A model for the onset of breakdown in an axisymmetric compressible vortex

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A simple inviscid model to predict the onset of breakdown in an axisymmetric vortex is proposed. Three problems are considered: the shock-induced breakdown of a compressible vortex, the breakdown of a free compressible vortex, and the breakdown of a free incompressible vortex. The same physical reasoning is used in all three problems to predict the onset of breakdown. It is hypothesized that breakdown is the result of the competing effects of adverse pressure rise and streamwise momentum flux at the vortex centerline. Breakdown is assumed to occur if the pressure rise exceeds the axial momentum flux. A formula with no adjustable constants is derived for the critical swirl number in all three problems. The dependence of the critical swirl number on parameters such as upstream Mach number, excess/deficit in centerline axial velocity, and shock oblique angle is explored. The predictions for the onset of shock-induced breakdown and free incompressible breakdown are compared to experiment and computation, and good agreement is observed. Finally, a new breakdown map is proposed. It is suggested that the adverse pressure rise at the vortex axis be plotted against the axial momentum flux to determine the onset of breakdown. The proposed map allows the simultaneous comparison of data from flows ranging from incompressible breakdown to breakdown induced by a shock wave. © *1996 American Institute of Physics.* [S1070-6631(96)02112-5]

I. INTRODUCTION

A large body of information exists (e.g. see the review articles by Hall,[1] Leibovich,[2] Wedemeyer,[3] Escudier,[4] Stuart,[5] and Delery[6]) on the breakdown of incompressible streamwise vortices. Less is known about vortex breakdown at high speeds. An interesting example of supersonic vortex breakdown is the breakdown induced by the interaction of the vortex with a shock wave. The flow in supersonic engine inlets and over high-speed delta wings constitute technologically important examples of this phenomenon, which is termed ''shock-induced vortex breakdown.''

Gustintsev *et al.*[7] and Zatoloka *et al.*[8] appear to have conducted the earliest investigations into shock-induced vortex breakdown. The qualitative similarity of the flow to that of a separated boundary layer was noted in these experiments. Subsequently, Horowitz,[9] Delery *et al.*,[10] Metwally *et al.*,[11] and Cattafesta and Settles[12] have experimentally studied vortex breakdown induced by a normal shock. The interaction between streamwise vortices and wedge-attached oblique shock waves was experimentally investigated by Kalkhoran and Sforza.[13]

Horowitz[9] and Delery *et al.*[10] were the first to quantitatively characterize the nature of the breakdown. Their experiments studied normal shocks of strength equal to Mach 1.6, 1.75, 2 and 2.28. At each Mach number, they varied the swirl in the incident vortex and identified a critical swirl number above which the vortex would break down. The results were plotted on a ''breakdown map'' of swirl number against Mach number, where it was observed that the critical swirl number decreased as the Mach number of the shock increased. A companion numerical study using the Euler equations supported the experimentally observed trends. The experiments by Metwally *et al.*[11] and Cattafesta and Settles[12] extended the range of available data to Mach 4. Based on their visualization of the flow, Metwally *et al.*[11] proposed a qualitative picture of the flow-field resulting from the breakdown of the vortex.

Rizzetta[14] obtained numerical solutions to the Reynolds averaged Euler and Navier–Stokes equations, with the objective of predicting Kalkhoran and Sforza's[13] experimental measurements of pressure distribution on the wedge. The swirling supersonic flow in a circular duct was computed by Kandil *et al.*[15,16] who provided qualitative flow-field information on the breakdown. The most extensive computations of shock-induced vortex breakdown are the recent calculations by Erlebacher *et al.*[17] These workers studied the interaction between a streamwise vortex and a normal shock wave using the unsteady, axisymmetric, compressible Navier–Stokes equations. Mach numbers from 1.3 to 10 were computed. In the same spirit as Delery *et al.*,[10] a critical swirl number was numerically identified at each Mach number, and a breakdown map of swirl number against Mach number made. The trend observed by Delery *et al.*[10] was seen to extend to Mach 10; i.e., the critical swirl number decreased with increasing Mach number. Some interesting features of the flow field were also highlighted.

The only attempt to quantitatively predict some aspect of shock-induced breakdown appears to have been made by Cattafesta[18] who equated the ratio of swirl number (downstream to upstream) across the shock wave to the velocity ratio (upstream to downstream) across the shock. By comparing to experimental data, he obtained a value of 0.6 for the swirl number behind the shock wave. More recently, Erlebacher *et al.*[17] have proposed an empirical correlation between the critical swirl number and the Mach number of the shock wave, based on a curve fit to their data.

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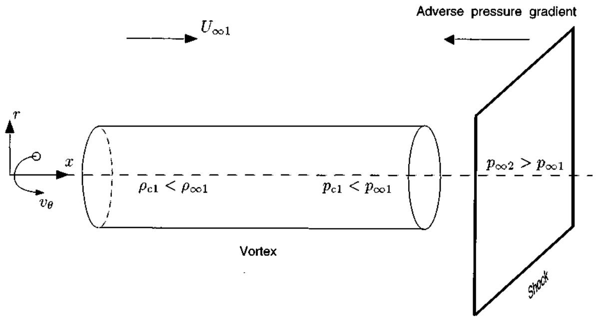

FIG. 1. Schematic of the interaction between a streamwise vortex and a normal shock wave.

