On the Hierarchical Scaling Behavior of Turbulent Wall-Flows

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Perry & Coworkers Hierarchical Model of Wall Turbulence

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A. E. Perry and M. S. Chong

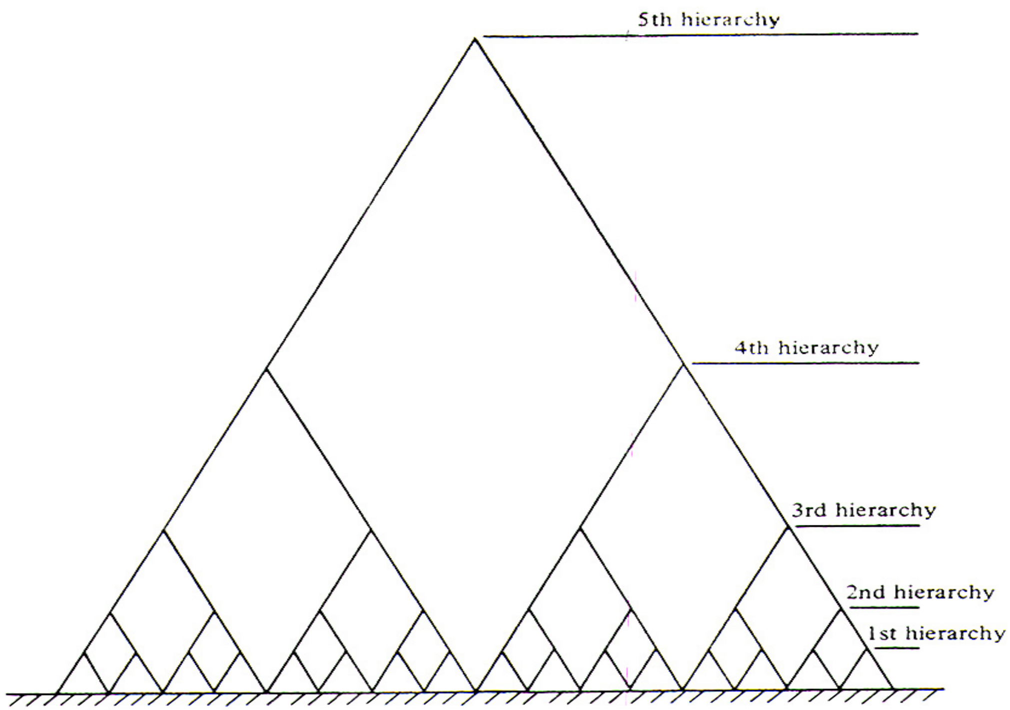

FIGURE 14. Symbolic representation of a discrete system of hierarchies.

Cumulative Construction of Mean Momentum Field

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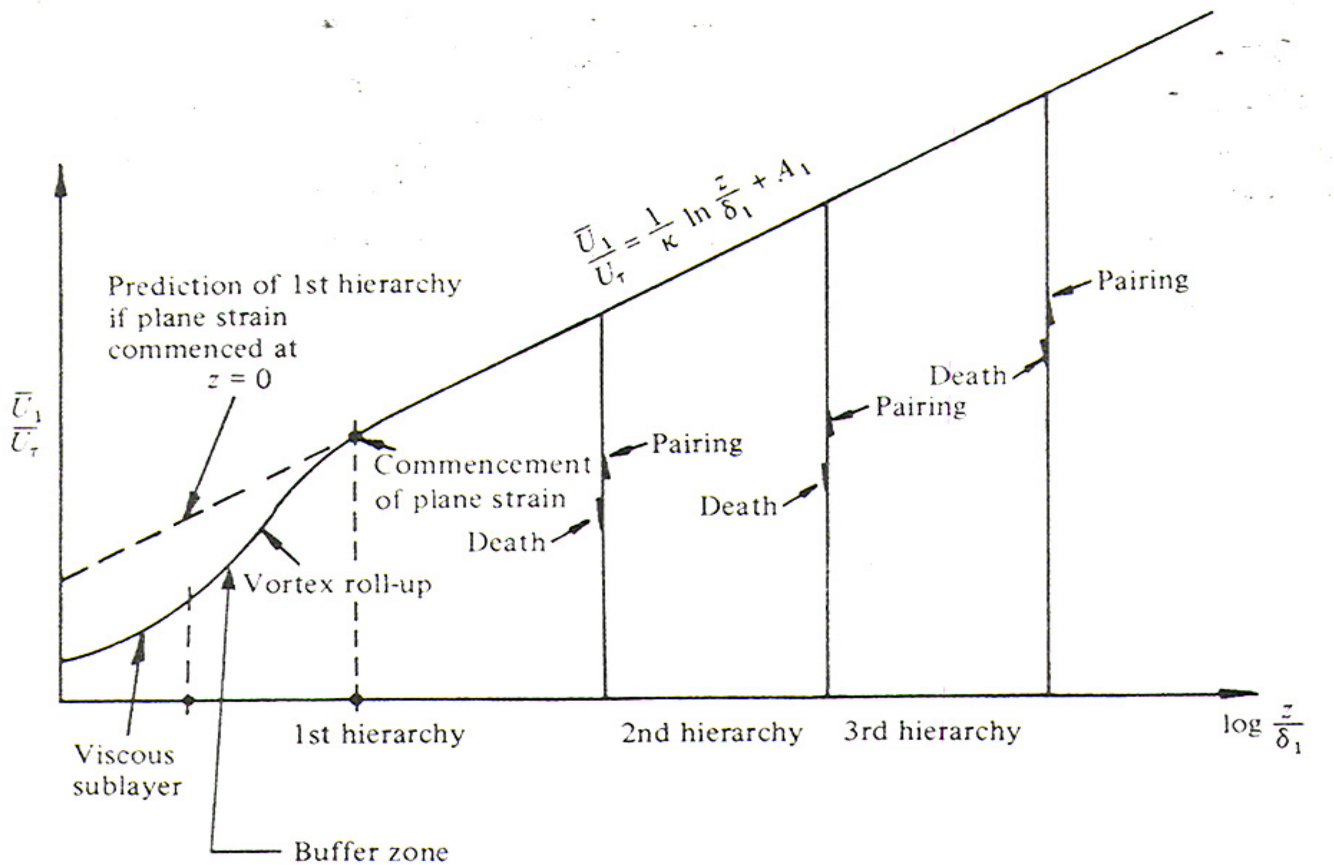


FIGURE 18. Mean-velocity profile interpreted in terms of quantum-jump model. A_1 is a universal constant.

