Dissipation integral method for turbulent boundary layers

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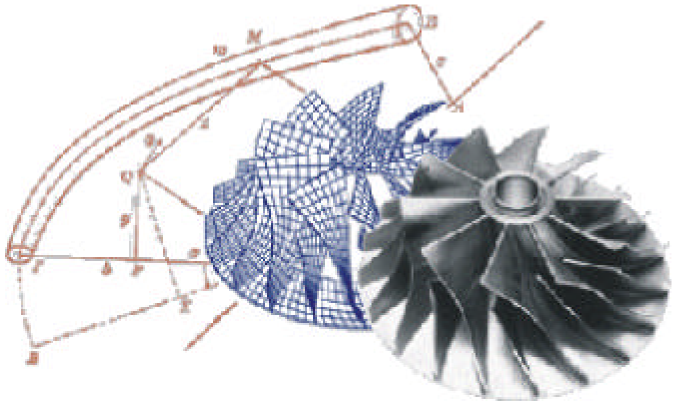
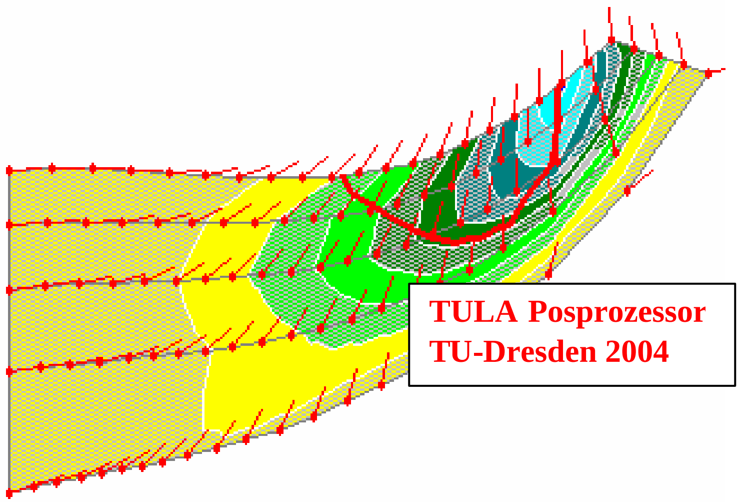
Outline

- Looking for a calculation procedure for three-dimensional fully turbulent boundary layers
- robust and applicable for engineering applications
- they should not compete with LES or other approaches





Combining an old a new idea

Wärme umgesetzt wird und so mechanisch verloren geht, folgt nun aus dem allgemeinen Energiesatz bei Berücksichtigung der üblichen Grenzschichtvernachlässigungen noch eine weitere gewöhnliche Differentialgleichung:

$$\frac{d}{dx}\int_0^\infty \varrho\, u\left(\frac{1}{2}U^2 - \frac{1}{2}u^2\right) dy = \mu \int_0^\infty \left(\frac{\partial u}{\partial y}\right)^2 dy$$

(Energiestromverlust je Längeneinheit = Dissipation in der Längeneinheit).

Diese Gleichung kann man ebenfalls leicht aus (1) ableiten, indem man nach Addition

Wieghardt, K,: Über einen Energiesatz zur Berechnung laminarer Grenzschichten, ZAMM 1948, pp. 231 ff.

Eine solche Energiegleichung besteht übrigens auch für turbulente Reibungsschichten, nur daß dort für die Dissipation statt $\mu \int_0^\infty \left(\frac{\partial u}{\partial y}\right)^2 dy$ zu schreiben ist:

$$\int_0^\infty \tau \frac{\partial u}{\partial y} dy = -\int_0^\infty \frac{\partial \tau}{\partial y} u\, dy, \qquad (9)$$

Perry , A. E. et al.: Wall turbulence closure based on classical similarity laws and the attached eddy hypothesis, Phys. Fluids 6 (2), 1994, pp. 1024-1035

that is

$$\frac{U_1 - U}{U_\tau} = f[\eta, \Pi]$$

$$= -\frac{1}{\kappa}\ln\eta + \frac{\Pi}{\kappa} W_c[1,\Pi] - \frac{\Pi}{\kappa} W_c[\eta,\ \Pi] \qquad (4)$$

where U_1 is the local free-stream velocity.

The mean continuity equation is

$$\frac{\partial U}{\partial x} + \frac{\partial W}{\partial z} = 0 \qquad (5)$$

where W is the mean velocity component normal to the wall. Note that two-dimensional mean flow is being assumed.

The mean momentum equation is

$$U\frac{\partial U}{\partial x} + W\frac{\partial U}{\partial z} = -\frac{1}{\rho}\frac{dp}{dx} + \frac{1}{\rho}\frac{\partial \tau}{\partial z} \qquad (6)$$

momentum balances and shear stress profiles to take the logarithmic law profile all the way to the wall.