In this paper, we propose a model to predict the onset of shock-induced vortex breakdown. The proposed model has no adjustable constants, and is compared to both experiment and computation. Also, the dependence of the critical swirl number on parameters such as the upstream Mach number, excess/deficit in centerline axial velocity, and shock oblique angle is explored. Two other problems are then considered: the breakdown of a free compressible vortex, and free incompressible vortex breakdown. The same breakdown criterion is used in all three problems to predict the onset of breakdown. Finally, a new breakdown map is proposed, that allows the simultaneous comparison of data from flows ranging from incompressible breakdown to breakdown induced by a shock wave.

This paper is organized as follows. A description of the upstream vortex is first provided in Section II A. This is followed in Section II B by a description of the proposed breakdown criterion and expressions for the critical swirl number. Section III compares the model predictions to computation and experiment. The influence of centerline excess/deficit in axial velocity, and obliquity of the shock wave is also discussed. The onset of breakdown in a free compressible vortex is discussed in Section IV. Incompressible vortex breakdown is briefly considered in Section V. A new breakdown map is then proposed in Section VI. The paper is concluded with a brief summary in Section VII.

II. STATEMENT OF PROBLEM

Figure 1 shows a schematic of the interaction between a streamwise vortex and a normal shock wave. The axial flow is from left to right. The variables x and r are used to denote the axial and radial coordinate respectively. The axial and swirl components of velocity are denoted by U and v_θ respectively, and p, ρ and T represent the pressure, density and temperature. The subscripts ''∞'' and ''c'' correspond to values in the free-stream and the centerline of the vortex, and the states upstream and downstream of the shock wave are respectively denoted by the subscripts ''1'' and ''2'' (e.g., $p_{\infty 2}$ denotes the free-stream pressure downstream of the shock wave).

A description of the incident vortex is first provided in Section II A. This is followed in Section II B by an outline of the model.

A. The upstream vortex

Studies of incompressible vortex breakdown (e.g. Darmofal[19]) suggest that the onset of breakdown is generally independent of viscosity for vortex Reynolds number (based on free-stream axial velocity and core radius) greater than about 300. As a result, viscosity is neglected in this paper. The upstream vortex is therefore governed by the axisymmetric, compressible Euler equations. It is readily seen that the profiles,

$$v_\theta = v_\theta(r), \quad U = U(r), \quad p = p(r), \quad \rho = \rho(r) \tag{1}$$

trivially satisfy the continuity, axial momentum and energy equations. The radial momentum equation,

$$\frac{dp}{dr} = \frac{\rho v_\theta^2}{r} \tag{2}$$

remains to be satisfied. Experiments[10,12] show that the swirl profile of the Burgers vortex is a good fit to experimental data. However, the Burgers profile makes analytical solution difficult. As a result, this paper uses the Rankine vortex as an approximation for the upstream vortex. Non-dimensionalizing the radial coordinate by the core radius (location where v_θ is maximum) and velocity by the peak value of the swirl velocity (denoted by $v_{\theta m}$), the swirl velocity profile of the upstream vortex is given by,

$$\begin{aligned} \tilde{v}_\theta &= \tilde{r}, \quad \tilde{r} \leqslant 1 \\ &= \frac{1}{\tilde{r}}, \quad \tilde{r} \geqslant 1, \end{aligned} \tag{3}$$

where the tilde is used to denote non-dimensional variables.

The density varies with radius for a compressible vortex. This paper considers two different idealizations of the thermodynamic field in the upstream vortex: spatially uniform stagnation temperature and spatially uniform entropy. The assumption of uniform stagnation temperature is prompted by experimental data. Delery *et al.*[10] note that the total temperature in the upstream vortex in their experiments is approximately uniform. Measurements in a Mach 3 vortex by Metwally *et al.*[11] and Cattafesta and Settles[12] seem to support this approximation. Cattafesta and Settles' data (Fig. 7 of their paper) show a deficit of about 4% of free-stream in total temperature at the centerline. The idealization of uniform entropy is prompted by past theoretical and computational studies on compressible vortices (e.g. Colonius *et al.*[20]).

Expressions for the centerline pressure and density for the uniform stagnation temperature vortex and uniform entropy vortex are obtained below. Defining the non-dimensional variables,

$$\tilde{p} = \frac{p}{p_\infty}, \quad \tilde{\rho} = \frac{\rho}{\rho_\infty}, \quad \tilde{T} = \frac{T}{T_\infty} \tag{4}$$

the radial momentum equation becomes,

$$\frac{d\tilde{p}}{dr} = \gamma(\Gamma M_\infty)^2 \tilde{\rho} \frac{\tilde{v}_\theta^{\,2}}{\tilde{r}}. \tag{5}$$

The variable γ denotes the ratio of specific heats and is taken as 1.4 in this paper. Γ is the swirl number of the vortex, and is defined as $\Gamma = v_{\theta m}/U_\infty$. M_∞ is the free-stream Mach number, defined as $M_\infty = U_\infty/c_\infty$. ΓM_∞ will be recognized as the swirl Mach number, $v_{\theta m}/c_\infty$.