Physical Evidence: Vortex Packets (Adrian et al. 2000, plus others)

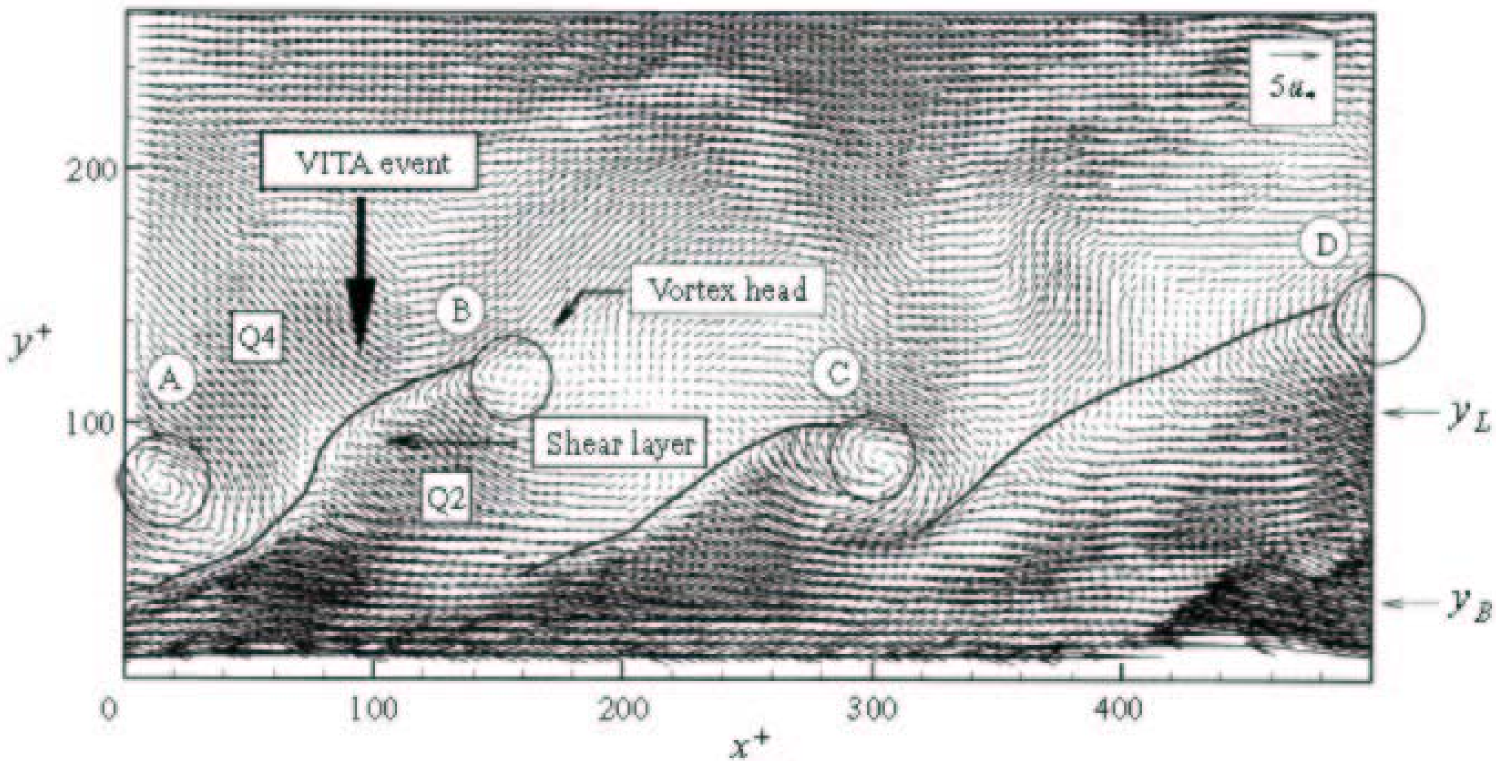


FIGURE 11. Near-wall realization at $Re_\theta = 930$ showing four hairpin vortex signatures aligned in the streamwise direction. Instantaneous velocity vectors are viewed in a frame-of-reference moving at $U_c = 0.8U_\infty$ and scaled with inner variables. Vortex heads and inclined shear layers are indicated schematically, along with the elements triggering a VITA event.