Substituting (4) and (5) into (6) and making use of relations derived from the logarithmic law of the wall and momentum integral equation we obtain, after much algebra

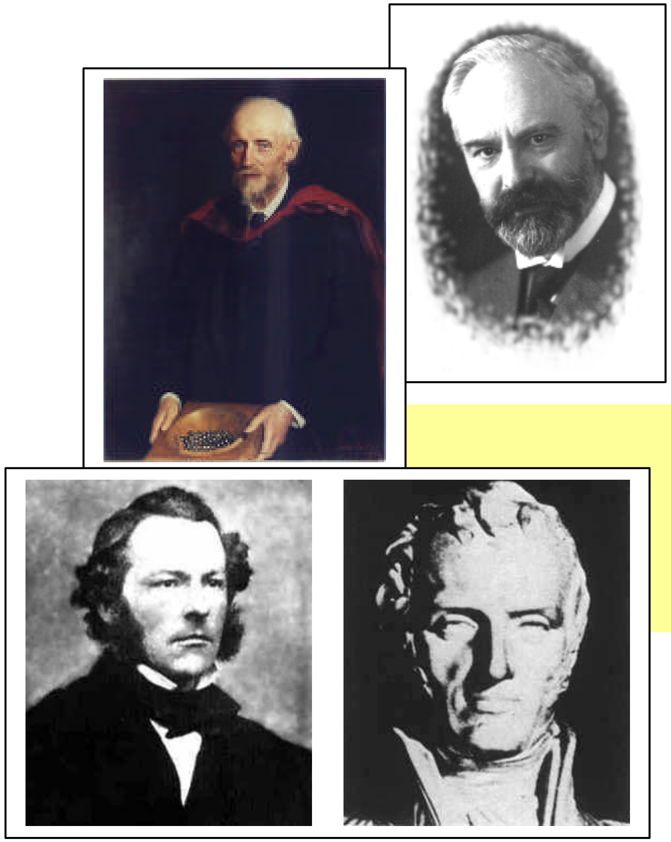
$$\frac{\tau}{\tau_0} = f_1[\eta,\Pi,S] + f_2[\eta,\Pi,S]\delta_c \frac{d\Pi}{dx} + f_3[\eta,\Pi,S]\frac{\delta_c}{U_1}\frac{dU_1}{dx}. \qquad (8)$$

Here

$$S = U_1/U_\tau = \sqrt{2/C_f'} \qquad (9)$$

where C_f' is the local skin friction coefficient, given by

Boundary layer equations



$$\frac{u}{h_1}\frac{\partial u}{\partial x}+\frac{v}{h_2}\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}-v^2\frac{1}{h_1h_2}\frac{\partial h_2}{\partial x}+uv\frac{1}{h_1h_2}\frac{\partial h_1}{\partial y}=-\frac{1}{\rho}\frac{1}{h_1}\frac{\partial p_\infty}{\partial x}+\frac{1}{\rho}\frac{\partial \tau_x}{\partial z}$$

$$\frac{u}{h_1}\frac{\partial v}{\partial x}+\frac{v}{h_2}\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}-u^2\frac{1}{h_1h_2}\frac{\partial h_1}{\partial y}+uv\frac{1}{h_1h_2}\frac{\partial h_2}{\partial x}=-\frac{1}{\rho}\frac{1}{h_2}\frac{\partial p_\infty}{\partial y}+\frac{1}{\rho}\frac{\partial \tau_y}{\partial z}$$

$$\tau_x=\mu\frac{\partial u}{\partial z}-\rho\overline{u'v'}$$

$$\tau_y=\mu\frac{\partial v}{\partial z}-\rho\overline{u'w'}$$

$$\frac{\partial\left(h_2u\right)}{\partial x}+\frac{\partial\left(h_1v\right)}{\partial y}+h_1h_2\frac{\partial w}{\partial z}=0$$

Two steps to obtain the integral equations

Step 1: **Continuity equation and the momentum equations are multiplied with weighting functions. To obtain the general dissipation integral equations these weighting functions have to be chosen as power functions of the mean velocity profiles.**

Step 2: **Integration of the momentum equation versus the wall normal coordinate. The integration boundaries are the wall and the outer edge of the boundary layer. The velocity component perpendicular to the wall is eliminated using the continuity.**

	x-component	y-component
continuity equation	$\frac{u^{(k+1)}}{(k+1)}$	$\frac{v^{(k+1)}}{(k+1)}$
momentum equations	u^k	v^k

General form of dissipation integral equations

$$\frac{1}{h_1}\frac{\partial \phi_k}{\partial x}+\frac{1}{h_2}\frac{\partial \mu_k}{\partial y}+\phi_k\left[\frac{1}{h_1}\frac{(k+2)}{u_e}\frac{\partial u_e}{\partial x}+\frac{1}{h_1 h_2}\frac{\partial h_2}{\partial x}\right]+\mu_k\left[\frac{1}{h_2}\frac{(k+2)}{u_e}\frac{\partial u_e}{\partial y}+\frac{(k+2)}{h_1 h_2}\frac{\partial h_1}{\partial y}\right]$$

$$+\lambda_k\left[\frac{1}{h_2}\frac{(k+1)}{u_e}\frac{\partial u_\delta}{\partial y}-\frac{(k+1)}{h_1 h_2}\frac{\partial h_2}{\partial x}\left(\frac{v_\delta}{u_e}\right)+\frac{(k+1)}{h_1 h_2}\frac{\partial h_1}{\partial y}\left(\frac{u_\delta}{u_e}\right)\right]$$

$$+\gamma_k\frac{1}{h_1}\frac{(k+1)}{u_e}\frac{\partial u_\delta}{\partial x}-\theta_k\frac{(k+1)}{h_1 h_2}\frac{\partial h_2}{\partial x}=\frac{(k+1)}{\rho}\frac{1}{u_e^2}\int_0^\delta\left(\frac{u}{u_e}\right)^k\frac{\partial \tau_x}{\partial z}dz$$