Uniform entropy vortex: If the entropy is spatially uniform,

$$\tilde{p} = \tilde{\rho}^{\gamma}. \tag{6}$$

Expressing the density in terms of the pressure in the radial momentum equation and integrating yields the following expressions for the centerline pressure and density:

$$\tilde{p}_c = [1 - (\gamma - 1)\Gamma^2 M_\infty^2]^{\gamma/(\gamma-1)}, \quad \tilde{\rho}_c = [1 - (\gamma - 1)\Gamma^2 M_\infty^2]^{1/(\gamma-1)}. \tag{7}$$

Uniform stagnation temperature vortex: The spatial uniformity of stagnation temperature requires that

$$T + \frac{U^2 + v_\theta^2}{2C_p} = T_\infty + \frac{U_\infty^2}{2C_p}. \tag{8}$$

Delery *et al.*'s[10] experiments show that the axial velocity in the upstream vortex was nearly uniform; i.e., $U = U_\infty$. Cattafesta and Settles[12] on the other hand, observe a wake-like profile. This paper assumes uniform axial velocity for the uniform stagnation temperature vortex. This yields the following expression for the non-dimensional temperature in the vortex:

$$\tilde{T} = 1 - \frac{\gamma - 1}{2}(\Gamma M_\infty)^2 \tilde{v}_\theta^{\,2}. \tag{9}$$

The equation of state implies that $\tilde{p} = \tilde{\rho}\,\tilde{T}$. Substituting for $\tilde{\rho}$ and $\tilde{T}$ in the radial momentum equation and integrating yields the following expressions for the density and pressure at the centerline of the uniform stagnation temperature vortex:

$$\tilde{p}_c = \left[1 - \frac{(\gamma - 1)}{2}\Gamma^2 M_\infty^2\right]^{2\gamma/(\gamma-1)}, \quad \tilde{T}_c = 1, \quad \tilde{\rho}_c = \tilde{p}_c. \tag{10}$$

B. A criterion for shock-induced breakdown

A simple criterion for breakdown of the upstream vortex is first proposed. The properties of the upstream vortex (Section II A) are then used to obtain an expression for the critical swirl number above which the vortex would break down. The breakdown criterion is based upon an approximation to the axial momentum equation at the centerline of the vortex. Note that as a result of axisymmetry, the radial velocity at the centerline would be zero. When combined with the swirl velocity being zero at the centerline, this suggests that the flow near the vortex centerline would largely be in the streamwise direction. The one-dimensional momentum equations may therefore be used to model the flow around the vortex centerline. $p + \rho U^2$ would therefore be constant across a region of rapid streamwise variation.

Consider the vortex impinging upon the shock wave. On account of the rotation, the pressure at the center of the vortex is less than the free-stream value; i.e., $p_{c1} < p_{\infty 1}$. Pressure rises across a shock wave; i.e., $p_{\infty 2} > p_{\infty 1}$. The vortex therefore experiences an adverse streamwise pressure rise, which may be quantified by the pressure difference, $p_{\infty 2} - p_{c1}$. The fluid in the vortex has a certain inertia in the streamwise direction, which may be quantified by the streamwise momentum flux, $\rho_{c1} U_{c1}^2$. Breakdown is assumed to occur if the axial pressure rise exceeds the upstream streamwise momentum flux, thereby stagnating the flow; i.e., if

$$p_{\infty 2} - p_{c1} \geqslant \rho_{c1} U_{c1}^2 \geqslant \rho_{c1} U_{\infty 1}^2 \left(1 + \frac{\Delta U}{U_{\infty 1}}\right)^2 \tag{11}$$

where ΔU denotes the upstream excess in axial velocity at the centerline. If the axial velocity is uniform, then $\Delta U = 0$. The threshold for breakdown is therefore given by the relation,

$$p_{\infty 2} - p_{c1} = \rho_{c1} U_{\infty 1}^2 \left(1 + \frac{\Delta U}{U_{\infty 1}}\right)^2. \tag{12}$$

The axial velocity is assumed to be uniform through most of this paper. The effect of non-uniform axial velocity is separately discussed in Section III B. Equation (12) may be rewritten in non-dimensional form for uniform axial velocity as,

$$\tilde{p}_{\infty 2} - \tilde{p}_{c1} = \gamma \tilde{\rho}_{c1} M_{\infty 1}^2. \tag{13}$$

We have already obtained expressions for $\tilde{p}_{c1}$ and $\tilde{\rho}_{c1}$ in terms of Γ and $M_{\infty 1}$. The Rankine–Hugoniot equations for a normal shock express $\tilde{p}_{\infty 2}$ in terms of the upstream Mach number, $M_{\infty 1}$. Substituting for $\tilde{p}_{c1}$, $\tilde{\rho}_{c1}$ and $\tilde{p}_{\infty 2}$ into the above breakdown criterion will therefore yield an expression for the critical swirl number Γ_{crit} in terms of Mach number of the shock wave for a vortex with uniform axial velocity. This expression is derived below.