Physical Model

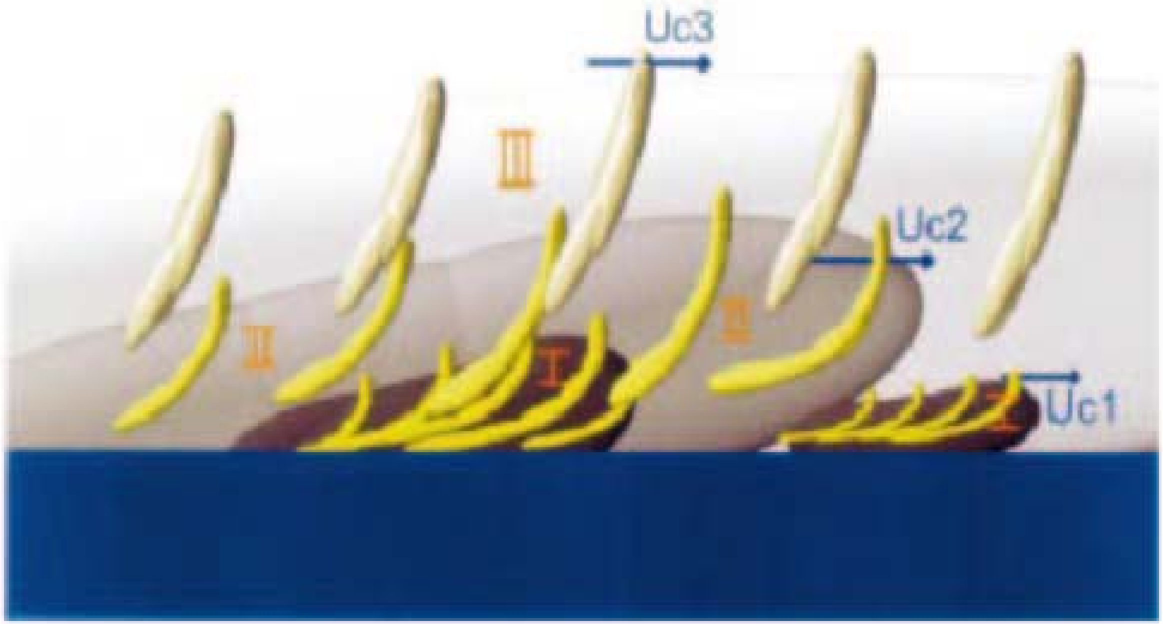


FIGURE 22. Idealized model of hairpin packets nesting within larger hairpin packets, and travelling at different velocities. The nested hierarchy creates the appearance of multiple uniform-momentum zones, and a progressively lower velocity as one approaches the wall.

Some Interesting Questions

Do the scaling behaviors of the mean dynamical equations,

- naturally favor hierarchical models?
- reflect the instantaneous observations?

Scaling and Theory

1) Wei, T., Fife, P., Klewicki, J. and McMurtry, P., "Properties of the mean momentum balance in turbulent boundary layer, pipe and channel flows," *J. Fluid Mech.* (under consideration) 2004.

2) Fife, P., Wei, T., Klewicki, J. and McMurtry, P., "Stress Gradient Balance Layers and Scaling Hierarchies in Wall-Bounded Turbulent Flows," *J. Fluid Mech*. (under consideration) 2004.

Primary Assumptions

1) RANS equations describe the mean dynamics

2) Monotonicity: velocity is monotone increasing and the velocity gradient is monotone decreasing with distance from the wall

Scaling Patch

A "scaling patch" exists when:

i) the scaled independent variable is O(1)

ii) the variation in the scaled dependent variable is O(1), and

iii) the derivatives of the scaled dependent variable are all O(1)

(These conditions are satisfied when the relevant terms in the scaled equations are free of large/small parameters)

Mean Momentum Balance Data

Fully developed channel flow,

$$-\frac{1}{\delta^+} = \frac{d^2U^+}{dy^{+2}} - \frac{d\overline{uv}^+}{dy^+}$$

2-D boundary layer flow,

$$U^+\frac{\partial U^+}{\partial x^+} + V^+\frac{\partial U^+}{\partial y^+} = \frac{\partial^2 U^+}{\partial y^{+2}} - \frac{\partial \overline{uv}^+}{\partial y^+}$$