$$\frac{1}{h_1}\frac{\partial \sigma_k}{\partial x}+\frac{1}{h_2}\frac{\partial \rho_k}{\partial y}+\sigma_k\left[\frac{1}{h_1}\frac{(k+2)}{u_e}\frac{\partial u_e}{\partial x}+\frac{(k+2)}{h_1 h_2}\frac{\partial h_2}{\partial x}\right]+\rho_k\left[\frac{1}{h_2}\frac{(k+2)}{u_e}\frac{\partial u_e}{\partial y}+\frac{1}{h_1 h_2}\frac{\partial h_1}{\partial y}\right]$$

$$+\pi_k\left[\frac{1}{h_1}\frac{(k+1)}{u_e}\frac{\partial v_\delta}{\partial y}-\frac{(k+1)}{h_1 h_2}\frac{\partial h_1}{\partial y}\left(\frac{u_\delta}{u_e}\right)+\frac{(k+1)}{h_1 h_2}\frac{\partial h_2}{\partial x}\left(\frac{v_\delta}{u_e}\right)\right]$$

$$+\nu_k\frac{1}{h_2}\frac{(k+1)}{u_e}\frac{\partial v_\delta}{\partial y}-\tau_k\frac{(k+1)}{h_1 h_2}\frac{\partial h_1}{\partial y}=\frac{(k+1)}{\rho}\frac{1}{u_e^2}\int_0^\delta\left(\frac{v}{u_e}\right)^k\frac{\partial \tau_y}{\partial z}dz$$

Dissipation integral equations of interest

Momentum balance in x-direction

(Momentum balance in y-direction)

$$\frac{1}{h_1}\frac{\partial \delta_{11}}{\partial x} + \frac{1}{h_2}\frac{\partial \delta_{12}}{\partial y} + \delta_{11}\left[\frac{1}{h_1}\frac{2}{u_e}\frac{\partial u_e}{\partial x} + \frac{1}{h_1 h_2}\frac{\partial h_2}{\partial x}\right] + \delta_{12}\left[\frac{1}{h_2}\frac{2}{u_e}\frac{\partial u_e}{\partial y} + \frac{2}{h_1 h_2}\frac{\partial h_1}{\partial y}\right]$$

$$+\delta_2\left[\frac{1}{h_2}\frac{1}{u_e}\frac{\partial u_\delta}{\partial y} - \frac{1}{h_1 h_2}\frac{\partial h_2}{\partial x}\left(\frac{v_\delta}{u_e}\right) + \frac{1}{h_1 h_2}\frac{\partial h_1}{\partial y}\left(\frac{u_\delta}{u_e}\right)\right]$$

$$+\delta_1\frac{1}{h_1}\frac{1}{u_e}\frac{\partial u_\delta}{\partial x} - \delta_{22}\frac{1}{h_1 h_2}\frac{\partial h_2}{\partial x} = \frac{c_{fx}}{2}$$

Balance of kinetic energy in x-direction

(Balance of kinetic energy in y-direction)

$$\frac{1}{h_1}\frac{\partial \Delta_{11}}{\partial x} + \frac{1}{h_2}\frac{\partial \Delta_{12}}{\partial y} + \Delta_{11}\left[\frac{1}{h_1}\frac{3}{u_e}\frac{\partial u_e}{\partial x} + \frac{1}{h_1 h_2}\frac{\partial h_2}{\partial x}\right] + \Delta_{12}\left[\frac{1}{h_2}\frac{3}{u_e}\frac{\partial u_e}{\partial y} + \frac{3}{h_1 h_2}\frac{\partial h_1}{\partial y}\right]$$

$$+\Delta_2\left[\frac{1}{h_2}\frac{2}{u_e}\frac{\partial u_\delta}{\partial y} - \frac{2}{h_1 h_2}\frac{\partial h_2}{\partial x}\left(\frac{v_\delta}{u_e}\right) + \frac{2}{h_1 h_2}\frac{\partial h_1}{\partial y}\left(\frac{u_\delta}{u_e}\right)\right]$$

$$-\Delta_3\frac{2}{h_1 h_2}\frac{\partial h_2}{\partial x} = \frac{2}{\rho u_e^2}\int_0^\delta \left(\frac{u}{u_e}\right)\frac{\partial \tau_x}{\partial z}\,dz$$

Two-dimensional case

k = 0 momentum balance	$\frac{d\delta_2}{dx} + (2 + H_{12})\frac{\delta_2}{u_\delta}\frac{du}{dx} = c_f$
k = 1 balance of mechanical energy	$\frac{d\delta_3}{dx} + 3\frac{\delta_3}{u_\delta}\frac{du}{dx} = -2\left(\frac{u_\tau}{u_\delta}\right)^3 \int_0^1 \left(\frac{u}{u_\tau}\right)\frac{\partial(\tau/\tau_w)}{\partial(y/\delta)}\, d\frac{y}{\delta} = c_{D,2}$
k = 2	$\frac{d\delta_4}{dx} + (4 - 3H_{42})\frac{\delta_4}{u_\delta}\frac{du}{dx} = -3\left(\frac{u_\tau}{u_\delta}\right)^4 \int_0^1 \left(\frac{u}{u_\tau}\right)^2 \frac{\partial(\tau/\tau_w)}{\partial(y/\delta)}\, d\frac{y}{\delta} = c_{D,3}$

$$\frac{u(z)}{u_\tau} = \frac{1}{\kappa}\ln\left[z^+\right] + C + w(\eta), \quad \eta = \frac{z^+}{K_\tau}$$

$$w(\eta,\Pi) = 2\eta^2(3 - 2\eta) - \frac{1}{\Pi}\eta^2(1 - \eta)(1 - 2\eta)$$

$$\frac{\tau}{\tau_w} = f_1(\eta, \Pi, S) + f_2(\eta, \Pi, S)\,\delta\frac{d\Pi}{dx} + f_3(\eta, \Pi, S)\frac{\delta}{u_\delta}\frac{du_\delta}{dx}$$