Uniform stagnation temperature vortex: For a uniform stagnation temperature vortex, we have $\tilde{\rho}_{c1} = \tilde{p}_{c1}$. Substitution into the criterion for breakdown [Eq. (13)] yields,

$$\tilde{p}_{c1} = \frac{\tilde{p}_{\infty 2}}{1 + \gamma M_{\infty 1}^2}, \tag{14}$$

where $\tilde{p}_{\infty 2}$ is given by the Rankine–Hugoniot equations as,

$$\tilde{p}_{\infty 2} = 1 + \frac{2\gamma}{\gamma + 1}(M_{\infty 1}^2 - 1). \tag{15}$$

Substituting for $\tilde{p}_{c1}$ from Eq. (10) and $\tilde{p}_{\infty 2}$ from Eq. (15) into Eq. (14), we get,

$$\left[1 - \frac{\gamma - 1}{2}\Gamma_{\text{crit}}^2 M_{\infty 1}^2\right]^{2\gamma/(\gamma-1)} = \frac{1}{1 + \gamma M_{\infty 1}^2}\left[1 + \frac{2\gamma}{\gamma + 1}(M_{\infty 1}^2 - 1)\right], \tag{16}$$

which upon rearrangement yields the following expression for the critical swirl number as a function of the Mach number of the shock:

$$\Gamma_{\rm crit}=\frac{1}{M_{\infty 1}}\sqrt{\frac{2}{\gamma-1}\left\{1-\left(\frac{1}{1+\gamma M_{\infty 1}^2}\left[1+\frac{2\gamma}{\gamma+1}(M_{\infty 1}^2-1)\right]\right)^{(\gamma-1)/2\gamma}\right\}}. \tag{17}$$

Uniform entropy vortex: Expressions for the centerline density and temperature for a uniform entropy vortex are given by Eq. (7). Substitution into Eq. (13) yields the following implicit equation for the critical swirl number as a function of the Mach number:

$$1+\frac{2\gamma}{\gamma+1}(M_{\infty 1}^2-1)-[1-(\gamma-1)\Gamma_{\rm crit}^2 M_{\infty 1}^2]^{\gamma/(\gamma-1)} = \gamma M_{\infty 1}^2[1-(\gamma-1)\Gamma_{\rm crit}^2 M_{\infty 1}^2]^{1/(\gamma-1)}. \tag{18}$$

The Newton–Raphson method was used to solve the above equation for $\Gamma_{\rm crit}$ as a function of the Mach number of the shock wave.

III. RESULTS: SHOCK-INDUCED VORTEX BREAKDOWN

A. Uniform axial velocity

Results for the critical swirl number are presented for the case where the axial velocity is uniform. Figure 2 shows the predicted values of the critical swirl number as a function of the Mach number of the shock. The predicted values are compared to the experimental values reported by Delery *et al.*[10] (the data were obtained from Fig. 35 of their paper) for Mach numbers of 1.75, 2 and 2.28. Also shown are results from the computations by Erlebacher *et al.*[17] (the data were obtained from Table 3 of their report). Note that the computational data at Mach 1.7 were very close to the experimental value at Mach 1.75 (0.331 as compared to 0.33). This made the experimental data hard to discern when both experimental and computational results were plotted. As a result, the computational value at Mach 1.7 is not plotted in Fig. 2.

The predicted values are seen to be in good agreement with both experiment and computation. The critical swirl number is predicted to decrease with increasing Mach number as observed. According to the proposed criterion [Eqs. (7), (10) and (13)], this decrease in $\Gamma_{\rm crit}$ is due to a combination of two factors: increase in the adverse pressure rise (due to $\tilde{p}_{\infty 2}$ increasing while $\tilde{p}_{c1}$ decreases) and decrease in streamwise momentum flux (due to $\tilde{\rho}_{c1}$ decreasing) with increasing Mach number.

The ability of the model to predict the onset of shock-induced breakdown is further evaluated in Fig. 3, where data from Metwally *et al.*[11] are plotted (obtained from Fig. 6 of their paper). The "strong interactions" observed experimentally are seen to lie in the region where the model predicts breakdown, while the "weak interaction" regions lie in the predicted region of non-breakdown. Note that the curve of $\Gamma_{\rm crit}$ in Fig. 3 assumes uniform axial velocity. Metwally *et al.*[11] point out that the Mach 3 and Mach 3.5 vortices had noticeable deficit in centerline velocity for the breakdown cases. As will be seen in Section III B, the critical swirl number is predicted to decrease as the centerline velocity decreases; i.e., the filled symbols for the Mach 3 and Mach 3.5 cases would move further into the breakdown region if the deficit in centerline velocity were accounted for in Fig. 3.

B. Non-uniform axial velocity

The influence of an excess/deficit in the centerline axial velocity on the critical swirl number is next considered. For

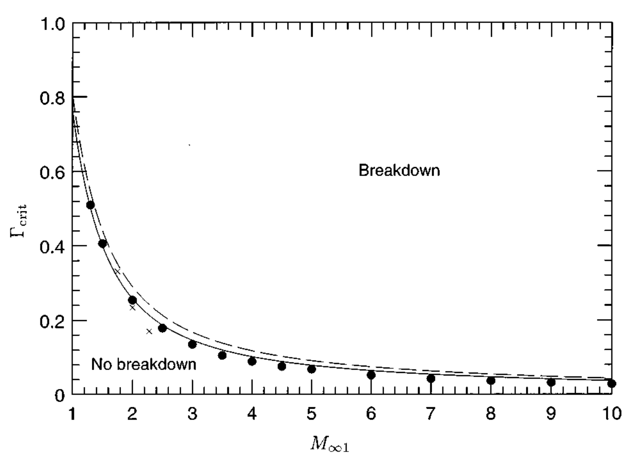

FIG. 2. Comparison of predicted critical swirl number to experiment and computation of shock-induced vortex breakdown. — (Prediction: uniform stagnation temperature), --- (prediction: uniform entropy), ● (computation—Ref. 17), × (experiment—Ref. 10).