$$\mathrm{A} \qquad = \quad \mathrm{B} \quad + \quad \mathrm{C}$$

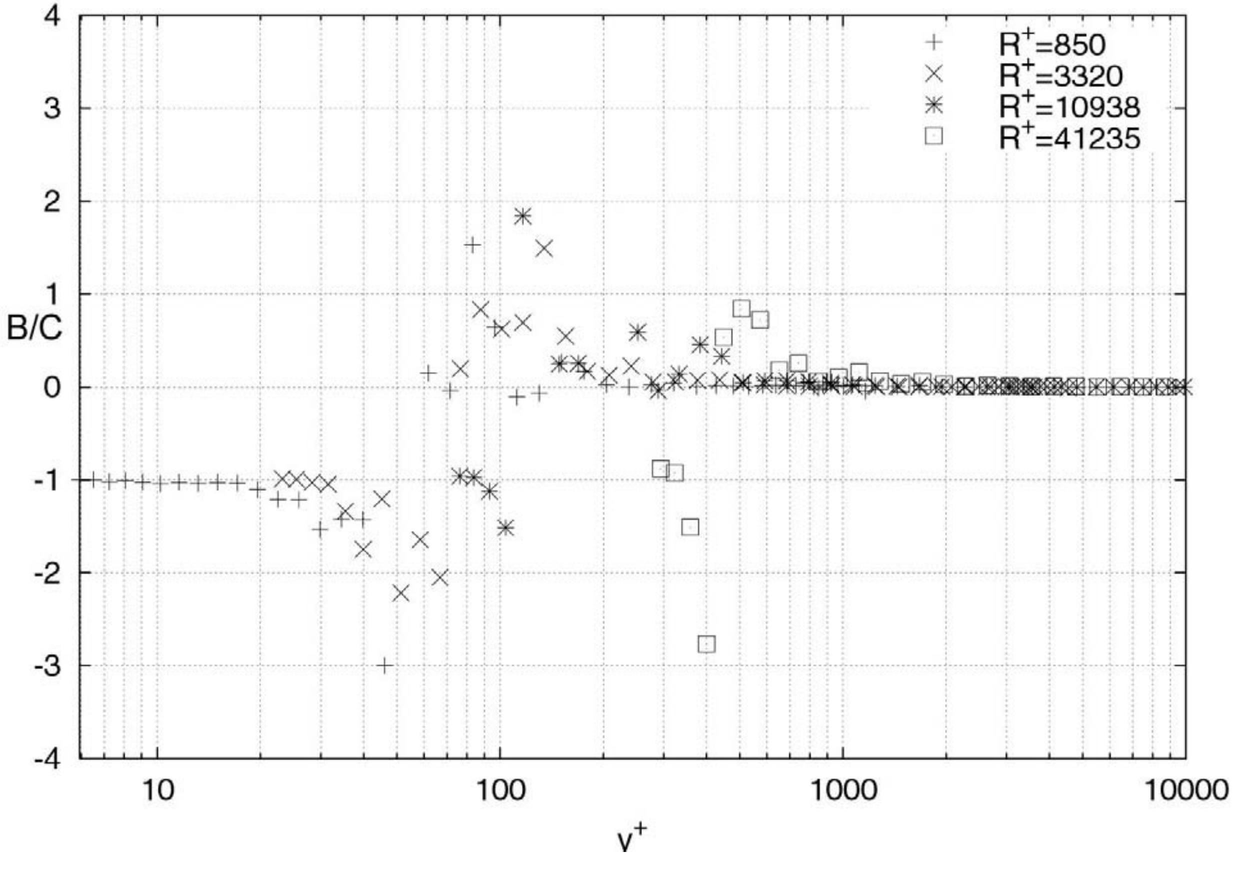
Stress Gradient Ratios: Limiting Cases

$$|B/C| = |\frac{d^2U^+}{dy^{+2}} / \frac{d\overline{uv}^+}{dy^+}| \ll 1, \quad \text{inertial force} \approx \text{mean advection}$$

$$|B/C| = |\frac{d^2U^+}{dy^{+2}} / \frac{d\overline{uv}^+}{dy^+}| \simeq 1, \quad \text{viscous force} \approx \text{inertial force}$$

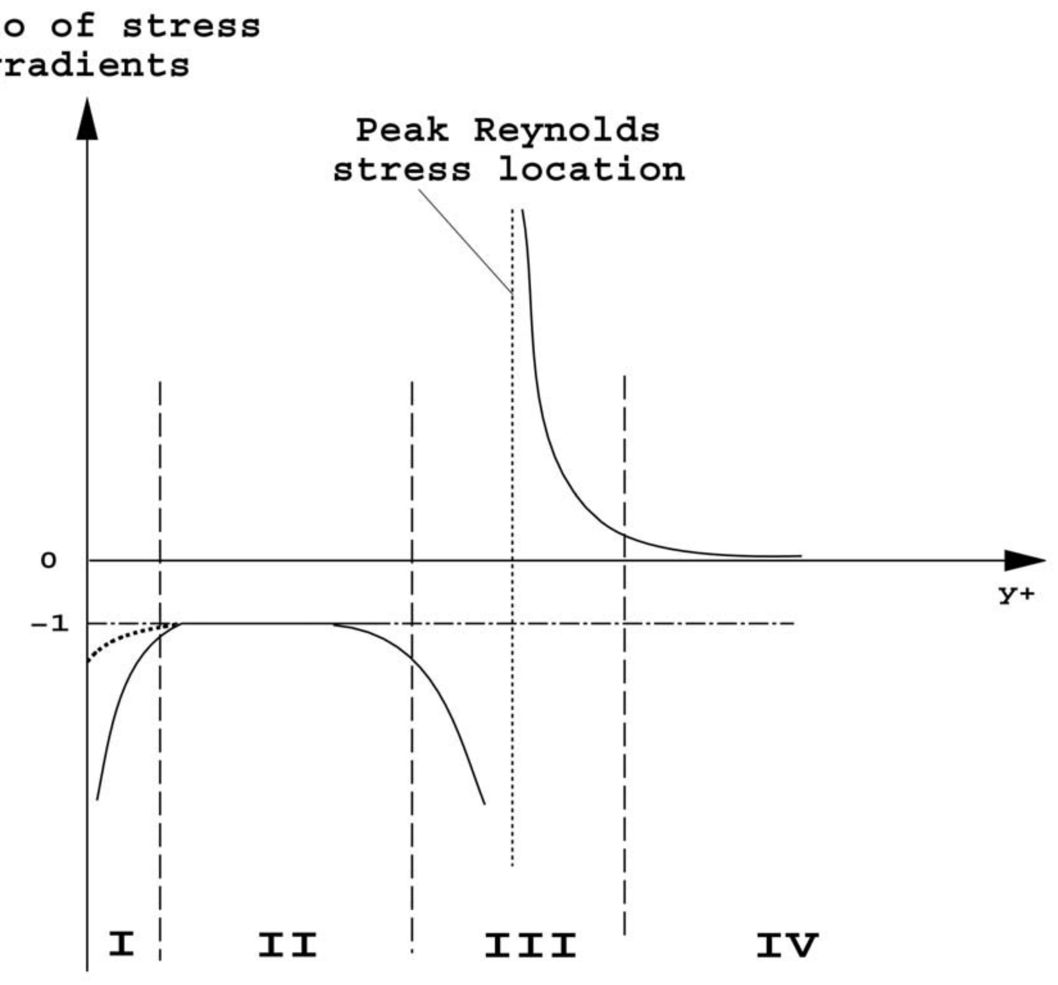
$$|B/C| = |\frac{d^2U^+}{dy^{+2}} / \frac{d\overline{uv}^+}{dy^+}| \gg 1, \quad \text{viscous force} \approx \text{mean advection}$$

