(h)}$ and a short calculation shows that

$$R^{(h)} H^{(h)} \nabla' \psi = \nabla' \Delta_3^s \left(\frac{1}{h} Y^{(h)} \right) - \Delta_3^s \widetilde{F}^{(h)} \nabla' \eta,$$

where $\widetilde{F}^{(h)}(z) = z_3 F^{(h)}(z)$. Since $H^{(h)} \rightharpoonup H,\ F^{(h)} \to R,\ R^{(h)} \to R$ and

$$\Delta_3^s \frac{1}{h} Y^{(h)} = \frac{1}{s} \int_0^s \frac{1}{h} Y_{,3}^{(h)}(\cdot + \sigma e_3)\, \mathrm{d}\sigma = \frac{1}{s} \int_0^s \big(F^{(h)} \eta\big)(\cdot + \sigma e_3)\, \mathrm{d}\sigma \to R\eta$$

we get $RH\nabla'\psi = \nabla'(R\eta) - R\nabla'\eta = RS_{M,u}\nabla'\psi$, where $S_{M,u}$ is the relative Weingarten map introduced in (6), (8). Thus $H' = S_{M,u}$ and $G''(z', z_3) = G_0'(z') + z_3 S_{M,u}$. Expanding the quadratic form Q_2 we easily deduce (12) from (16).

Attainment of lower bound. Let $u \in \mathcal{A},\ Y = u \circ \psi : \Omega' \to \mathbb{R}^3,\ b = R\eta : \Omega' \to \mathbb{R}^3$ where $R(z) \in \mathrm{SO}(3)$ is the unique extension of $(\nabla_{\tan} u)(\psi(z))$ from $T_{\psi(z)}M$ to $\mathbb{R}^3$. Then b is the unit normal, $b = Y_{,1} \wedge Y_{,2} / |Y_{,1} \wedge Y_{,2}|$. As in the case of plates [6], one can define suitable approximations $Y_\lambda \in W^{2,\infty}$ and $q_\lambda \in W^{1,\infty}$ which agree with Y and b, respectively, on a large set. Then the ansatz

$$Y^{(h)}(z', z_3) = Y_{\lambda_h}(z') + h z_3 q_{\lambda_h}(z') + \frac{1}{2} h^2 z_3^2 d_h(z'), \qquad u^{(h)} = Y^{(h)} \circ \big(\psi^{(h)}\big)^{-1},$$

where $\lambda_h = c/h,\ d_h \in W_0^{1,\infty}(\Omega'; \mathbb{R}^3),\ h\|d_h\|_{W^{1,\infty}} \to 0, d_h \to d$ in $L^2(\Omega'; \mathbb{R}^3)$ yields the desired assertion similarly to [6]. The map d is chosen such that $Q_3(S_{M,u} + R^{\mathrm{T}} d \otimes \mu) = Q_2(x, S_{M,u})$.

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