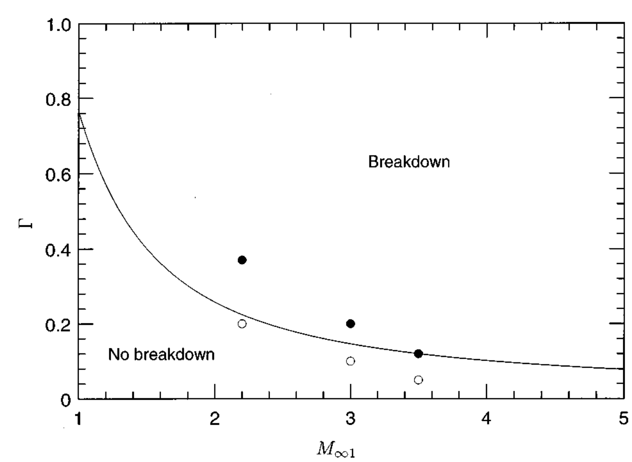

FIG. 3. Evaluation of model in predicting the onset of shock-induced vortex breakdown. — (Predicted $\Gamma_{\rm crit}$ [Eq. (17)]: uniform stagnation temperature), ● (experiment—Ref. 11: breakdown), ○ (experiment—Ref. 11: no breakdown).

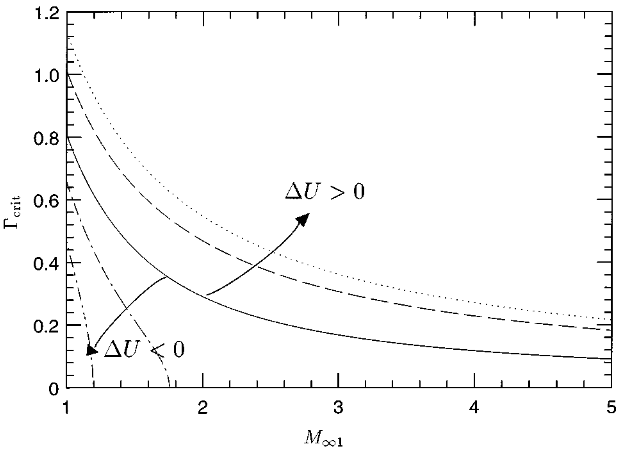

FIG. 4. Influence of axial velocity on the onset of vortex breakdown induced by a shock. –·– ($\Delta U/U_{\infty 1}=-0.5$), ––– ($\Delta U/U_{\infty 1}=-0.25$), — ($\Delta U/U_{\infty 1}=0$), --- ($\Delta U/U_{\infty 1}=0.5$), ··· ($\Delta U/U_{\infty 1}=1$).

convenience, results are shown only for the uniform entropy vortex. The breakdown criterion [Eq. (12)] may be divided through by $p_{\infty 1}$ to yield the following non-dimensional criterion:

$$\tilde{p}_{\infty 2}-\tilde{p}_{c1}=\gamma\tilde{\rho}_{c1}M_{\infty 1}^2\left(1+\frac{\Delta U}{U_{\infty 1}}\right)^2. \tag{19}$$

Substituting for $\tilde{p}_{c1}$ and $\tilde{\rho}_{c1}$ from Eqs. 7, we get the following equation for Γ_{crit} as a function of $M_{\infty 1}$ and $\Delta U/U_{\infty 1}$:

$$\tilde{p}_{\infty 2}-[1-(\gamma-1)\Gamma_{crit}^2M_{\infty 1}^2]^{\gamma/(\gamma-1)} = \gamma M_{\infty 1}^2\left(1+\frac{\Delta U}{U_{\infty 1}}\right)^2[1-(\gamma-1)\Gamma_{crit}^2M_{\infty 1}^2]^{1/(\gamma-1)}. \tag{20}$$

The Newton–Raphson method was used to solve the above equation for Γ_{crit}, after expressing $\tilde{p}_{\infty 2}$ in terms of $M_{\infty 1}$. Figure 4 shows the variation of the critical swirl number with Mach number for different values of $\Delta U/U_{\infty 1}$. Note that $\Delta U>0$ corresponds to a jet-like axial velocity profile of the upstream vortex while $\Delta U<0$ corresponds to a wake-like profile. The predicted results show a strong sensitivity to the excess/deficit in centerline axial velocity. Jet-like profiles of the axial velocity are observed to delay breakdown, while a wake-like profile makes the vortex more susceptible to breakdown. The same trend is known to apply in the breakdown of an incompressible vortex, where axial blowing is often used to alleviate the breakdown.[6]