Viscous to Reynolds Stress Gradient Ratio
(Pipe Flow, Zagarola and Smits 1997)



Four Layer Structure of Boundary Layer Pipe and Channel Flows

(At any fixed Reynolds number)



Layer Structure Prescribed by the Mean Dynamics

Layer I: *Inner Viscous/Advection Balance Layer*

(traditional viscous sublayer)

Layer II: *Stress Gradient Balance Layer*

Layer III: *Meso Viscous/Advection/Inertial Balance Layer*

Layer IV: *Inertial/Advection Balance Layer*

Reynolds Number Scalings: Inner

Layer I: $\Delta_I U^+ \sim O(1)$ $(\simeq 3)$, $\quad \Delta_I y^+ \sim O(1)$ $(\simeq 3)$

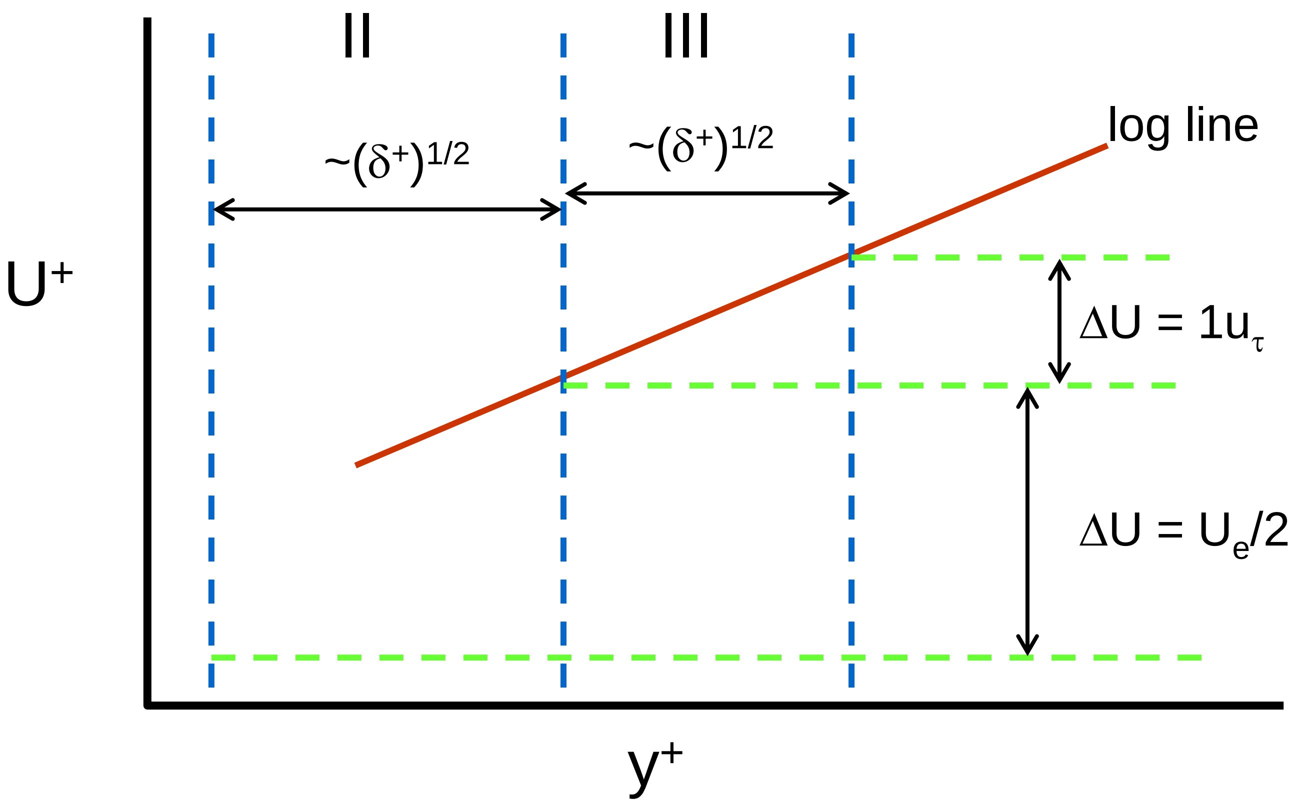
Layer II: $\Delta_{II} U^+ \sim O(U_\infty^+)$ $(\simeq \frac{U_\infty^+}{2})$, $\quad \Delta_{II} y^+ \sim \sqrt{\delta^+}$

Layer III: $\Delta_{III} U^+ \sim O(1)$ $(\simeq 1)$, $\quad \Delta_{III} y^+ \sim \sqrt{\delta^+}$

Layer IV: $\Delta_{IV} U^+ \sim O(U_\infty^+)$ $(\rightarrow \frac{U_\infty^+}{2})$, $\quad \Delta_{IV} y^+ \rightarrow \delta^+$

(Note that layer IV properties are apparently asymptotic as $R_\theta \rightarrow \infty$)

Layers II and III



Multiscale Analysis of Channel Flow

Streamwise Momentum Equation:

Inner variables (y^+, U^+, and $T \equiv -\overline{uv}$)

$$\frac{d^2U^+}{dy^{+2}} + \frac{dT^+}{dy^+} + \epsilon^2 = 0,$$

where $\epsilon^2 = \frac{1}{\delta^+}$ such that $\epsilon \to 0$ as $Re \to \infty$.