Dissipationintegral for two-dimensional case

$$c_D = -2\left(\frac{\mathbf{u}_\tau}{\mathbf{u}_\delta}\right)^3 \int_0^1 \left(\frac{\mathbf{u}}{\mathbf{u}_\tau}\right) \frac{\partial(\tau/\tau_w)}{\partial(\mathbf{y}/\delta)}\, \mathbf{d}\frac{\mathbf{y}}{\delta}$$

$$c_D = \mathbf{f}(\mathbf{S}, \Pi, \zeta, \beta)$$

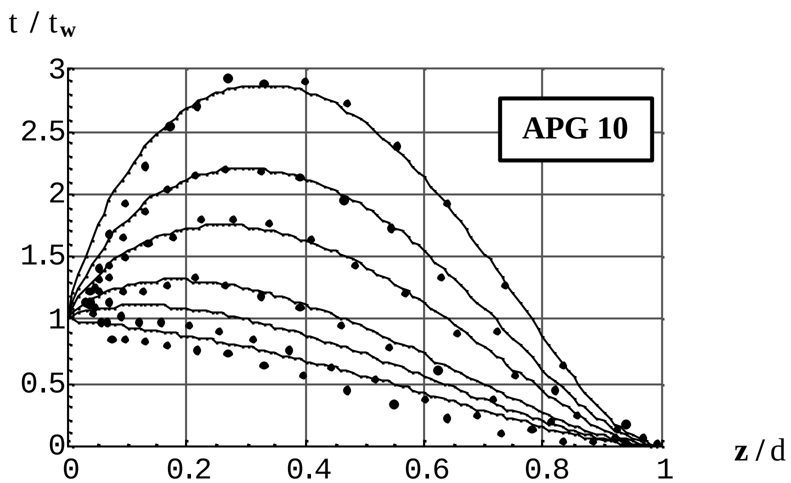
$$\mathbf{S} = \frac{\mathbf{u}}{\mathbf{u}_\tau} \qquad \Pi$$

$$\zeta = \delta\, \mathbf{S}\, \frac{\partial \Pi}{\partial \mathbf{x}} \qquad \beta = -\rho\, \delta_1\, \mathbf{S}\, \frac{\partial \mathbf{u}_e}{\partial \mathbf{x}}$$

$$c_D = \frac{S - [1.353(4.569 + 8.419\Pi + 6.316\Pi^2 + 1.859\Pi^3 - 1.248S - 2.218\Pi S - 1.525\Pi^2 S - 0.381\Pi^3 S + 0.128S^2 + 0.202\Pi S^2 + 0.0945\Pi^2 S^2)]}{(8.437\times10^6 + 1.334\times10^7\Pi + 6.240\times10^6\Pi^2 - 3.459\times10^6 S - 5.469\times10^6\Pi S - 2558400\Pi^2 S + 694253S^2 + 706020\Pi S^2)} +$$

$$\zeta* = \frac{-[7.669\times10^6(-32.772 - 180.230\Pi - 347.099\Pi^2 - 313.663\Pi^3 - 146.036\Pi^4 - 29.008\Pi^5 + 24.177S + 99.770\Pi S + 139.145\Pi^2 S + 87.542\Pi^3 S + 21.956\Pi^4 S - 4.570S^2 - 15.695\Pi S^2 - 15.616\Pi^2 S^2 - 5.266\Pi^3 S^2 - 0.305S^3 + 0.800\Pi S^3 + 0.407\Pi^2 S^3)]}{[S(8.437\times10^6 + 1.334\times10^7\Pi + 6.24\times10^6\Pi^2 - 3.45917\times10^6 S - 5.469\times10^6\Pi S - 2.5584\times10^6\Pi^2 S + 694253S^2 + 706020\Pi S^2)]} +$$

$$\beta* = \frac{[1.572\times10^6(0.385 + 9.454\Pi + 25.361\Pi^2 + 26.410\Pi^3 + 13.506\Pi^4 + 2.900\Pi^5 - 2.657S - 9.128\Pi S - 12.672\Pi^2 S - 8.256\Pi^3 S - 2.157\Pi^4 S + 0.5286S^2 + 1.371\Pi S + 1.239\Pi^2 S^2 + 0.397\Pi^3 S^2)]}{[(0.240 + 0.244\Pi)(8.437\times10^6 + 1.334\times10^7\Pi + 6.240\times10^6\Pi^2 - 3.45917\times10^6 S - 5.469\times10^6\Pi S - 2.558\times10^6\Pi^2 S + 694253S^2 + 706020\Pi S^2)]}$$