Figure 4 shows that for vortices with a wake-like axial velocity, the critical swirl number becomes zero at a finite Mach number; i.e., breakdown is predicted at and beyond this cut-off Mach number, even in the absence of swirl . This result may be explained as follows. In the absence of swirl, the ''vortex'' reduces to an axisymmetric wake (or jet). This wake (or jet) can undergo reverse flow accompanied by radial outflow upon experiencing a strong enough adverse pressure gradient. We have assumed that breakdown occurs when the adverse pressure rise at the vortex centerline exceeds the centerline axial momentum flux. We noted that on account of its rotation, the centerline pressure rise, $p_{\infty 2}-p_{c1}$ is greater than the free-stream rise, $p_{\infty 2}-p_{\infty 1}$. Also, rotation results in the centerline density (ρ_{c1}) being lower than the free-stream density. As a result, the centerline momentum flux, $\rho_{c1}U_{c1}^2$ is less than its value computed using the free-stream density. Thus, swirl ''amplifies'' (in the terminology of Hall[1]) the adverse pressure rise experienced by the vortex while suppressing the axial momentum flux. Both factors make the vortex more susceptible to breakdown. This implies that if the free-stream pressure rise exceeds the axial momentum flux computed using the free-stream density, then the presence of swirl is not needed for ''breakdown.'' The flow at and above the cut-off Mach number corresponds to this scenario. The cut-off Mach number (denoted by M_{cut}) can therefore be predicted by the following criterion:

$$p_{\infty 2}-p_{\infty 1}=\rho_{\infty 1}U_{c1}^2 \tag{21}$$

which yields,

$$\tilde{p}_{\infty 2}-1=\gamma M_{cut}^2\left(1+\frac{\Delta U}{U_{\infty 1}}\right)^2. \tag{22}$$

Substituting for $\tilde{p}_{\infty 2}$ from Eq. (15), we get the following equation for the cut-off Mach number in terms of the velocity excess/deficit:

$$\frac{2\gamma}{\gamma+1}(M_{cut}^2-1)=\gamma M_{cut}^2\left(1+\frac{\Delta U}{U_{\infty 1}}\right)^2 \tag{23}$$

which may be rearranged to obtain the following expression for the cut-off Mach number:

$$M_{cut}=\sqrt{\frac{2\gamma}{\gamma+1}\left[\frac{2\gamma}{\gamma+1}-\gamma\left(1+\frac{\Delta U}{U_{\infty 1}}\right)^2\right]^{-1}}. \tag{24}$$

C. Breakdown induced by an oblique shock wave

If the shock wave were oblique, the onset of breakdown would be expected to depend on the oblique angle. Although the interaction of an oblique shock with an axisymmetric vortex is not axisymmetric, it is envisioned that the onset of breakdown can be predicted by extending the arguments of the previous section. Reiterating the criterion for breakdown for uniform axial velocity, we require that $\tilde{p}_{\infty 2}-\tilde{p}_{c1}=\gamma\tilde{\rho}_{c1}M_{\infty 1}^2$. The influence of shock obliquity is modeled as follows. The properties of the upstream vortex ($\tilde{p}_{c1},\tilde{\rho}_{c1}$) depend solely upon the free-stream Mach number and swirl number. However the pressure behind the shock ($\tilde{p}_{\infty 2}$) is determined by the normal Mach number, $M_{\infty 1}\sin\alpha$ (α denotes the angle the shock makes with the streamwise direction). Replacing $M_{\infty 1}$ in Eq. (15) by $M_{\infty 1}\sin\alpha$ to obtain $\tilde{p}_{\infty 2}$ and substituting as before for $\tilde{p}_{c1}$ and $\tilde{\rho}_{c1}$ yields the following expressions for the critical swirl number.

Uniform stagnation temperature vortex:

$$\Gamma_{\rm crit}=\frac{1}{M_{\infty1}}\sqrt{\frac{2}{\gamma-1}\left\{1-\left(\frac{1}{1+\gamma M_{\infty1}^2}\left[1+\frac{2\gamma}{\gamma+1}([M_{\infty1}\,\sin\,\alpha]^2-1)\right]\right)^{(\gamma-1)/2\gamma}\right\}}. \tag{25}$$

Uniform entropy vortex:

$$1+\frac{2\gamma}{\gamma+1}([M_{\infty1}\,\sin\,\alpha]^2-1)-[1-(\gamma-1)\Gamma_{\rm crit}^2M_{\infty1}^2]^{\gamma/(\gamma-1)}=\gamma M_{\infty1}^2[1-(\gamma-1)\Gamma_{\rm crit}^2M_{\infty1}^2]^{1/(\gamma-1)}. \tag{26}$$

It is readily seen that for the same upstream Mach number, $\Gamma_{\rm crit}$ is predicted to increase as the shock becomes increasingly oblique. This prediction may be explained by noting that the pressure rise across an oblique shock is lower than that for a normal shock at the same Mach number. As a result, the adverse pressure rise that the vortex experiences is smaller, thereby delaying the onset of breakdown.

IV. SHOCK-FREE BREAKDOWN OF A COMPRESSIBLE VORTEX

Section III discussed vortex breakdown induced by a shock wave. The breakdown of a free axisymmetric vortex, i.e. breakdown in the absence of an externally imposed pressure gradient, is considered in this section. Incompressible streamwise vortices at sufficiently high swirl number are known to break down, even in the absence of an externally applied adverse pressure gradient. It is to be expected that their high-speed counterparts would exhibit similar behavior. The critical swirl number in high-speed flow would be a function of the Mach number. This section derives an expression for the critical swirl number in terms of the free-stream Mach number; i.e., we consider the influence of compressibility on the breakdown of a free vortex. The arguments used are identical to those in breakdown induced by a shock. The only difference is that while the adverse pressure rise was set equal to $p_{\infty2}-p_{c1}$ for shock-induced breakdown, it is set equal to $p_{\infty1}-p_{c1}$ for the shock-free breakdown. The rationale for this assumption is that in the absence of the shock, the vortex discharges into the atmosphere. As a result, the vortex sees a pressure equal to $p_{\infty1}$ ahead of it, as well as in the free-stream. The difference between atmospheric pressure ($p_{\infty1}$), and the pressure at the vortex centerline (p_{c1}) provides the adverse pressure rise that causes breakdown. Breakdown of the vortex is therefore assumed to occur when