Outer variables (η, U^+, and $T \equiv -\overline{uv}$)

$$\frac{dT^+}{d\eta} + 1 + \epsilon^2 \frac{d^2U^+}{d\eta^2} = 0$$

Layer I

$$y^+ = O(1), \quad T^+(y^+) \simeq 0$$

and thus the momentum equation reduces to,

$$\frac{d^2U^+}{dy^{+2}} \simeq 0$$

and so

$$U^+ \simeq y^+$$

(traditional viscous sublayer)

Layer II

For a region outside the sublayer, both the viscous and Reynolds stress gradients are $O(1) \gg O(\epsilon^2)$. Thus, the inner-normalized momentum equation becomes,

$$\frac{d^2U^+}{dy^{+2}} + \frac{dT^+}{dy^+} = 0$$

(Stress Gradient Balance Layer)

Balance Breaking and Exchange From Layer II to Layer III

Across layer II, the viscous and Reynolds stress gradients balance to within $O(\epsilon^2)$. Both, however, are decreasing functions of y^+. Thus, at some Reynolds number dependent wall-normal position, ${y_2}^+(\epsilon)$, they will become $O(\epsilon^2)$ eventhough their *formal* appearance in the full momentum equation,

$$\frac{d^2U^+}{dy^{+2}} + \frac{dT^+}{dy^+} + \epsilon^2 = 0,$$

does not indicate this fact.

Layer III Rescaling

In layer III all the terms are of the same order of magnitude (except right at y_m), and thus the goal is to find a scaling that renders all of the terms $O(1)$. To this end, let,

$$dy^+ = \alpha d\hat{y}, \text{ and } dT^+ = \beta d\hat{T},$$

and require that,

$$\frac{d^2U^+}{d\hat{y}^2} \text{ and } \frac{d\hat{T}}{d\hat{y}} \text{ be } O(1)$$

This yields,

$$d\hat{y} = \epsilon dy^+, \text{ and } dT^+ = \epsilon d\hat{T}$$

Layer III Rescaling (continued)

With this transformation, the mean momentum equation becomes,

$$\frac{d^2U^+}{d\hat{y}^2} + \frac{d\hat{T}}{d\hat{y}} + 1 = 0$$

with

$$y^+ = {y_m}^+ + \frac{1}{\epsilon}\hat{y}, \quad T^+ = T_m^+ + \epsilon\hat{T}$$

(i.e., the “hat” variables are centered around the peak in T^+)

Layer III Properties

Overall, the characteristics of layer III include:

$$|\hat{y}| \leq O(1), \;\; y^+ = O(\tfrac{1}{\epsilon}), \;\; \frac{dU^+}{dy^+} = O(\epsilon), \;\; \frac{dU^+}{d\hat{y}} = O(1),$$

and the higher order derivatives of $\frac{dU^+}{d\hat{y}}$ and $\hat{T}$ are $O(1)$.

Furthermore, note that:

$$\hat{y} \sim \sqrt{\eta y^+}$$

(geometric mean of the inner and outer scales)

$$\hat{y} \equiv \frac{(y - y_m)}{\sqrt{\nu\delta/u_\tau}}$$

(meso normalized distance)

Layer IV

In this layer the $O(\epsilon^2)$ term in the outer-normalized form of the momentum equation may be neglected. This results in,

$$\frac{dT^+}{d\eta} + 1 = 0$$

Integration yields

$$T^+(\eta) = 1 - \eta$$

(traditional outer layer result, valid only for $y > y_m$)

Multiscale Analysis of Couette Flow

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Because Couette flow is entirely composed of a stress gradient balance layer,

$$\frac{d^2U^+}{dy^{+2}} + \frac{dT^+}{dy^+} = 0,$$

Its study should be especially instructive in elucidating how the mean dynamics of such a layer are established.

A Remarkable Transformation

Define an adjusted Reynolds stress,

$$\tilde{T} = T^{+} - \epsilon^{2} y^{+}$$

Then the streamwise momentum equation becomes,

$$\frac{d^{2}U^{+}}{dy^{+2}} + \frac{d\tilde{T}}{dy^{+}} + \epsilon^{2} = 0,$$

(i.e., is identical in form to the equation for channel flow)

Generalized Adjusted Reynolds Stresses

Consider now the generalization of this transformation,

$$T^{\rho} = T^{+}(y^{+}) - \rho y^{+},$$

where ρ is a small positive number. (Note that ρ is a superscript not an exponent.)