Three-dimensional case

Momentum balance
in s-direction

(momentum balance in n-direction,

balance of kinetic energy in s-direction,

balance of kinetic energy in n-direction,

additional equations)

$$\delta\frac{\partial \mathbf{f}_{11}}{\partial p}\frac{\partial p}{\partial \mathbf{s}}+\delta\frac{\partial \mathbf{f}_{11}}{\partial \mathbf{S}}\frac{\partial \mathbf{S}}{\partial \mathbf{s}}+\mathbf{f}_{11}\frac{\partial \delta}{\partial \mathbf{s}}+\delta\frac{\partial \mathbf{f}_{12}}{\partial p}\frac{\partial p}{\partial \mathbf{n}}+\delta\frac{\partial \mathbf{f}_{12}}{\partial \mathbf{S}}\frac{\partial \mathbf{S}}{\partial \mathbf{n}}+\delta\frac{\partial \mathbf{f}_{12}}{\partial \mathbf{B}}\frac{\partial \mathbf{B}}{\partial \mathbf{n}}+\mathbf{f}_{12}\frac{\partial \delta}{\partial \mathbf{n}}=$$

$$\frac{\mathbf{c}_{\mathbf{fs}}}{\mathbf{2}}-\theta_{11}\left[\frac{\mathbf{2}}{\mathbf{u}_{\mathbf{e}}}\frac{\partial \mathbf{u}_{\mathbf{e}}}{\partial \mathbf{s}}+\frac{\mathbf{1}}{\mathbf{h}_2}\frac{\partial \mathbf{h}_2}{\partial \mathbf{s}}\right]-\theta_{12}\left[\frac{\mathbf{2}}{\mathbf{u}_{\mathbf{e}}}\frac{\partial \mathbf{u}_{\mathbf{e}}}{\partial \mathbf{n}}+\frac{\mathbf{1}}{\mathbf{h}_1}\frac{\partial \mathbf{h}_1}{\partial \mathbf{n}}\right]$$

$$-\theta_{1}\frac{\mathbf{1}}{\mathbf{u}_{\mathbf{e}}}\frac{\partial \mathbf{u}_{\mathbf{e}}}{\partial \mathbf{s}}-\theta_{2}\left[\frac{\mathbf{1}}{\mathbf{u}_{\mathbf{e}}}\frac{\partial \mathbf{u}_{\mathbf{e}}}{\partial \mathbf{n}}+\frac{\mathbf{1}}{\mathbf{h}_1}\frac{\partial \mathbf{h}_1}{\partial \mathbf{n}}\right]+\theta_{22}\frac{\mathbf{1}}{\mathbf{h}_2}\frac{\partial \mathbf{h}_2}{\partial \mathbf{s}}$$

$$\begin{pmatrix} \mathbf{A}_{12} & \mathbf{A}_{12} & \mathbf{0} & \mathbf{A}_{14} & \mathbf{A}_{15} & \mathbf{A}_{16} & \mathbf{A}_{17} & \mathbf{A}_{18} \\ \mathbf{A}_{21} & \mathbf{A}_{22} & \mathbf{A}_{23} & \mathbf{A}_{24} & \mathbf{A}_{25} & \mathbf{A}_{26} & \mathbf{A}_{27} & \mathbf{A}_{28} \\ \mathbf{A}_{31} & \mathbf{A}_{32} & \mathbf{0} & \mathbf{A}_{34} & \mathbf{A}_{35} & \mathbf{A}_{36} & \mathbf{A}_{37} & \mathbf{A}_{38} \\ \mathbf{A}_{41} & \mathbf{A}_{42} & \mathbf{0} & \mathbf{A}_{44} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \partial p/\partial \mathbf{s} \\ \partial \mathbf{S}/\partial \mathbf{s} \\ \partial \mathbf{B}/\partial \mathbf{s} \\ \partial \delta/\partial \mathbf{s} \\ \partial p/\partial \mathbf{n} \\ \partial \mathbf{S}/\partial \mathbf{n} \\ \partial \mathbf{B}/\partial \mathbf{n} \\ \partial \delta/\partial \mathbf{n} \end{pmatrix} = \begin{pmatrix} \mathbf{K}_1 \\ \mathbf{K}_2 \\ \mathbf{K}_3 \\ \mathbf{K}_4 \end{pmatrix}$$

Dissipationintegral for three-dimensional case

$$-\frac{2\delta}{\tau_w/\frac{\rho}{2}u_e^2}\left[\left(\frac{u}{u_e}\right)\int_0^\eta \frac{\partial(u/u_e)}{\partial x}\,d\eta - \frac{1}{h_1}\int_0^\eta \frac{\partial(u/u_e)^2}{\partial x}\,d\eta + \frac{1}{h_1 h_2}\frac{\partial h_2}{\partial x}\int_0^\eta\left(\left(\frac{v}{u_e}\right)^2 - \left(\frac{v_\delta}{u_e}\right)^2\right)d\eta\right]$$

$$-\frac{2\delta}{\tau_w/\frac{\rho}{2}u_e^2}\left[\left(\frac{u}{u_e}\right)\int_0^\eta \frac{\partial(v/u_e)}{\partial y}\,d\eta - \frac{1}{h_2}\int_0^\eta \frac{\partial(uv/u_e^2)}{\partial y}\,d\eta - \frac{1}{h_1 h_2}\frac{\partial h_1}{\partial y}\int_0^\eta\left(\left(\frac{uv}{u_e^2}\right) - \left(\frac{u_\delta v_\delta}{u_e^2}\right)\right)d\eta\right]$$