$$p_{\infty1}-p_{c1}\geqslant\rho_{c1}U_{c1}^2. \tag{27}$$

The criterion for shock-free breakdown is therefore given by,

$$1-\tilde{p}_{c1}=\gamma\tilde{\rho}_{c1}M_{\infty1}^2 \tag{28}$$

which is identical to the expression obtained when $\tilde{p}_{\infty2}$ is set to 1 in Eq. (13). The corresponding expressions for the critical swirl number are given below.

Uniform stagnation temperature vortex:

$$\Gamma_{\rm crit}=\frac{1}{M_{\infty1}}\sqrt{\frac{2}{\gamma-1}\left[1-\left(\frac{1}{1+\gamma M_{\infty1}^2}\right)^{(\gamma-1)/2\gamma}\right]}. \tag{29}$$

Uniform entropy vortex:

$$1-[1-(\gamma-1)\Gamma_{\rm crit}^2M_{\infty1}^2]^{\gamma/(\gamma-1)}$$
$$=\gamma M_{\infty1}^2[1-(\gamma-1)\Gamma_{\rm crit}^2M_{\infty1}^2]^{1/(\gamma-1)}. \tag{30}$$

Figure 5 shows the predicted values of the critical swirl number as a function of the free-stream Mach number. Also shown (for supersonic flow) are the values obtained for breakdown induced by a shock wave at the same Mach number. Compressibility is seen to make the vortex more susceptible to breakdown. A similar trend was noted by Keller.[21] This trend may be explained by noting [Eqs. (7) and (10)] that increase in the free-stream Mach number decreases the centerline pressure and density, thereby increasing the adverse pressure rise while decreasing the axial momentum flux. The predicted values of $\Gamma_{\rm crit}$ in the absence of the shock are seen to be greater than those predicted for shock-induced breakdown. This trend can be explained by noting that the pressure rise across the shock wave produces a larger adverse pressure rise for the same upstream momentum flux.

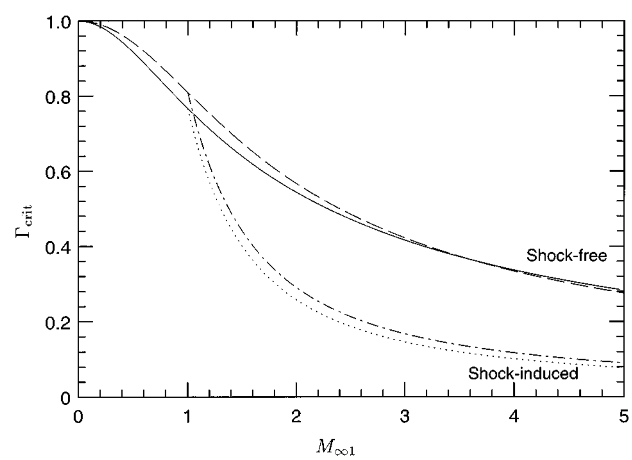

FIG. 5. Predicted critical swirl number for shock-free vortex breakdown compared to the prediction for shock-induced breakdown. — (Shock-free: uniform stagnation temperature), --- (shock-free: uniform entropy), ⋯ (shock-induced: uniform stagnation temperature), –·– (shock-induced: uniform entropy).

TABLE I. Prediction of critical swirl number for incompressible vortex breakdown compared to other approaches. All data other than the present reproduced from review article by Delery (Ref. 6).

	S_{crit}
Quasi-cylindrical	1.41
Axisymmetric N-S	1.35
Bossel	1.12
Squire	1.4
Benjamin	1.4
Num. simulation	1.28
Spall *et al.*	1.37
Present	**1.4**

V. INCOMPRESSIBLE VORTEX BREAKDOWN

Figure 5 shows that as $M_{\infty 1}$ tends towards 0, Γ_{crit} tends towards 1. An incompressible vortex in the absence of externally imposed adverse pressure gradients, is therefore predicted to undergo breakdown at a critical swirl number of one. The same result can of course be derived, by setting $\rho = \rho_\infty$ in the radial momentum equation and integrating to obtain the centerline pressure ($p_{c1} = p_{\infty 1} - \rho_\infty v_{\theta m}^2$), which is then substituted into the breakdown criterion [Eq. (27)]. In a recent review article, Delery[6] documents (Section 3.4.5 of his paper) critical swirl numbers for incompressible vortex breakdown as predicted by different theories. He considers a Burgers vortex, and defines a swirl parameter S as

$$S = \frac{C}{r_c U_\infty}, \tag{31}$$

where the variables C and r_c denote the circulation and core radius respectively. For a Burgers vortex, the swirl velocity is given by (Eq. 1 in Delery's[6] paper)

$$v_\theta = \frac{C}{r}[1 - e^{-1.256(r/r_c)^2}]. \tag{32}$$

This implies that the swirl parameter S is related to the swirl number Γ by,

$$S = \frac{\Gamma}{1 - e^{-1.256}} = 1.398\Gamma. \tag{33}$$

Thus $\Gamma_{crit} = 1$ corresponds to $S_{crit} = 1.398 \approx 1.4$. We reproduce in Table I, from Delery's[6] paper, the critical swirl numbers predicted by different approaches. Most approaches are seen to predict values very close to that predicted by our simple criterion.