Adjusted Reynolds Stresses

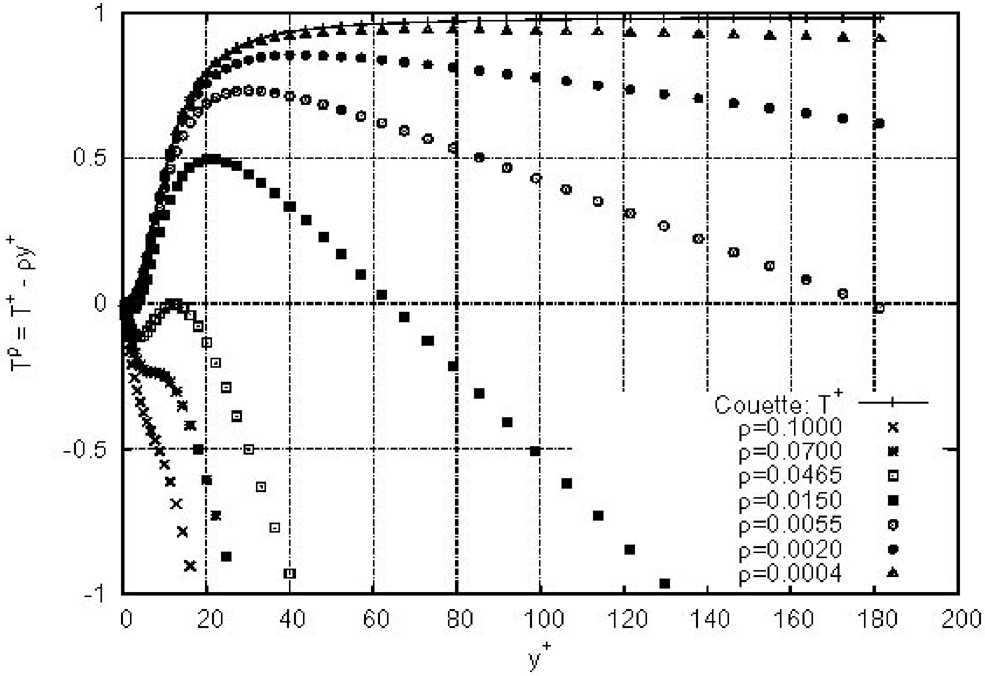

FIGURE 3. Adjusted Reynolds stress profile for various values of ρ. The case $\rho = \epsilon^4$ corresponds to within $O(\epsilon^2)$ to the genuine Reynolds stress for Couette flow (see Section 3.4.6), and $\rho = \epsilon^2$ is an approximation to that for pressure driven channel flow. The DNS data is from Kawamura, Abe & Shingai (2000), $\delta^+ = Re_\tau = 181.3$ and $\epsilon = .074$.

Hierarchy Equations

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Under this generalized transformation (for each value of ρ) the momentum equation becomes,

$$\frac{d^2U^+}{dy^{+2}} + \frac{dT^\rho}{dy^+} + \rho = 0$$

Scaling Layer Hierarchy

For each value of ρ, these equations undergo the same balance exchange as described previously (associated with the peaks of T^{ρ})

For each ρ this defines a layer, L_{ρ}, comprising a stress gradient balance layer/meso layer structure (i.e., an intermediate scaling patch)

Balance Exchange for Each T$^{\rho}$

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As before, for each T^{ρ} a transformation is derived,

$$y^{+} = y_{m}^{+}(\rho) + \rho^{-1/2}\hat{y}, \quad T^{\rho} = T_{m}^{\rho} + \rho^{1/2}\hat{T}^{\rho},$$

yielding,

$$\frac{d^2U^{+}}{d\hat{y}^2} + \frac{d\hat{T}^{\rho}}{d\hat{y}} + 1 = 0,$$

and thus verifying the existence of a scaling patch for each ρ.

Hierarchy Properties

- A continuum of layers exists, and within each layer $\frac{dU^+}{d\hat{y}} = O(1)$; $\frac{dU^+}{dy^+} = O(\rho^{1/2})$; and the higher order derivatives of $\frac{dU^+}{d\hat{y}}$ and $\hat{T}$ are less than or equal to $O(1)$ (i.e., each layer is a scaling patch)

- The thickness of each layer is proportional to the distance of the center of its corresponding "mesolayer" from the wall

- The spatial extent of the hierarchy is given by:
  $(20 \text{ to } 36) < y_m^+(\rho) < 1/\epsilon^2$ when,
  $(0.0035 \text{ to } 0.14) > \rho > O(\epsilon^4)$

Logarithmic Dependence

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Within each L_ρ, the corresponding T^ρ reaches a maximum at $\hat{y} = 0$ (i.e., at $y^+ = y_m^+(\rho)$). At that location, define:

$$A(\rho) = -\frac{d^2\hat{T}^\rho}{d\hat{y}^2}(0)$$

Logarithmic Dependence (continued)

It can be shown that $A(\rho) = O(1)$ function that may on some sub-domains equal a constant

If A = const., a logarithmic mean profile is identically recovered (i.e., is rigorously analytically proven)

If A varies slightly, then the profile is bounded above and below by logarithmic functions.

Behavior of $A(y^+)$

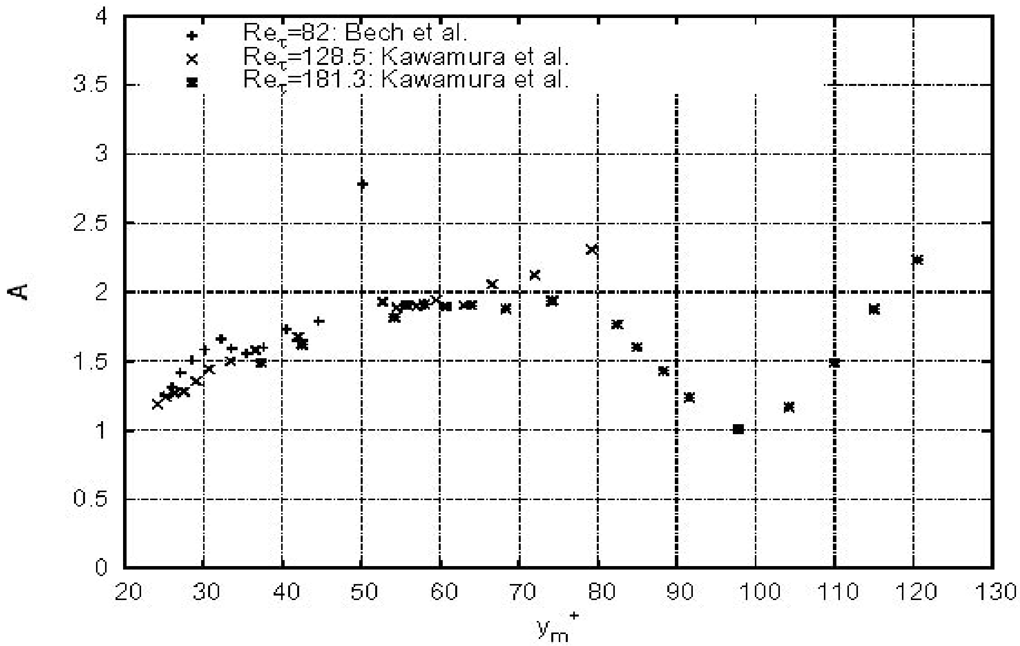


FIGURE 5. $A(y^+)$ for different Reynolds numbers as estimated by finite difference of $T^+(y^+)$. These estimates indicate a trend to larger internal intervals of relatively constant A for larger Re, thus agreeing with the present theory. The total range of the function A also increases with Re. The values of A were calculated from finite differencing DNS data of Bech *et al.* (1995) and Kawamura *et al.* (2000), (3.35), with locations y_m^+ determined from Fig. 3.