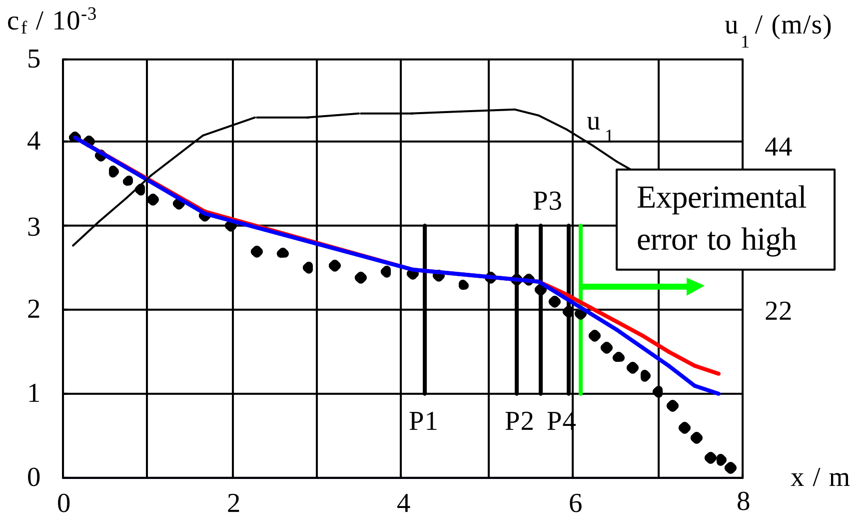
$$-\rho\delta\frac{u_e}{\tau_w}\frac{\partial u_e}{\partial x}\left[\left(\frac{u}{u_e}\right)\int_0^\eta\left(\frac{u}{u_e}\right)d\eta - \frac{2}{h_1}\int_0^\eta\left(\frac{u}{u_e}\right)^2 d\eta\right] - \rho\delta\frac{u_e}{\tau_w}\frac{\partial u_e}{\partial y}\left[\left(\frac{u}{u_e}\right)\int_0^\eta\left(\frac{v}{u_e}\right)d\eta - \frac{2}{h_2}\int_0^\eta\left(\frac{uv}{u_e^2}\right)^2 d\eta\right]$$

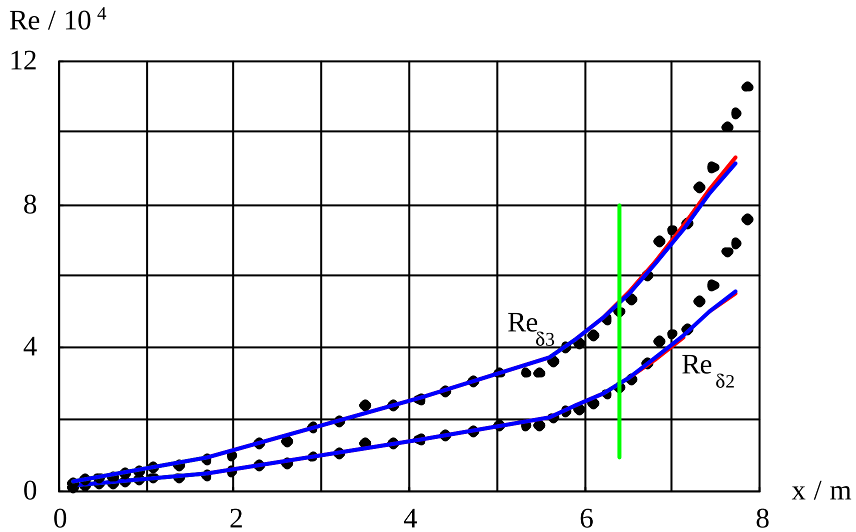
$$-\rho\delta\eta\frac{u_e}{\tau_w}\left[\frac{1}{h_1}\frac{u_\delta}{u_e}\frac{\partial u_\delta}{\partial x} + \frac{1}{h_2}\frac{v_\delta}{u_e}\frac{\partial u_\delta}{\partial y}\right] + \frac{\tau_{wx}}{\tau_w} = \frac{\tau_x}{\tau_w}$$

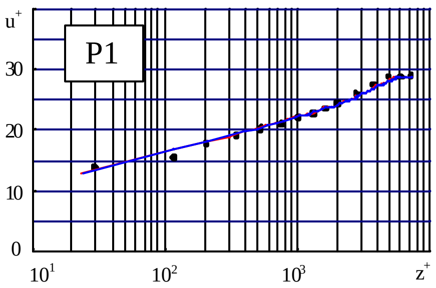
$$c_{Dx} = \frac{2}{\rho}\frac{1}{u_e^2}\int_0^\delta\left(\frac{u}{u_e}\right)\frac{\partial(\tau_x/\tau_w)}{\partial\eta}\,d\eta \quad \rightarrow \quad \text{MATHEMATICA}^{®}$$

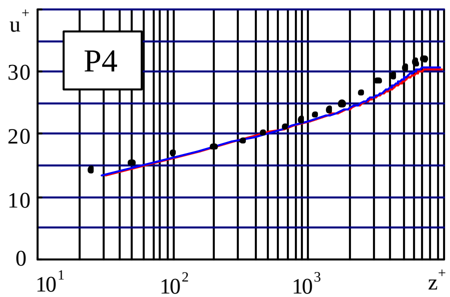
Two-dimensional test case

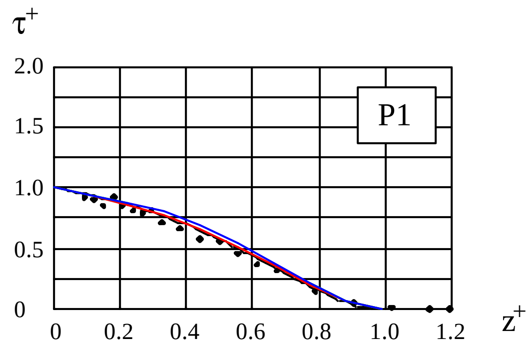
IDENT 2100 - G. B. Schubauer, P. S. Klebanoff