VI. A "UNIVERSAL" BREAKDOWN MAP

The preceding sections presented results for the onset of vortex breakdown by plotting the critical swirl number as a function of Mach number. The curve $\Gamma_{crit} = \Gamma_{crit}(M_{\infty 1})$ defined the boundary between the regimes of breakdown and non-breakdown. However, it is clear that the critical swirl number is not universal (as also observed by Delery[6]). For example, Section III B (Fig. 4) showed that Γ_{crit} depended on the velocity excess/deficit at the centerline. If the breakdown were precipitated by an oblique shock wave as opposed to a normal shock, then Γ_{crit} was noted to depend on the inclination angle of the shock. Similarly, if the breakdown were that of a free vortex instead of being shock induced, yet another curve for the critical swirl number was obtained.

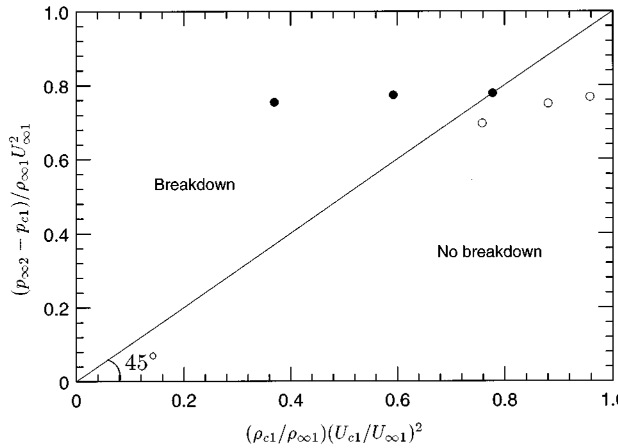

FIG. 6. Evaluation of the proposed breakdown map in predicting the onset of vortex breakdown. ● (Experiment: breakdown), ○ (experiment: no breakdown).

In this section, we propose a breakdown map that allows a common breakdown boundary to be defined for all of the above mentioned problems. The proposed map is based on the breakdown criterion that was proposed in Section II B; i.e.,

$$p_{\infty 2} - p_{c1} \geqslant \rho_{c1} U_{c1}^2. \tag{34}$$

Recall that the same criterion with $p_{\infty 2}$ appropriately defined, was applied to all the breakdown problems discussed in this paper. This suggests that a plot of $p_{\infty 2} - p_{c1}$ against $\rho_{c1} U_{c1}^2$ could be used to map the onset of vortex breakdown. The proposed map could even be used for incompressible vortex breakdown, and would be expected to adequately represent the onset of breakdown induced by pressure gradients acting over distances that are small as compared to a characteristic length scale of the vortex. The curve $p_{\infty 2} - p_{c1} = \rho_{c1} U_{c1}^2$ (the 45° line) would act as the boundary between the breakdown and non-breakdown regimes. Note that the proposed map does not require any additional data to be measured. Experimental information on parameters such as $\Gamma, \Delta U/U_\infty, M_\infty$ and shock angle could be used to obtain both the pressure rise and the axial momentum flux using the equations in Section II A. The proposed map is illustrated in Fig. 6. Note that the pressure rise and momentum flux are nondimensionalized by $\rho_{\infty 1} U_{\infty 1}^2$ to allow incompressible data to be plotted. Data from Metwally [11] (the same data shown in Fig. 3) are also shown. The data from Fig. 3 are combined with Eq. 10 to determine the pressure rise and axial momentum flux. The breakdown and non-breakdown cases are seen to be appropriately delineated.

VII. SUMMARY

A simple inviscid model was proposed to predict the onset of breakdown in an axisymmetric vortex. Three problems were considered: the shock-induced breakdown of a compressible vortex, the breakdown of a free compressible vortex, and the breakdown of a free incompressible vortex.

The same physical reasoning was used to predict the onset of breakdown in all three problems. It was hypothesized that breakdown is the result of the competing effects of adverse pressure rise and streamwise momentum flux at the vortex centerline. Breakdown was assumed to occur if the pressure rise exceeded the axial momentum flux. A formula with no adjustable constants was derived for the critical swirl number in all three problems. The dependence of the critical swirl number on parameters such as upstream Mach number, excess/deficit in centerline axial velocity, and shock oblique angle was explored. The predictions for the onset of shock-induced breakdown and free incompressible breakdown were compared to experiment and computation, and good agreement was observed. Finally, a new breakdown map was proposed as an alternative to the map of critical swirl number against free-stream Mach number. The new map was based on the observation that the same breakdown criterion was used in all the problems considered in this paper. To determine the onset of breakdown, it was suggested that the adverse pressure rise at the vortex centerline, be plotted against the axial momentum flux. The proposed map allows the simultaneous comparison of data from flows ranging from incompressible breakdown to breakdown induced by a shock wave.

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