Relation to Channel Flow

Consider now the channel-to-Couette transformation, combined with the adjusted Reynolds stress transformation,

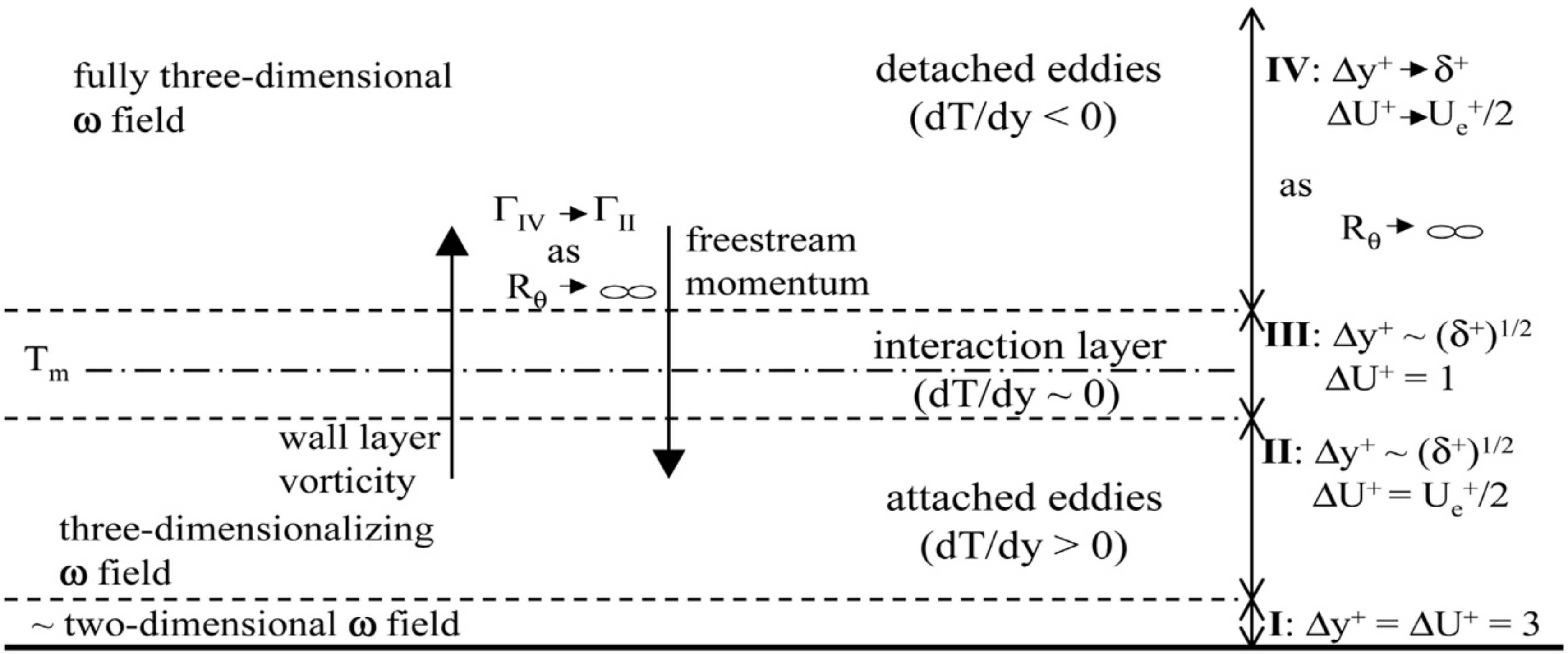
$$T^{\rho}(y^+) = T^+(y^+) + \epsilon^2 y^+ - \rho y^+.$$

This transforms the channel flow momentum equation to,

$$\frac{d^2U^+}{dy^{+2}} + \frac{dT^{\rho}}{dy^+} + \rho = 0$$

This is identically the "hierarchy equation" for Couette flow, and thus all the results for the Couette flow are recovered for the channel – the hierarchy extends to $y/\delta \approx 0.5$.

Physical Model of Boundary Layer Dynamics



Conclusions

Under the monotonicity assumption (and completely independent of any inner/outer overlap ideas), rigorous analysis of the RANS equations reveals that,

- Turbulent channel and Couette flows intrinsically contain a hierarchical layer structure
- The hierarchy constitutes a continuum of scaling patches and adjusts with Reynolds number to connect the traditional inner and outer scaling patches
- The hierarchy provides a firm analytical basis for the often invoked distance-from-the-wall scaling
- The question of a logarithmic mean profile depends on the properties of the hierarchy defined function, $A(\rho)$.

Questions?

DATE 5/02/04 | SCORER | ATTESTED

Maureen Krenicki

HOLES	1	2	3	4	5	6	7	8	9	TOTAL
MEN'S TEE	205	255	265	335	315	133	145	170	185	2008
MEN'S HANDICAP STROKES	6	4	5	1	2	9	7	8	3	
MEN'S PAR	3	4	4	4	4	3	3	3	3	31
Joe	4^2	4^2	4^2	4^2	5^2	1^0	3^2	3^1	3^2	31
+/−										
WOMEN'S PAR	4	4	4	4	4	3	3	3	3	32
WOMEN'S HANDICAP STROKES	6	4	5	1	2	9	7	8	3	
WOMEN'S YARDAGE	180	230	250	305	275	120	115	146	180	1801