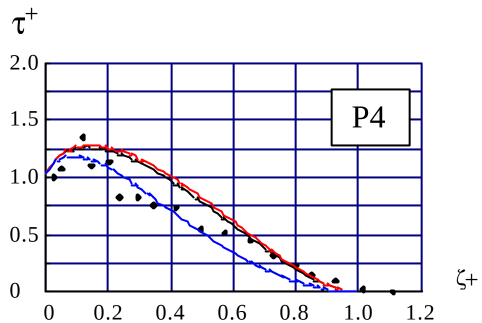






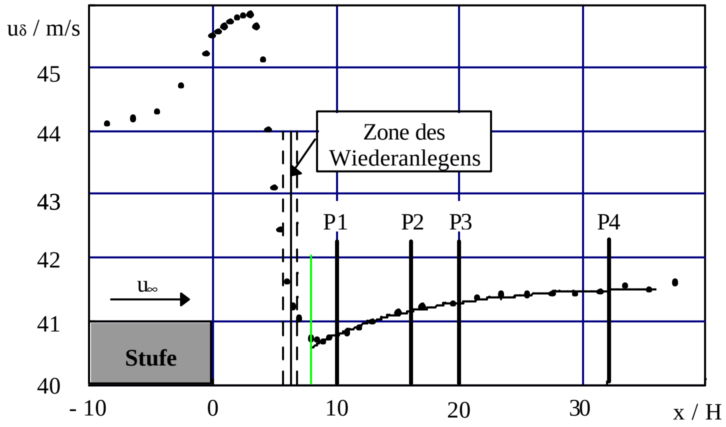


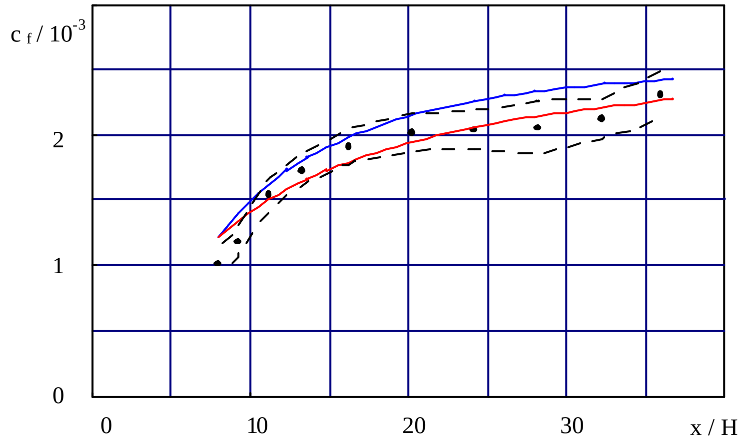


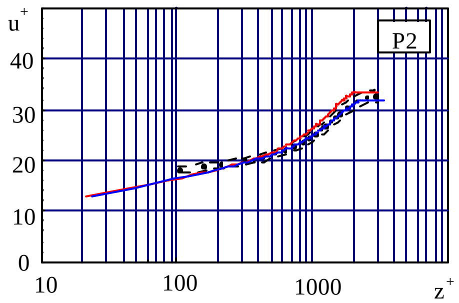


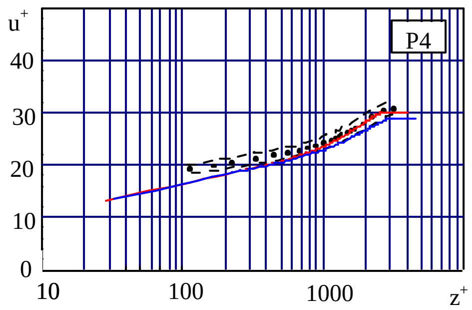
Two-dimensional test case

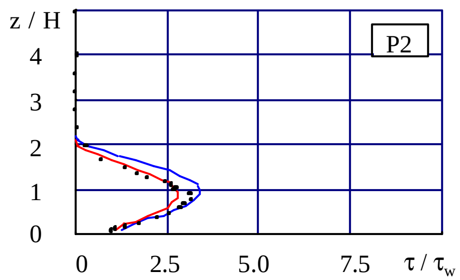
Backward facing Step - M. D. Driver, H. L. Seegmiller

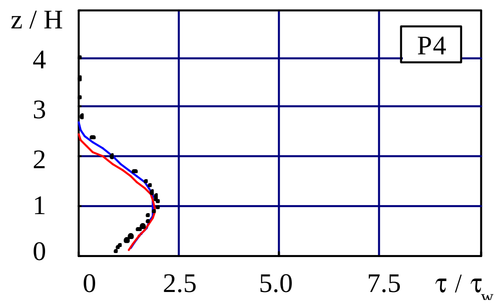












Sixteen two-dimensional test cases

$$W(c_f) = 100\% \frac{\sum_{j=1}^{n} \left| c^*_{f,Cal} - c^*_{f,Exp} \right|}{\sum_{j=1}^{n} c^*_{f,Exp}}$$

$$V(\gamma) = \frac{100\%}{m} \sum_{i=1}^{m} \left| \frac{\gamma_{i,Cal} - \gamma_{i,Exp}}{\gamma_{i,Exp}} \right|$$

c^*_f … wall skin friction at the last position of test case

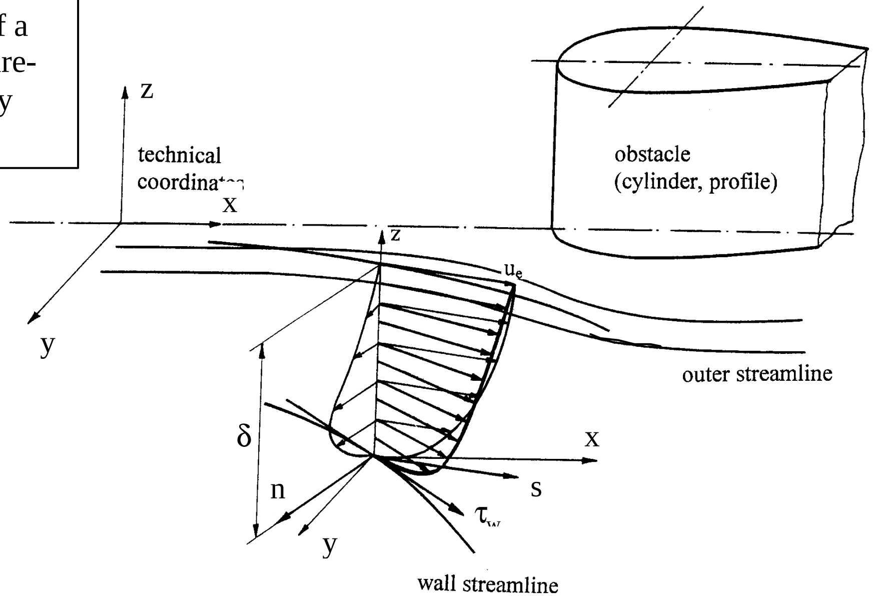
m … number of experimental points

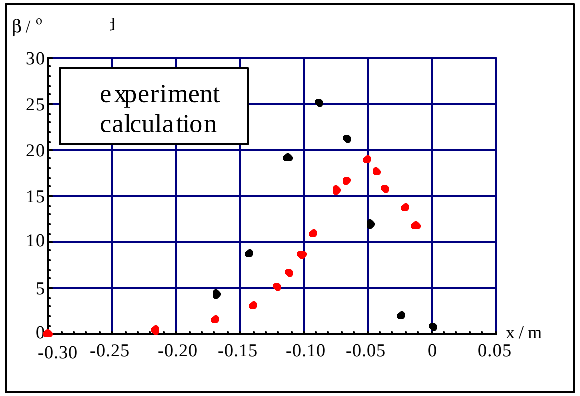
	$V(Re_{\delta 2})$	$V(Re_{\delta 3})$	$V(H_{12})$	$W(c_f)$
A1	4.6 %	4.7 %	2.7 %	8.5 %
A2	5.0 %	4.9 %	2.6 %	8.2 %
k, ε - model*				36 -58 %
k, ω - model*				4.0 -6.0 %

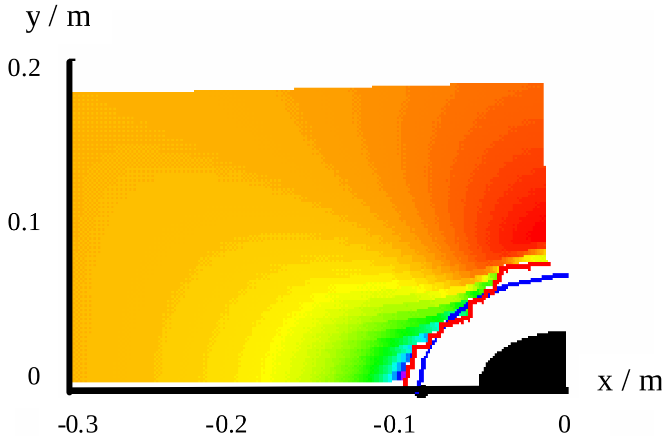
*D. C. Wilcox: AIAA-Journal, vol. 31, 1993

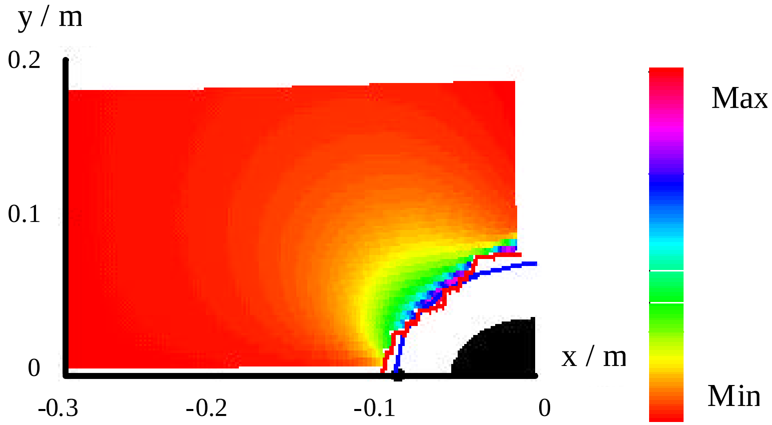
Three-dimensional test case

Ölcmen, Simpson
An experimental study of a three-dimensional pressure-driven turbulent boundary layer, JFM **290** 1995









wake parameter p **of mean velocity profile**
separation line **experiment** **calculation**

cross-flow angle b_0

Summary and open issues

- **The general form of dissipation integral algorithms for three-dimensional turbulent boundary layers is derived**
- **The approach can be used for engineering applications**
- **The two-dimensional problem can be improved by using an inverse formulation**
- **The three-dimensional problem can be improved by using profiles which are mor “universal” and using an inverse formulation**

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